Combinatorial Linear Algebra

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This minisymposium will highlight recent advances in the use of linear algebra to reveal the intrinsic combinatorial structure of matrices described by graphs and digraphs; and the use of graph theory in developing deeper algebraic and analytic theory for matrices that incorporates the underlying structure of the matrix.

The spectral radius and the diameter of connected graphs

Sebastian M. Cioabă, University of Delaware
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Mon 11:00, Room C

In this talk, I will discuss the problem of determining the minimum spectral radius of order n and diameter D. I will focus on the cases when D is constant and when D grows linearly with n.

[1] S.M. Cioabă, E. van Dam, J. Koolen and J.H. Lee, Asymptotic results on the spectral radius and the diameter of graphs, *Linear Algebra and its Applications*, **432** 722-737, (2010).
[2] S.M. Cioabă, E. van Dam, J. Koolen and J.H. Lee, A lower bound for the spectral radius of graphs with fixed diameter, *European Journal of Combinatorics*, to appear.

Joint work with Edwin van Dam (Tilburg University), Jack Koolen (POSTECH) and Jae-Hoo Lee (University of Wisconsin-Madison)

Majorization permutahedra and $(0,1)$-matrices

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Tue 11:50, Room C

Let $x_{[j]}$ denote the jth largest component of a real vector x. For vectors $x, v \in \Re^n$ one says that x is *majorized* by v ([1], [2]), denoted by $x \preceq v$, provided that $\sum_{j=1}^k x_{[j]} \leq \sum_{j=1}^k v_{[j]}$ for $k = 1, \ldots, n$ where there is equality for $k = n$. A *majorization permutahedron* $M(v)$ is a polytope associated with a majorization $x \preceq v$ in $\Re^n$, defined by $M(v) = \{x \in \Re^n : x \preceq v\}$. By Rado's theorem ([2]) $M(v)$ is the convex hull of all permutations of v. Several properties of these polytopes are investigated and a connection to discrete convexity is established. These results are used to obtain a generalization of the Gale-Ryser theorem for $(0,1)$-matrices with given line sums.

[1] R.A. Brualdi, *Combinatorial Matrix Classes*, Encyclopedia of Mathematics, Cambridge University Press. 2006.
[2] A.W. Marshall and I. Olkin, *Inequalities: Theory of Majorization and Its Applications*, Academic Press, New York, 1979.

Why are minimum rank of graph problems interesting? (In my opinion)

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Mon 15:00, Room C

Given a graph $G = (V, E)$ on n vertices, we may associate a number of collections of matrices whose zero-nonzero pattern is constrained in some fashion by the edges of G. For example, let $S(G)$ be the set of all real symmetric matrices $A = [a_{ij}]$, such that if $i \neq j$, then $a_{ij} \neq 0$ iff $\{i, j\} \in E$. The parameter $mr(G)$ defined as the $\min\{rank(A) : A \in S(G)\}$ is known as the *minimum rank of* G (with respect to $S(G)$). Other notions of "minimum rank" may be defined in a similar manner. It is striking that for many different classes of graphs, notions of minimum rank are intimately connected with purely combinatorial graph parameters for that class. I intend to survey a number of results (new and old) along these lines to offer my perspective on why I think minimum rank parameters are interesting and worthwhile.

On the connection between weighted graphs and independence number

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Tue 12:15, Room C

In this paper we generalize the concept of the Merris index of a graph by considering weighted Laplacians and obtain a better upper bound for the independence number, namely, the minimum, on all possible weights, of the Merris index. We refer to this bound as weighted Merris index and show that in many cases, for example, regular bipartite graphs, it is equal to the independence number. Complete graphs are an example of a strict inequality. In order to construct graphs for which equality holds we study what happens to such graphs when an edge or a vertex are added, and find sufficient conditions for equality for the new graphs.. We also give some insights, using the independence number, on the vertices that are contained in the maximal independence set.

[1] Bojan Mohar, "Graph Theory, Combinatorics, and Applications", Vol. 2, Ed. Y. Alavi, G. Chartrand, O. R. Oellermann, A. J. Schwenk, Wiley, 1991, pp. 871-898.
[2] Felix Goldberg and Gregory Shapiro, "The Merris index of a graph", Electronic Journal of Linear Algebra, vol. 10 (2003), pp. 212-222.
[3] Kinkar Ch. Das, R.B. Bapat, " A sharp upper bound on the largest Laplacian eigenvalue of weighted graphs", Linear Algebra and its Applications 409 (2005) 153-165
[4] Roger A.Horn and Charles R.Johnson, Matrix Analysis,1985 pp.181
[5] Russell Merris. Laplacian matrices of graphs: a survey. Linear Algebra Appl., 197/8:143-176, 1994.
[6] W.N. Anderson, T.D. Morley, "Eigenvalues of the Laplacian of a graph", Lin. Multilin. Algebra 18 (1985) 141-145.

Joint work with Abraham Berman (Technion - Israel Institute of Technology)

Graphs cospectral with Kneser graphs.

Willem Haemers, Tilburg University
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Mon 11:25, Room C

An important problem in spectral graph theory is to decide which graphs are determined by the spectrum. In this talk we consider the famous Kneser graphs $K(n, k)$. The main result is the construction of graphs cospectral but nonisomorphic to $K(n, k)$ when $n = 3k - 1$, $k > 2$ and for infinitely many other pairs (n, k). We also consider related graphs in the Johnson association scheme.

Joint work with Farzaneh Ramezani(IPM)

Average minimum rank of graphs of fixed order

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The minimum rank of a simple graph G is defined to be the smallest possible rank over all real symmetric matrices whose ijth entry (for $i \neq j$) is nonzero whenever $\{i,j\}$ is an edge in G and is zero otherwise. The average minimum rank over all labeled graphs of order n is investigated by determining bounds for the expected value of minimum rank of $G(n, \frac{1}{2})$, the usual Erdős-Rényi random graph on n vertices with edge probability $\frac{1}{2}$.

Joint work with Tracy Hall (Brigham Young University), Ryan Martin (Iowa State University), Bryan Shader (University of Wyoming)

Eigenvalues, Multiplicities and Graphs: An Update

Charles R. Johnson, College of William and Mary
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We survey recent results about the possible lists of multiplicities occurring among the eigenvalues of a Hermitian matrix with a given graph. Some interesting problems will be mentioned.

The minimum rank of a graph containing a k-clique

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Tue 11:00, Room C

Let G be an undirected graph on n vertices and let F be a field. We denote by $S(F,G)$ the set of all $n \times n$ symmetric matrices with entries in F and whose graph is G, and by $mr(F,G)$ the minimum rank of all matrices in $S(F,G)$. In this talk we consider $mr(F,G)$ when G contains a k-clique. It is known that if F is an infinite field then $mr(F,G) \leq n-k+1$. The validity of this upper bound for $mr(F,G)$ when F is a finite field is discussed.

Cut-norms and spectra of matrices

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In 1997, Frieze and Kannan introduced and studied the cut-norm $\|A\|_\square$ of an $m \times n$ matrix $A = [a_{ij}]$, defined by

$$\|A\|_\square = \max_{X\subset[m],\ Y\subset[n]} \frac{1}{mn} \left| \sum_{i\in X, j\in Y} a_{ij} \right|.$$

Ever since then this parameter kept getting new attention. This talk presents inequalities between two versions of the cut-norm and the two largest singular values of arbitrary complex matrices. These results extend, in particular, the well-known graph-theoretical Expander Mixing Lemma and give a hitherto unknown converse of it. Furthermore, they imply a solution of a problem of Lovász, and give a spectral sampling theorem, which informally states that almost all principal submatrices of a real symmetric matrix are spectrally similar to it.

Eigenvalues and ordered multiplicities for matrices associated with a graph

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For a given tree T let $\mathcal{S}(T)$ denote the set of all symmetric matrices whose graph is T. A question for a given tree T is the following Inverse Eigenvalue Problem: If T has n vertices, exactly which sets of n real numbers (including multiplicities) occur as the spectrum of A for some $A \in \mathcal{S}(T)$. Another problem for T is the characterization of the lists of multiplicities, ordered by numerical order of the underlying eigenvalues (ordered multiplicities), that occur among matrices in $\mathcal{S}(T)$. Though the solution for these two problems is known for certain classes of trees (see [1], [2], [3]), the problem is, in general, open. When T is either a generalized star or a double generalized star, such two problems are equivalent, i.e., the only constraint on existence of a matrix in $\mathcal{S}(T)$ with prescribed spectrum is the existence of the corresponding list of ordered multiplicities.

The following simple observation is the purpose of this presentation: Given the spectrum of an n-by-n symmetric matrix B whose graph is a tree, there is a double generalized star T and a matrix $A \in \mathcal{S}(T)$ with the same spectrum as B.

[1] C. R. Johnson and A. Leal-Duarte. On the possible multiplicities of the eigenvalues of an Hermitian matrix whose graph is a given tree. *Linear Algebra and Its Applications* 348:7–21 (2002).
[2] C. R. Johnson, A. Leal-Duarte and C. M. Saiago. Inverse eigenvalue problems and lists of multiplicities of eigenvalues for matrices whose graph is a tree: the case of generalized stars and double generalized stars. *Linear Algebra and Its Applications* 373:311–330 (2003).
[3] Francesco Barioli and Shaun Fallat. On the eigenvalues of generalized and double generalized stars. *Linear Multilinear Algebra* 53(4):269–291 (2005).

Integral, square-integral graphs and perfect state transfer

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Mon 11:50, Room C

It has been shown earlier that the necessary condition for the existence of a perfect state transfer in a quantum spin network is that the whole adjacency spectrum of the underlying graph has the form $a_1\sqrt{b}, a_2\sqrt{b}, \ldots, a_n\sqrt{b}$, for some integers $a_1, \ldots, a_n$ and b. We will call such graphs the *square-integral* graphs. Note that for $b = 1$ we get the usual integral graphs.

In the talk we will survey the known results on graphs with perfect state transfer, and then determine which of the known 4-regular integral graphs and the semiregular bipartite square-integral graphs with small vertex degree have perfect state transfer. We will also determine the conditions which ensure that the perfect state transfer property is preserved under NEPS of graphs.