Nonlinear Eigenvalue Problems

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A variety of applications in science and engineering lead to eigenvalue problems that are nonlinear in the eigenvalue parameter. This includes polynomial, rational, as well as genuinely nonlinear eigenvalue problems. In recent years, tremendous progress has been made in addressing such eigenvalue problems, both on the theoretical and the computational side. The purpose of this minisymposium is to survey these developments and point out new directions in this area. A range of topics will be covered, including linearization, perturbation theory, structure preservation, numerical methods and emerging applications such as photonic band structure calculation.

Classification of Hermitian Matrix Polynomials with Real Eigenvalues of Definite Type

M. Al-Ammari, The University of Manchester, UK
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Fri 11:00, Room Fermi

The spectral properties of Hermitian matrix polynomials with real eigenvalues have been extensively studied, through classes such as the definite or definitizable pencils, definite, hyperbolic, or quasihyperbolic matrix polynomials, and overdamped or gyroscopically stabilized quadratics. We give a unified treatment of these and related classes that uses the eigenvalue type (or sign characteristic) as a common thread. Equivalent conditions are given for each class in a consistent format. We show that these classes form a hierarchy, all of which are contained in the new class of quasidefinite matrix polynomials. As well as collecting and unifying existing results, we propose a new characterization of hyperbolicity in terms of the distribution of the eigenvalue types on the real line. By analyzing their effect on eigenvalue type, we show that homogeneous rotations allow results for matrix polynomials with nonsingular or definite leading coefficient to be translated into results with no such requirement on the leading coefficient, which is important for treating definite and quasidefinite polynomials.

[1] M. Al-Ammari and F. Tisseur. Hermitian matrix polynomials with real eigenvalues of definite type- part I: Classification, MIMS Eprint 2010.9, Manchester Institute for Mathematical Sciences, The University of Manchester, UK, Jan. 2010. 24 pp.
[2] D. S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann. Vector spaces of linearizations for matrix polynomials. SIAM J. Matrix Anal. Appl., 28(4):971–1004, 2006.

Joint work with F. Tisseur (The University of Manchester)

Eigenvalue enclosures for the Dirac operator

L. Boulton, Heriot-Watt University, United Kingdom
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Fri 15:50, Room Fermi

The variational formulation of spectral problems associated to relativistic and non-relativistic atomic structures leads to rigorous upper bounds for the energy eigenvalues. This approach has proven to be highly valuable in the numerical estimation of these eigenvalues by finite-basis projection methods. Less effort has been devoted to the investigation of rigourous lower bounds for the spectrum of hamiltonians. This is due, in part, to the fact that most available techniques require a priori information, not usually at hand, about the problem being considered. Moreover, these procedures are often several orders of magnitude less accurate than their “upper-bound” counterparts. In this talk we will report on rigourous methods for computation of eigenvalue enclosures of Dirac operators. We will demonstrate the applicability of these techniques and will show how matrix polynomials and functions arise naturally in them. We will also report on outcomes of various numerical experiments performed on benchmark potentials.

Joint work with J. Dolbeault (Université Paris Dauphine)

Linearizations of rectangular matrix polynomials

Fernando De Terán, Universidad Carlos III de Madrid, Spain
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Thu 17:10, Room Fermi

Linearizations of regular matrix polynomials have been widely studied and they have shown to be a useful tool in several areas including the Polynomial Eigenvalue Problem. Also, linearizations of square singular matrix polynomials have been recently studied by the authors in a series of papers. By contrast, very little is known about linearizations of rectangular matrix polynomials. In this talk, we will present some results regarding general linearizations of rectangular matrix polynomials and we will also introduce a new family of strong linearizations of rectangular polynomials extending the family of *Fiedler pencils*. This family, which includes the *first* and *second companion forms*, was introduced in [1] for regular matrix polynomials and later extended in [2] to square singular polynomials. We will show that this family of linearizations enjoys several interesting properties that may be useful for future applications.

[1] E. N. Antoniou and S. Vologiannidis, A new family of companion forms of polynomial matrices, Electron. J. Linear Algebra, 11 (2004), pp. 78–87.
[2] F. De Terán, F. M. Dopico, and D. S. Mackey, Fiedler companion linearizations and the recovery of minimal indices, submitted to SIAM J. Matrix Anal. Appl.

Joint work with Froilán M. Dopico (Universidad Carlos III de Madrid) and D. Steven Mackey (Western Michigan University)

Generic spectral perturbation results for matrix polynomials

Froilán M. Dopico, Universidad Carlos III de Madrid, Leganés, Spain
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Thu 17:35, Room Fermi

In this talk, we deal with two spectral perturbation problems for matrix polynomials. In the first one, we consider the change of the elementary divisors of a regular matrix polynomial under a perturbation of low rank, while in the second one, we consider the first order perturbation term in the perturbation expansions of the eigenvalues of a square singular matrix polynomial. A common feature of these two problems is that, although the behavior of the considered magnitudes under an arbitrary perturbation may be very complicated, the “generic” behavior can be described in a compact and sharp way, where by "generic" behavior we understand the one that holds for all perturbations except those in a proper algebraic manifold of zero measure in the set of perturbations.

Joint work with Fernando De Terán (Universidad Carlos III de Madrid)

On nonlinear eigenvalue problems with applications to absorptive photonic crystals

C. Engström, ETH Zurich, Switzerland
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Fri 15:00, Room Fermi

Dielectric and metallic photonic crystals are promising materials for controlling and manipulating electromagnetic waves [1]. For frequency independent material models considerable mathematical progress has been made [2]. In the frequency dependent case, however, the nonlinearity of the spectral problem complicates the analysis. We study the spectrum of a scalar operator-valued function with periodic coefficients, which after application of the Floquet transform become a family of spectral problems on the torus. The frequency dependence of the material parameters lead to spectral analysis of a family of holomorphic operator-valued functions. We show that the spectrum for a passive material model consists of isolated eigenvalues of finite geometrical multiplicity. These eigenvalues depend continuously on the quasi momentum and all non-zero eigenvalues have a non-zero imaginary part whenever losses (absorption) occur [3].

Lorentz permittivity model, which is a common model for solid materials, lead to a rational eigenvalue problem. We study both the self-adjoint case and the non-self-adjoint case. Moreover, a high-order discontinuous Galerkin method is used to discretize the operator-valued function, and the resulting matrix problem is transformed into a linear eigenvalue problem. Finally, we use an implicitly restarted Arnoldi method to compute approximate eigenpairs of the sparse matrix problem.

[1] K. Sakoda, *Optical properties of photonic crystals*, Springer-Verlag, Heidelberg, 2001.
[2] P. Kuchment, *Floquet theory for partial differential equations*, Birkhäuser, Basel, 1993.
[3] C. Engström, *On the spectrum of a holomorphic operator-valued function with applications to absorptive photonic crystals*, To appear.

Computation and continuation of invariant pairs for polynomial and nonlinear eigenvalue problems

D. Kressner, ETH Zurich, Switzerland
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Fri 15:25, Room Fermi

We consider matrix eigenvalue problems that are polynomial or genuinely nonlinear in the eigenvalue parameter. One of the most fundamental differences to the linear case is that distinct eigenvalues may have linearly dependent eigenvectors or even share the same eigenvector. This can be a severe hindrance in the development of general numerical schemes for computing several eigenvalues of a polynomial or nonlinear eigenvalue problem, either simultaneously or subsequently. The purpose of this talk is to show that the concept of invariant pairs offers a way of representing eigenvalues and eigenvectors that is insensitive to this phenomenon. We will demonstrate the use of this concept with a number of numerical examples and discuss continuation methods for invariant pairs.

[1] T. Betcke and D. Kressner. Perturbation, Computation and Refinement of Invariant Pairs for Matrix Polynomials. Technical report 2009-21, Seminar for applied mathematics, ETH Zurich, July 2009. Revised February 2010.
[2] D. Kressner. A block Newton method for nonlinear eigenvalue problems. Numer. Math., 114(2):355–372, 2009.

Joint work with Timo Betcke (University of Reading)

Leave it to Smith: Canonical Forms for Structured Matrix Polynomials, Part II

Niloufer Mackey, Western Michigan University, Kalamazoo, USA
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Thu 15:25, Room Fermi

Polynomial eigenvalue problems arise in many applications, and often the underlying matrix polynomial P is structured in some way. A much used computational approach to such problems starts with a linearization such as the companion form of P, and then applies a general purpose algorithm to the linearization. But when P is structured, it can be advantageous to use a linearization with the same structure as P, if one can be found. It turns out that there are structured polynomials for which a linearization with the same structure does not exist. Using the Smith form as the central tool, we describe which matrix polynomials from the classes of alternating, palindromic, and skew-symmetric polynomials allow a linearization with the same structure.

[1] D. S. Mackey, N. Mackey, C. Mehl, V. Mehrmann, Jordan Structures of Alternating Matrix Polynomials, Linear Alg. Appl., v. 432:4, pp. 867–891, 2010.
[2] D. S. Mackey, N. Mackey, C. Mehl, V. Mehrmann, Smith Forms of Palindromic Matrix Polynomials, In preparation.

Joint work with D. Steven Mackey (Western Michigan University), Christian Mehl (Technische Universität Berlin), Volker Mehrmann (Technische Universität Berlin).

Spectral Equivalence and the Rank Theorem for Matrix Polynomials

D. Steven Mackey, Western Michigan University, Kalamazoo, MI, USA
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Thu 16:45, Room Fermi

We investigate the extent to which two matrix polynomials of different sizes and degrees, regular or singular, square or rectangular, can have the same (scalar) spectral data, i.e., the same finite and infinite elementary divisors. The classical example of this phenomenon is the well-known concept of strong linearization. Taking this example as a prototype, we introduce the notion of spectral equivalence, describe its basic properties, and give a variety of examples. For matrix polynomials $P(\lambda)$ that are singular, the minimal indices of $P(\lambda)$ are another type of scalar spectral-like data that encode important properties of the left and right nullspaces of $P(\lambda)$. When two singular matrix polynomials are spectrally equivalent, what are the possible relationships between their minimal indices? For example, can they be equal? In trying to answer these questions, we prove the Rank Theorem for Matrix Polynomials, a simple but fundamental relation between elementary divisors, minimal indices, and rank that holds for any matrix polynomial.

Earlier results analyzing the relationship between the minimal indices of a singular polynomial and those of several particular classes of strong linearization can be found in the recent papers [1] and [2].

[1] F. De Terán, F.M. Dopico, and D.S. Mackey, Linearizations of singular matrix polynomials and the recovery of minimal indices, Electron. J. Linear Alg., 18 (2009), pp. 371–402.
[2] F. De Terán, F.M. Dopico, and D.S. Mackey, Fiedler companion linearizations and the recovery of minimal indices, submitted to SIAM J. Matrix Anal. Appl.

Joint work with Fernando De Terán (Universidad Carlos III de Madrid) and Froilán M. Dopico (Universidad Carlos III de Madrid).

Leave it to Smith: Canonical Forms for Structured Matrix Polynomials, Part I

Christian Mehl, Technische Universität Berlin, Germany
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Thu 15:00, Room Fermi

Polynomial eigenvalue problems arise in many applications, and often the underlying matrix polynomial P is structured in some way. A much used computational approach to such problems starts with a linearization such as the companion form of P, and then applies a general purpose algorithm to the linearization. But when P is structured, it can be advantageous to use a linearization with the same structure as P, if one can be found. It turns out that there are structured polynomials for which a linearization with the same structure does not exist. Using the Smith form as the central tool, we describe which matrix polynomials from the classes of alternating, palindromic, and skew-symmetric polynomials allow a linearization with the same structure.

[1] D. S. Mackey, N. Mackey, C. Mehl, V. Mehrmann, Jordan Structures of Alternating Matrix Polynomials, Linear Alg. Appl., v. 432:4, pp. 867–891, 2010.
[2] D. S. Mackey, N. Mackey, C. Mehl, V. Mehrmann, Smith Forms of Palindromic Matrix Polynomials, In preparation.

Joint work with D. Steven Mackey (Western Michigan University), Niloufer Mackey (Western Michigan University), Volker Mehrmann (Technische Universität Berlin).

Nonlinear eigenvalue problems in acoustic field computation

V. Mehrmann, TU Berlin, Germany
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Fri 12:15, Room Fermi

We will discuss the numerical solution of large scale parametric eigenvalue problems arising in acoustic field problems. In current industrial applications a few eigenvalues in a specified region of the complex plane have be computed for nonlinear eigenvalue problems with several million degrees of freedom within an optimization loop. Based on geometric, topological or material changes the acoustic field within modern cars is then optimized on the basis of the eigenvalue computations.

We will discuss the currently used methods and their properties from a numerical and computational point of view. In particular we discuss homotopy methods and eigenvalue continuation techniques.

Joint work with T. Baumgarten (TU Berlin), C. Schröder (TU Berlin)

A "shift-and-deflate" technique for matrix polynomials

B. Meini, University of Pisa, Italy
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Fri 11:25, Room Fermi

Let $P(z)=\sum_{i=0}^n A_i z^i$ be a regular $k\times k$ matrix polynomial of degree n, and let $\lambda\in\mathbb{C}$, $u\in\mathbb{C}^k$, $u\neq 0$ such that $P(\lambda)u=0$. Given $\mu\in\mathbb{C}\cup\{\infty\}$, the shift technique introduced in [2] allows to trasform the matrix polynomial $P(z)$ into a new $k\times k$ matrix polynomial $\tilde P(z)$ of degree n such that $\tilde P(\mu)u=0$. That is, the root λ of $P(z)$ is *shifted* to the root μ of $\tilde P(z)$, and the remaining roots are kept unchanged. In [1] the authors show how to deflate a couple of known roots of a quadratic matrix polynomial $P(z)$, by trasforming $P(z)$ into a $(k-1)\times(k-1)$ matrix polynomial $\hat P(z)$, having as roots the unknown roots of $P(z)$. The aim of this talk is to show how the shift technique can be used to the same purpose in a much simpler way. Moreover, if $P(z)$ has a specific structure, like symmetric matrix coefficients, or palindromic structure, then the matrix polynomial $\hat P(z)$ can be constructed with the same structure of the original polynomial.

[1] D. Garvey, C. Munro, and F. Tisseur. Deflating Quadratic Matrix Polynomials with Structure Preserving Transformations. MIMS EPrint 2009.22, March 2009.
[2] C. He, B. Meini, and N. H. Rhee. A shifted cyclic reduction algorithm for quasi-birth-death problems. *SIAM J. Matrix Anal. Appl.*, 23(3):673–691, 2001/02.

Structure Preserving Transformations for Quadratic Matrix Polynomials

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Thu 15:50, Room Fermi

A structure preserving transformations (SPT) is a map transforming a quadratic matrix polynomial $Q(\lambda)=\lambda^2A_2+\lambda A_1+A_0$ into a new quadratic $\widetilde Q(\lambda)=\lambda^2\widetilde A_2+\lambda\widetilde A_1+\widetilde A_0$ isospectral to $Q(\lambda)$ (i.e., Q and $\widetilde Q$ have the same Jordan canonical form). Two essential points are

1. An SPT does not act on the polynomial Q: it is defined as an action on a linearization L of Q.
2. Computationally, an SPT can be applied by working only with the $n\times n$ coefficient matrices of Q, avoiding computations on the larger pencil L.

In this talk we describe the concept of SPTs and present recent developments involving them. SPTs are a novel and promising approach to solving quadratic eigenproblems.

Nonlinear low rank modification of a symmetric eigenvalue problem

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Fri 11:50, Room Fermi

In a recent report Huang, Bai and Su [1] studied existence and uniqueness results and interlacing properties of nonlinear rank-one modifications of symmetric eigenvalue problem. In this talk we generalize the uniqueness conditions and we discuss generalizations to low rank modifications. Based on approximation properties of the Rayleigh functional we design numerical methods the local convergence of which are quadratic or even cubic. Numerical examples demonstrate their efficiency. We further consider low rank modifications of hyperbolic quadratic eigenvalue problems and more general nonlinear eigenvalue problems allowing for a minmax characterization of their eigenvalues.

[1] X. Huang, Z. Bai, Y. Su, Nonlinear rank-one modification of a symmetric eigenvalue problem, Technical Report, UC Davis, 2009, to appear in Math. Comp.
[2] H. Voss, K. Yildiztekin, Nonlinear low rank modification of a symmetric eigenvalue problem, Technical Report, Hamburg University of Technology, 2009, Submitted to SIAm J. Matrix Anal. Appl.

Joint work with K. Yildiztekin (Hamburg University of Technology)