# Linear Algebra in Quantum Information Theory

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The past two decades have witnessed a wide range of fundamental discoveries in quantum information science. These range from protocols revolutionizing public-key cryptography to novel algorithms and tools for communication, information processing, and simulation of physical systems. Even if the mathematical context of quantum information science is wide and multidisciplinary, linear algebra covers a major role, if not ubiquitous. In fact, by the standard formulation of quantum mechanics, physical states and their dynamics are both represented by matrices. The classification of quantum states, schemes for error-correcting codes, methods for allocating quantum resources, promising models of implementable computation, all need a vast number of linear algebraic notions and techniques. This minisymposium is intended as a workshop for strengthening communication between quantum information scientists and the linear algebra community. The minisymposium is a great occasion to present open problems and foster collaborations.

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**Characterization of circulant graphs having perfect state transfer**

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*Fri 15:50, Auditorium*

In this paper we answer the question of when circulant quantum spin networks with nearest-neighbor couplings can give perfect state transfer. The network is described by a circulant graph $G$, which is characterized by its circulant adjacency matrix $A$. Formally, we say that there exists a *perfect state transfer* (PST) between vertices $a, b \in V(G)$ if $|F(\tau)_{ab}| = 1$, for some positive real number $\tau$, where $F(t) = \exp(\imath At)$. Saxena, Severini, Shparlinski in [3]proved that $|F(\tau)_{aa}| = 1$ for some $a \in V(G)$ and $\tau \in \mathbb{R}^+$ if and only if all eigenvalues of $G$ are integer (that is, the graph is integral). The integral circulant graph $\mathrm{ICG}_n(D)$ has the vertex set $Z_n = \{0, 1, 2, \ldots, n-1\}$ and vertices $a$ and $b$ are adjacent if $\gcd(a-b, n) \in D$, where $D \subseteq \{d : d \mid n,\ 1 \le d < n\}$. These graphs are highly symmetric and have important applications in chemical graph theory. We show that $\mathrm{ICG}_n(D)$ has PST if and only if $n \in 4\mathbb{N}$ and $D = D_3 \cup D_2 \cup 2D_2 \cup 4D_2 \cup \{n/2^a\}$, where $D_3 \subseteq \{d : d \mid n,\ n/d \in 8\mathbb{N}\}$, $D_2 \subseteq \{d : d \mid n,\ n/d \in 8\mathbb{N}+4\} \setminus \{n/4\}$ and $a \in \{1, 2\}$. We have thus answered the question of complete characterization of perfect state transfer in integral circulant graphs raised in [1]Furthermore, we also calculate perfect quantum communication distance (distance between vertices where PST occurs) and describe the spectra of integral circulant graphs having PST. For $n \in 4\mathbb{N}$ classes of $\mathrm{ICG}_n(D)$ such that PST exists between non-antipodal vertices are characterized. This answers a question posed by Godsil in [2]. We conclude by giving a closed form expression calculating the number of integral circulant graphs of a given order having PST.

[1] R.J. Angeles-Canul, R.M. Norton, M.C. Opperman, C.C. Paribello, M.C. Russell, C. Tamonk, *Perfect state transfer, integral circulants and join of graphs*, Quantum Information and Computation, Vol. 10, No. 3&4 (2010) 0325–0342.
[2] C.D. Godsil, *Periodic Graphs*, arXiv:0806.2074v1 [math.CO] 12 Jun 2008.
[3] N. Saxena, S. Severini, I. Shparlinski, *Parameters of integral circulant graphs and periodic quantum dynamics*, International Journal of Quantum Information 5 (2007), 417–430.

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**Indirect Hamiltonian Estimation**

Daniel Burgarth, Imperial College London

*Fri 15:00, Auditorium*

It is well known that certain matrices with band structure are uniquely determined by their spectrum and only a few components of their eigenstates [1]. Recently, these methods have been applied to quantum spin models, demonstrating that Hamiltonian tomography can be performed indirectly [2-5]. From the perspective of quantum information this is useful because the standard process tomography is very inefficient. Further graph theoretical criteria were developed that tell us which components of the eigenstates need to be known in order to infer the full matrix (i.e., Hamiltonian) [4,6]. We review such methods and show how they can be generalized to arbitrary quadratic Hamiltonians of bosons or fermions [7].

[1] G. M. L. Gladwell, Inverse Problems in Vibration (Kluver, Dordrecht, 2004).
[2] D. Burgarth, K. Maruyama and F. Nori, Phys. Rev. A 79, 020305(R) (2009).
[3] C. Di Franco, M. Paternostro, and M. S. Kim, Phys. Rev. Lett. 102, 187203 (2009).
[4] D. Burgarth and K. Maruyama, New J. Phys. 11, 103019 (2009).
[5] M. Wiesniak and M. Markiewicz, arXiv:0911.3579v1.
[6] F. Barioli, W. Barrett, S. M. Fallat, H. T. Hall, L. Hogben, B. Shader, P. V.
D. Driessche and H. V. D. Holst, to appear in Lin. Alg. App.
[7] D. Burgarth, K. Maruyama and F. Nori, in preparation.

Joint work with K. Maruyama and F. Nori (RIKEN, Japan)

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**A quantum algorithm for linear systems of equations**

Aram Harrow, University of Bristol and Massachusetts Institute of Technology

*Thu 16:45, Auditorium*

Solving linear systems of equations is a common problem that arises both on its own and as a subroutine in more complex problems: given a matrix $A$ and a vector $b$, find a vector $x$ such that $Ax = b$. We consider the case where one doesn't need to know the solution $x$ itself, but rather an approximation of the expectation value of some operator associated with $x$, e.g., $x'Mx$ for some matrix $M$. In this case, when $A$ is sparse, $N$ by $N$ and has condition number $\kappa$, classical algorithms can find $x$ and estimate $x'Mx$ in $O(N\sqrt{(\kappa)})$ time. Here, we exhibit a quantum algorithm for this task that runs in $\mathrm{poly}(\log N, \kappa)$ time, an exponential improvement over the best classical algorithm. This talk is based on arXiv:0811.3171

Joint work with Avinatan Hassidim and Seth Lloyd

**Higher-order functions in Quantum Theory**
PAOLO PERINOTTI, University of Pavia

*Thu 17:10, Auditorium*

I will introduce the theory of higher-order functions in Quantum Theory. The main theorems and their application to circuit optimisation problems will be reviewed. I will show how the theory of Quantum Combs provides a proper framework to describe and optimise all quantum algorithms explored so far, but does not exhaust the theory of Quantum Computation. I will exhibit the primitive of quantum switch, and show the problems that such a simple task poses to the characterisation of the full hierarchy of higher order maps.

Joint work with G. Chiribella and G. M. D'Ariano

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**Perfect state transfer in integral circulant graphs**
MARKO PETKOVIC, University of Nis

*Fri 15:25, Auditorium*

The existence of perfect state transfer (PST) in quantum spin networks has been proposed by Christandl et. al. (2004) where they considered simple paths as a potential candidates for the network topology. Furthermore, Saxena, Severini and Shparlinski (2007) considered the networks based on circulant graphs. We extend the result of Saxena, Severini and Shparlinski (2007) and give the simple condition for characterizing all integral circulant graphs (ICGs) having the PST in terms of its eigenvalues. In this paper, it is proven that there exist integral circulant graph with $n$ vertices having perfect state transfer if and only if $4 \mid n$. There are found several classes of integral circulant graphs having perfect state transfer for values of $n$ divisible by 4. Moreover we proved the non-existence of PST for several other classes of integral circulant graphs whose order is divisible by 4. These classes cover the class of graphs where divisor set contains exactly two elements. Obtained results provides the first of two steps in solving the general problem: Which integral circulant graphs have PST?

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**Complex Hadamard matrices and combinatorial designs**
FERENC SZOLLOSI, Central European University, Budapest

*Thu 15:50, Auditorium*

In the first part of the talk we present a design theoretical approach to construct new, previously unknown complex Hadamard matrices of prime orders. Our methods generalize and extend the earlier results of de la Harpe–Jones [1] and Munemasa–Watatani [2] and offer a theoretical explanation for the existence of some sporadic examples of complex Hadamard matrices in the existing literature. In the second part we obtain equiangular tight frames of square orders from complex Hadamard matrices settling a recent question of Bodmann et al [3].

[1] P. de la Harpe and V.F.R. Jones, "Paires de sous-algèbres semi-simples et graphes fortement réguliers," C.R. Acad. Sci. Paris 311, Série I, (1990), 147-150.
[2] A. Munemasa, Y. Watatani, Orthogonal pairs of *-subalgebras and association schemes, C.R. Acad. Sci. Paris 314, Série I, 329-331 (1992).
[3] B.G. Bodmann, V.I. Paulsen and M. Tomforde, Equiangular tight frames from complex seidel matrices containing cube roots of unity, Linear Algebra Appl. 430 (2009), pp. 396–417.

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**Continuous families of complex (generalized) Hadamard matrices**
WOJCIECH TADEJ, Cardinal Stefan Wyszyński University, Warsaw, Poland
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*Thu 15:25, Auditorium*

An $N \times N$ complex Hadamard matrix is a matrix with orthogonal rows and columns (after rescaling a unitary) with all entries of modulus equal to one. The search for these is a special case of the search for unitary preimages of doubly stochastic matrices, which is of importance in particle physics. Complex Hadamard matrices have additional applications of their own, in particular in quantum information theory.

In this talk we present the currently known classification of complex Hadamard matrices of small size, which includes various continuous families. Also, results and hypotheses concerning construction of so called affine Hadamard families will be presented. By affine we mean a family where matrices of parametrizing phases form a linear subspace of the space of all real $N \times N$ matrices.

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**Zero-error communication via quantum channels, non-commutative graphs and a quantum Lovasz theta function**
ANDREAS WINTER, University of Bristol and National University of Singapore

*Thu 17:35, Auditorium*

We present the quantum channel version of Shannon's zero-error capacity problem. Motivated by recent progress on this question, we propose to consider a certain operator space as the quantum generalisation of the adjacency matrix, in terms of which the plain, quantum and entanglement-assisted capacity can be formulated, and for which we show some new basic properties. Most importantly, we define a quantum version of Lovasz' famous theta function, as the norm-completion (or stabilisation) of a "naive" generalisation of theta. We go on to show that this function upper bounds the number of entanglement-assisted zero-error messages, that it is given by a semidefinite programme, whose dual we write down explicitly, and that it is multiplicative with respect to the natural (strong) graph product. We explore various other properties of the new quantity, which reduces to Lovasz' original theta in the classical case, give several applications, and propose to study the operator spaces associated to channels as "non-commutative graphs", using the language of Hilbert modules. The talk is based on arXiv:1002.2514v2 [quant-ph]

Joint work with Runyao Duan, Simone Severini

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**Generalized numerical range as a versatile tool in the theory of quantum information**
KAROL ZYCZKOWSKI, Jagiellonian University, and Center for Theoretical Physics, Warsaw

*Thu 15:00, Auditorium*

We study operators acting on a composite Hilbert space and investigate their product numerical range, product spectral radius and product $C$–spectral radius. For any Hermitian operator $X$ acting on a bi-partite Hilbert space its product numerical range is formed by the set of all possible expectation values of $X$ among pure product states, $\langle\phi| \otimes \langle\psi|X|\psi\rangle \otimes |\phi\rangle$. Concrete bounds for the product numerical range for Hermitian operators are derived. Product numerical range of a

non-Hermitian operator forms a subset of the standard numerical range. While the latter set is convex, the product range needs not to be convex nor simply connected. Product numerical range of a tensor product is equal to the Minkowski product of numerical ranges of individual factors. As an exemplary application of these algebraic tools in the theory of quantum information we study block positive matrices and entanglement witnesses. Furthermore, we apply product numerical range to solve the problem of local distinguishability of a family of two unitary gates. Product $C$–spectral radius is useful for finding local fidelity between two states of a composite system, while higher order product numerical range can be used to design local quantum dark spaces and local error correction codes.