Transient Analysis of Fluid Models via Elementary Level Crossing Arguments

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Abstract

An analysis of the time dependent evolution of the canonical Markov modulated fluid flow model is presented using elementary level crossing arguments.

Key words : Fluid-flow, Transient Results, matrix-geometric method.

1 Introduction

We consider the canonical Markov modulated fluid flow (MMFF)

$$(\mathcal{F}, \mathcal{J}) = \{(F(t), J(t)) : \ t \geq 0\};$$

here, $F(t)$ is the level of fluid at time t in an unbounded fluid buffer, and $J(t)$ – also called the phase at time t – is the state at time $t+$ of a (finite state) continuous time Markov chain (CTMC) modulating the fluid process. The fluid level changes in a piece-wise linear fashion over time, and the instantaneous rate of change of the fluid level at time t depends on the state $J(t)$ of the modulating CTMC. Such models arise in a wide variety of areas like the theories of queues and dams, storage processes, risk theory, etc; see references [6], [7], [20] and citations therein. To keep the paper short, we restrict ourselves to the case where the fluid buffer is unbounded. For a rigorous treatment of the bounded case, we refer to Ahn & Ramaswami [4]; see Soares & Latouche [22] for a less formal treatment.

The transient (time-dependent) analysis of the fluid flow model has been a challenging problem. Early approaches to stochastic fluid flows have been based on matrix Wiener-Hopf factorizations (Rogers [18]), Siegmund duality (Asmussen [7]), spectral decomposition (Anick et al [6], Kobayashi & Ren [12]), and series expansions (Sericola [21]). Recently, Ahn and Ramaswami [2], [3] provided an analysis based on matrix-geometric methods [14], [15] in which a sequence of matrix-geometric queues are constructed as approximations to the fluid flow and then stochastic process limit theorems are used to obtain the (exact) results for the fluid model from those on the matrix-geometric queues. The approach of Ahn and Ramaswami was built on an earlier work of Ramaswami [17] which is the first systematic approach to fluid flows based on matrix-geometric methods, as well as on a stochastic discretization introduced by Adan & Resing [1]. Unfortunately, none of these approaches is elementary, particularly for class room use, and our main purpose here is to provide an elementary exposition. We shall also extend our earlier formulas derived under the assumption $F(0) = 0$ to an arbitrary initial fluid level.

With the power of hindsight, we can note that the approach presented here is none other than that originally envisaged by Ramaswami [17] for steady state analysis using level crossing methods and extended now to time dependent analysis. The elementary exposition here implicitly relies on (and hides) certain results on Markov renewal processes on arbitrary state spaces that are needed, and the work of Ahn and Ramaswami [3] can now be viewed as providing a rigorous development of those results using a stochastic discretization of the continuous component of the state variable. In that, the work in [3], [4] is comparable to that of Athreya & Ney [9] who demonstrated that when the state space is a metric space and the probability measure is sufficiently regular, Markov processes on such spaces can be handled using simple state space discretization methods and by appealing to discrete state space theory. Unlike [9] which uses a deterministic discretization, Ahn & Ramaswami [3], [4] have used stochastic discretization. In this context, it behooves us to bring to the attention of the reader a seminal paper by S.M. Samuels [19] that early on demonstrated that in "nice" spaces, despite popular belief to the contrary, conditional expectations and conditional probabilities can indeed be obtained by discretizing the space and passing on to limits; in light of that basic characterization of the Radon-Nikodym derivative, these results based on discretizations should offer no surprise despite the technicalities they involve.

Being primarily pedagogical in intent and noting that despite their simplicity the results here have gone unnoticed in the vast literature on fluid models, we shall highlight some of the results with remarks despite the fact that these should become obvious to the more advanced reader. These remarks also serve to highlight the technical subtleties assumed in this simple minded approach that necessitate a rigorous treatment as for instance in [3], [4].

2 Notations

Before we proceed, we set some basic notations that will be used throughout. We have used notations compatible with our prior work [2]-[5] so that readers motivated to review the rigorous counterpart of the development given here will not have much difficulty.

The continuous time Markov chain $\mathcal{J} = \{J(t), t \geq 0\}$ is assumed to have a finite state space $S = S_1 \cup S_2 \cup S_3$ and infinitesimal generator Q that, when partitioned according to the sets S_i, has the form

$$Q = \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{pmatrix}. \tag{1}$$

Specifically, the submatrix Q_{ij} contains the elements $Q(r,s)$ of the infinitesimal generator with $r \in S_i$ and $s \in S_j$. [Throughout, for any matrix A, we shall denote its elements by $A(i,j)$ or by $[A]_{ij}$ and reserve the notation A_{ij} for the submatrix of A with row indices in S_i and column indices in S_j. Similarly, for a matrix function $A(s)$, we shall denote its (i,j)-th element by $[A(s)]_{ij}$ and reserve the notation $A_{ij}(s)$ for the submatrix of $A(s)$ formed by row indices in S_i and column indices in S_j.]

We assume as given a set of positive constants $c(i),\ i \in S_1 \cup S_2$ and assume that the fluid process $F(\cdot)$ under consideration evolves in such a way that: during sojourn of $\mathcal{J}$ in state $i \in S_1$, the fluid level increases at rate $c(i)$; during sojourn of $\mathcal{J}$ in state $i \in S_2$, the fluid level decreases at (absolute) rate $c(i)$ provided the fluid level is positive; during sojourn of $\mathcal{J}$ in state $i \in S_3$, the fluid level remains constant. Note that when $J(t) = i \in S_2$, the instantaneous rate of change of $F(\cdot)$ at t is given by $-c(i) < 0$.

For later use, we define diagonal matrices $C_i,\ i = 1,2,3$ and C such that

$$C_1 = diag(c(i),\ i \in S_1),\ C_2 = diag(c(i),\ i \in S_2),\ C_3 = I_{|S_3|}, \tag{2}$$

where I_n is an identity matrix of order n, and

$$C = diag(C_1, C_2, C_3). \tag{3}$$

For a function $f(t,x)$, $\tilde{f}(s,x)$ shall denote the Laplace transform of f with respect to t. Unless otherwise stated, the argument s of such transforms is assumed to be such that $Re(s) > 0$.

We also use the notations $\mathcal{P}_{xi}$ and $\mathcal{E}_{xi}$ to denote the conditional probability and conditional expectation respectively, given that $F(0) = x$ and $J(0) = i$. Finally, ${}_y\mathcal{P}_{xi}$ and ${}_y\mathcal{E}_{xi}$ shall denote the taboo conditional probability and taboo conditional expectation respectively, given that $F(0) = x$ and $J(0) = i$ and taken over paths wherein the MMFF avoids the fluid levels $[0, y]$ (except possibly at time 0.)

3 Busy period analysis

From the structure of the process $(\mathcal{F}, \mathcal{J})$, it is obvious that the successive epochs when the fluid level returns to 0 form a semi-regenerative sequence (see [10] for a definition), whence the time dependent evolution of the MMFF can be determined from its behavior within passage times of the fluid level to zero. Thus, a basic quantity in the analysis of the fluid flow model is the matrix $\Psi(s)$ of order $|S_1| \times |S_2|$, whose (i,j)-th element, $i \in S_1,\ j \in S_2$, is the Laplace-Stieltjes transform (LST)

$$[\Psi(s)]_{ij} = \mathcal{E}_{0i}[e^{-s\tau}\chi\{J(\tau) = j\}],$$

where

$$\tau = \inf\{t > 0\ :\ F(t) = 0\},$$

and $\chi\{A\}$ denotes the indicator function of the set A. τ is often called the busy period of the fluid flow, and $\Psi(s)$ gives the joint distribution of the busy period duration and the state of the environment at the termination of the busy period, under the assumptions that the initial environment is known and that initially the fluid buffer is empty.

We have, in [5], developed a powerful algorithm to compute the matrix of transforms $\Psi(s)$. For the purpose of this paper, we shall assume $\Psi(s)$ as known and derive all quantities of interest in terms of that transform matrix.

3.1 Analysis of $\mathcal{F}$ in a Busy Period

Of interest here is the determination of the probabilities

$$[\mathbf{V}(t,x)]_{ij} = \ {}_0\mathcal{P}_{0i}[F(t) \leq x,\ J(t) = j], \quad i \in S_1,\ j \in S, \tag{4}$$

which is the probability that a busy period starting with the environment state $i \in S_1$ lasts at least t units of time and $(F(t), J(t)) \in [0,x] \times \{j\}$. Later, we shall use these to write down the transient distribution of the fluid model for arbitrary initial conditions and without regard to whether the time point t is in the (first) busy period or not. We also introduce the density $[\mathbf{v}(t,x)]_{ij} = \frac{\partial}{\partial x}[\mathbf{V}(t,x)]_{ij}$ and the associated Laplace transform

$$[\tilde{\mathbf{v}}(s,x)]_{ij} = \int_0^\infty e^{-st}[\mathbf{v}(t,x)]_{ij}\, dt.$$

These quantities define, in turn, the matrices $\mathbf{V}(t,x)$, $\mathbf{v}(t,x)$ and $\tilde{\mathbf{v}}(s,x)$ respectively which are all of dimension $|S_1| \times |S|$.

Remark 1 *A proof of the existence of the density $\mathbf{v}(t,x)$ has been given in [3] and is beyond the scope of an elementary presentation.*

The first important result we wish to prove is the following.

Theorem 1 *Define the matrices $\Theta(s)$ and $K(s)$ by the equations*

$$\begin{aligned} \Theta(s) &= C_1^{-1} Q_{13} (sI - Q_{33})^{-1} + \Psi(s) C_2^{-1} Q_{23} (sI - Q_{33})^{-1}. && (5) \\ K(s) &= C_1^{-1} (Q_{11} - sI) + \Psi(s) C_2^{-1} Q_{21} + \Theta(s) Q_{31}. && (6) \end{aligned}$$

Then, for all $x > 0$, when we partition the matrix $\tilde{\mathbf{v}}(s,x)$ as

$$\tilde{\mathbf{v}}(s,x) = [\tilde{\mathbf{v}}_{11}(s,x) \vdots \tilde{\mathbf{v}}_{12}(s,x) \vdots \tilde{\mathbf{v}}_{13}(s,x)], \tag{7}$$

according to the sets S_i, $1 \leq i \leq 3$, the submatrices appearing in (7) are given by the following formulae:

$$\begin{aligned} \tilde{\mathbf{v}}_{11}(s,x) &= e^{K(s)x}\, C_1^{-1}. && (8) \\ \tilde{\mathbf{v}}_{12}(s,x) &= e^{K(s)x}\, \Psi(s) C_2^{-1}. && (9) \\ \tilde{\mathbf{v}}_{13}(s,x) &= e^{K(s)x}\, \Theta(s). && (10) \end{aligned}$$

Remark 2 *It is easy to see from the discussion to follow that when S_3 is empty, as indeed occurs in many applications, all relevant results are obtained by simply dropping from the formulae all terms involving the set S_3. Also, when S_3 is nonempty, the invertibility of $(sI - Q_{33})$ for $Re(s) > 0$ follows from the fact that the eigenvalues of Q_{33} all have nonpositive real parts.*

The above theorem was established as Theorem 13(a) by us in [3] using a lengthy argument involving stochastic discretization and stochastic process limits. Our first goal is to establish this result (along with the interpretations of the matrices $K(s)$ and $\Theta(s)$ developed in [3]) in an elementary manner using simple arguments. We do that in the remaining subsections of this section. After that, we shall, in Section 4, take up the problem of determining the distribution of the fluid flow at an arbitrary time point for an arbitrary initial condition.

3.2 A Markov renewal kernel

For $x \geq 0$, we use the notation $N_j(t,x),\ j \in S$, to denote the number of epochs in the time interval $[0,t]$ at which a visit is made by the MMFF to (x,j). Let

$$[\Phi(t,x,x+y)]_{ij} = {}_x\mathcal{E}_{xi}[N_j(t,x+y)], \quad t \geq 0. \tag{11}$$

$$[\phi(t,x,x+y)]_{ij} = \frac{\partial}{\partial t}[\Phi(t,x,x+y)]_{ij}, \quad t > 0. \tag{12}$$

We recognize the matrix $\Phi(t,x,x+y)$ formed by the elements $[\Phi(t,x,x+y)]_{ij}$ to be a (taboo) Markov renewal kernel and $\phi(t,x,x+y)$ to be the associated Markov renewal density.

Remark 3 *For fixed x, y, i, j, the quantity $[\Phi(t,x,x+y)]_{ij}$ is the expected number of visits to $(x+y,j)$ in the time interval $[0,t]$ avoiding the set of fluid levels $[0,x]$ in the time interval $(0,t]$, given that the MMFF starts in the state (x,i). As a function of t, this is a non-decreasing function – i.e., a distribution function on $[0,\infty)$. We can prove the existence and differentiability of ϕ at each (t,x) using the discretization approach of Athreya & Ney [9] or from the results in Ahn & Ramaswami [3]. This elementary exposition, of course, will assume these technical facts without proving them rigorously.*

Remark 4 *From the standard results in Markov renewal theory, for $t > 0$, we may interpret the taboo renewal density $[\phi(t, x, x+y)]_{ij}\, dt$ as the elementary probability that starting in (x, i), the MMFF visits $(x+y, j)$ in the time interval $[t, t+dt)$ avoiding the set $[0, x]$ of fluid levels in the time interval $(0, t]$.*

Note that for $y > 0$, the submatrix $\Phi_{ij}(t, x, x+y) = 0$ for $i = 2, 3$ and $j = 1, 2, 3$, because if the MMFF starts off in an environment state in $S_2 \cup S_3$, then the process stays at the initial fluid level for a positive amount of time and the required taboo visits do not occur. Also, since the flow rates and transition rates do not depend on fluid level when $F(t) > 0$ (the spatial homogeneity property), we can also assert that

$$\Phi(t, x, x+y) = \Phi(t, 0, y). \tag{13}$$

In the next theorem, we obtain an interesting and useful relation between the functions $\phi(t, 0, x)$ and $\mathbf{v}(t, x)$.

Theorem 2 *For $t > 0$ and $x > 0$, we have*
(a) $\phi_{11}(t, 0, x) = \mathbf{v}_{11}(t, x) C_1$.
(b) $\phi_{12}(t, 0, x) = \mathbf{v}_{12}(t, x) C_2$.

Proof: Clearly

$${}_0\mathcal{P}_{0i}[N_j(t+h, x) - N_j(t, x) \geq 2] = o(h) \quad \text{as} \quad h \to 0,$$

since the event under consideration will require at least two changes of the environment in an interval of length h.

Now, we define $d(i) = -c(i)$ for $i \in S_1$ and $d(i) = c(i)$ for $i \in S_2$. Then, clearly, for $i \in S_1$ and $j \in S_1 \cup S_2$, and sufficiently small $h > 0$, we have,

$$\begin{aligned} &{}_0\mathcal{P}_{0i}[N_j(t+h, x) - N_j(t, x) \geq 1] \\ &= \int_0^h [\mathbf{v}(t, x + d(j)u)]_{ij} \{1 + [Q]_{jj} u\}\, du + \sum_{\substack{r \in S_1 \\ r \neq j}} \int_0^h [\mathbf{v}(t, x + d(r)u)]_{ij} [Q]_{rj} u\, du \\ &+ \sum_{\substack{r \in S_2 \\ r \neq j}} \int_0^h [\mathbf{v}(t, x + d(r)u)]_{rj} [Q]_{rj} u\, du + \sum_{\substack{r \in S_3 \\ r \neq j}} \int_0^h [\mathbf{v}(t, x)]_{ir} [Q]_{rj} u\, du + o(h). \end{aligned}$$

This is got by noting that an increment in $N_j(\cdot, x)$ occurs in the interval $(t, t+h]$ only if a visit is made to (x, j) at some $t + u$ with $u \in (0, h]$.

Now, divide by h and rearrange the terms to get

$$\begin{aligned}
&\frac{1}{h}\,{}_0\mathcal{P}_{0i}[N_j(t+h,x) - N_j(t,x) \geq 1] \\
&= \frac{1}{h}\int_0^h [\mathbf{v}(t,x+d(j)u)]_{ij}\,du + \sum_{r\in S_1}\frac{1}{h}\int_0^h [\mathbf{v}(t,x+d(r)u)]_{ir}[Q]_{rj}u\,du \\
&\quad + \sum_{r\in S_2}\frac{1}{h}\int_0^h [\mathbf{v}(t,x+d(r)u)]_{ir}[Q]_{rj}u\,du + \sum_{r\in S_3}\frac{1}{h}\int_0^h [\mathbf{v}(t,x)]_{ir}[Q]_{rj}u\,du \\
&\quad + o(h)/h.
\end{aligned}$$

In the above equation, the first term in the right of course goes to $[\mathbf{v}(t,x)]_{ij}c(j)$. If we consider the second term, then

$$\begin{aligned}
&\left|\sum_{r\in S_1}\frac{1}{h}\int_0^h [\mathbf{v}(t,x+d(r)u)]_{ir}[Q]_{rj}u\,du\right| \\
&\qquad \leq \sum_{r\in S_1}\int_0^h \left|[\mathbf{v}(t,x+d(r)u)]_{ir}[Q]_{rj}\right|du \to\ 0 \text{ as } \ h\to 0.
\end{aligned}$$

Similarly, the third and the fourth terms also go to 0 as h goes to 0. Thus, the theorem follows by noting that for a non-negative integer valued random variable X, $E[X] = \sum_{n\geq 1}\mathcal{P}[X\geq n]$. □

This theorem immediately yields the following corollary.

Corollary 1 *For $x > 0$,*
(a) $\tilde{\phi}_{11}(s,0,x) = \tilde{\mathbf{v}}_{11}(s,x)C_1$
(b) $\tilde{\phi}_{12}(s,0,x) = \tilde{\mathbf{v}}_{12}(s,x)C_2$

Remark 5 *Let $i,j\in S_1$. Clearly, $[\mathbf{v}(t,x)]_{ij}\,dx$ has the interpretation that starting in $(0,i)$, the process $(\mathcal{F},\mathcal{J})$ makes a visit at time t into the set $((x,x+dx),j)$ avoiding fluid level 0. When $J(t)=j$, note that the rate of increase dx in the fluid level at time t is $c(j)\,dt$. Operationally, replacing dx by $c(j)\,dt$ in $[\mathbf{v}(t,x)]_{ij}\,dx$, we thus can interpret $[\mathbf{v}(t,x)]_{ij}c(j)\,dt$ to be the probability of a visit into (x,j) in the time interval $(t,t+dt)$ avoiding level 0. But the latter probability is $[\phi(t,0,x)]_{ij}\,dt$. The above theorem and corollary encapsulate these highly intuitive operational results in a formal manner.*

3.3 Level crossing approach

Now, we use a level crossing argument to evaluate the kernels $\Phi(t,0,x)$.

The skip-free upward property of $F(\cdot)$ implies that, for $x, y > 0$, starting at level 0, the MMFF cannot reach $x + y$ without reaching x first. Thus,by conditioning on the last epoch of visit to level x (a Markov renewal argument), we can see that the submatrix ϕ_{11} of ϕ must satisfy the equation:

$$\phi_{11}(t, 0, x + y) = \int_0^t \phi_{11}(t - u, 0, x)\phi_{11}(u, x, x + y)\, du. \tag{14}$$

Remark 6 *Though highly intuitive, the above and other similar Markov renewal arguments to be used by us require a formal proof. Once again, the techniques of [3] may be viewed as an approach to such proofs using stochastic discretization and using standard Markov renewal calculations for the associated discrete state processes.*

We note that by the spatial homogeneity of the model, we can rewrite Equation (14) as

$$\phi_{11}(t, 0, x + y) = \int_0^t \phi_{11}(t - u, 0, x)\phi_{11}(u, 0, y)\, du. \tag{15}$$

Now, from Equation (15), we have that

$$\tilde{\phi}_{11}(s, 0, x + y) = \tilde{\phi}_{11}(s, 0, x)\tilde{\phi}_{11}(s, 0, y). \tag{16}$$

From the general results on semi-groups (see [11], Chapter 15, Sec. 11), one can deduce that $\tilde{\phi}_{11}(s, 0, x)$ must have the form

$$\tilde{\phi}_{11}(s, 0, x) = e^{K^\circ(s)x}, \tag{17}$$

for some matrix $K^\circ(s)$ which is a square matrix of order $|S_1|$. However, application of general semigroup theory here begs the nonsingularity of the matrices in question; and though provable, it is not elementary to do so. Therefore, below in Lemma 4, we will take a more direct path and demonstrate that not only does (17) hold, but also that it holds with $K^\circ(s) = K(s)$.

We can also obtain an expression for the submatrix $\tilde{\phi}_{12}$ of $\tilde{\phi}$; this matrix gives the elementary probabilities of a taboo visit to a higher fluid level avoiding the current level in the environment set S_2.

Lemma 1

$$\tilde{\phi}_{12}(s, 0, x) = \tilde{\phi}_{11}(s, 0, x)\Psi(s). \tag{18}$$

Proof: For $i \in S_1$ and $j \in S_2$, let $[\psi(t)]_{ij}$ denote the conditional density of the busy period duration $\tau\,\chi[J(\tau) = j]$, given $F(0) = 0,\ J(0) = i$. By conditioning on the last epoch of crossing level x from below in the busy period and letting $k \in S_1$ denote the environmental state at such an epoch, we can write again by a Markov renewal argument,

$$[\phi(t,0,x)]_{ij} = \sum_{k \in S_1} \int_0^t [\phi(t-u,0,x)]_{ik}\,[\psi(u)]_{kj}\,du; \tag{19}$$

here, we have used the fact that due to the spatial homogeneity of the model, $\psi(\cdot)$ also characterizes the return times to any level x, given that the MMFF starts in a state (x,k), $k \in S_1$. It follows that

$$\tilde{\phi}_{12}(s,0,x) = \tilde{\phi}_{11}(s,0,x)\Psi(s),$$

and that completes the proof. □

Recall that our primary interest is in the function $\mathbf{v}(t,x)$. First we show that it satisfies the following lemma.

Lemma 2 *For $t > 0$ and $x > 0$,*

$$\frac{\partial}{\partial t}\mathbf{v}_{11}(t,x) = \mathbf{v}_{11}(t,x)Q_{11} + \mathbf{v}_{12}(t,x)Q_{21}$$
$$\mathbf{v}_{13}(t,x)Q_{31} - \frac{\partial}{\partial x}\mathbf{v}_{11}(t,x)C_1. \tag{20}$$
$$\frac{\partial}{\partial t}\mathbf{v}_{13}(t,x) = \mathbf{v}_{11}(t,x)Q_{13} + \mathbf{v}_{12}(t,x)Q_{23} + \mathbf{v}_{13}(t,x)Q_{33}. \tag{21}$$

Furthermore, for all $x > 0$, we have $\mathbf{v}_{1i}(0,x) = 0$ for $i = 1,2,3$.

Proof: Equations (20) and (21) are indeed the Kolmogorov differential equations obtained by considering two time points t and $t + \Delta t$ and letting $\Delta t \downarrow 0$; we omit the details. Now, let $c^* = \max_{i \in S_1} c(i)$, and consider a fixed $x > 0$. Note that if $F(0) = 0$, then for all $t < x/c^*$, we have $F(t) < x$ a.s., whence we must have $\mathbf{v}_{1i}(0+,x) = 0$, $i = 1,2,3$. □.

Using Lemma 2, we can derive the following two lemmas.

Lemma 3 *We have,*

$$\tilde{\mathbf{v}}_{13}(s,x) = \tilde{\phi}_{11}(s,0,x)\Theta(s), \tag{22}$$

where $\Theta(s)$ is defined in Equation (5).

Proof: If we take Laplace transform on both sides of Equation (21), then

$$s\tilde{\mathbf{v}}_{13}(s,x) \quad = \quad \tilde{\mathbf{v}}_{11}(s,x)Q_{13} + \tilde{\mathbf{v}}_{12}(s,x)Q_{23} + \tilde{\mathbf{v}}_{13}(s,x)Q_{33}.$$

It follows immediately from Corollary 1 and Lemma 1 that

$$\tilde{\mathbf{v}}_{13}(s,x) = \tilde{\phi}_{11}(s,0,x)[C_1^{-1}Q_{13}(sI-Q_{33})^{-1} + \Psi(s)C_2^{-1}Q_{23}(sI-Q_{33})^{-1}],$$

and the proof is complete. □

Lemma 4 *For $x > 0$, we have*

$$\tilde{\phi}_{11}(s,0,x) = e^{K(s)x}, \tag{23}$$

where $K(s)$ is defined by Equation (6).

Proof: By taking Laplace transforms on both sides of Equation (20), we get

$$\begin{aligned} & s\tilde{\mathbf{v}}_{11}(s,x) \\ &= \tilde{\mathbf{v}}_{11}(s,x)Q_{11} + \tilde{\mathbf{v}}_{12}(s,x)Q_{21} + \tilde{\mathbf{v}}_{13}(s,x)Q_{31} - \frac{\partial}{\partial x}\tilde{\mathbf{v}}_{11}(s,x)C_1. \end{aligned}$$

Using Corollary 1, Lemma 1 and Lemma 3, this can be rewritten as

$$\begin{aligned} s\tilde{\phi}_{11}(s,0,x)C_1^{-1} \quad = \quad & \tilde{\phi}_{11}(s,0,x)C_1^{-1}Q_{11} + \tilde{\phi}_{11}(s,0,x)\Psi(s)C_2^{-1}Q_{21} \\ & + \tilde{\phi}_{11}(s,0,x)\Theta(s)Q_{31} - \frac{\partial}{\partial x}\tilde{\phi}_{11}(s,0,x), \end{aligned}$$

which is equivalent to the equation

$$\begin{aligned} \frac{\partial}{\partial x}\tilde{\phi}_{11}(s,0,x) &= \tilde{\phi}_{11}(s,0,x)[C_1^{-1}(Q_{11}-sI) + \Psi(s)C_2^{-1}Q_{21} + \Theta(s)Q_{31}] \\ &= \tilde{\phi}_{11}(s,0,x)K(s), \end{aligned}$$

where $K(s)$ is given by (6). Let us denote the Laplace-Stieltjes transform of $\Phi_{11}(t,0,x)$ by $\hat{\Phi}_{11}(s,0,x)$. Clearly, for $x > 0$, since $\Phi_{11}(0,0,x) = 0$, for $x > 0$, we have $\hat{\Phi}_{11}(s,0,x) = \tilde{\phi}_{11}(s,0,x)$. We also have the obvious condition $\hat{\Phi}_{11}(s,0,0) \equiv I$. Now the differential equation for $\tilde{\phi}_{11}$ is equivalent to the equation

$$\frac{\partial}{\partial x}\hat{\Phi}_{11}(s,0,x) = \hat{\Phi}_{11}(s,0,x)K(s).$$

This, combined with the initial condition $\hat{\Phi}_{11}(s,0,0) = I$ shows that

$$\hat{\Phi}_{11}(s,0,x) = e^{K(s)x},$$

and the proof is complete. □

Proof of Theorem 1: Theorem 1 follows trivially from Lemma 4, Lemma 1 and Lemma 3 in light of Corollary 1. □

4 Transient analysis

For $a \geq 0$, the quantities of interest are the transform matrices defined by the elements:

$$[\tilde{w}^a(s,x)]_{ij} = \int_0^\infty e^{-st}\mathcal{P}_{ai}[F(t) > x, J(t) = j]dt, \quad x \geq 0,\ i,j \in S, \qquad (24)$$

$$[\tilde{z}^a(s)]_{ij} = \int_0^\infty e^{-st}\mathcal{P}_{ai}[F(t) = 0, J(t) = j]dt,\ i,j \in S. \qquad (25)$$

These provide the joint distribution of the fluid level and phase at an arbitrary time t given that the MMFF starts at time 0 with an initial fluid level a and phase i.

To keep the discussion short, we will derive the formulae only for the case when $i \in S_1$; the corresponding formulae for the other cases can be deduced from these in an easy and obvious manner. Following our general convention with respect to partitioning, note that for $j = 1,2,3$, $\tilde{w}^a_{1j}(s,x)$ and $\tilde{z}^a_{1j}(s)$ are the submatrices of the respective matrices $\tilde{w}^a(s,x)$ and $\tilde{z}^a(s)$ with row indices restricted to S_1 and column indices restricted to S_j. These are the quantities we shall determine. We first deal with the case $a = 0$ which is somewhat easier, and then take up the general case later.

4.1 Case: Initial fluid level $\mathbf{F(0) = 0}$

Assume that $F(0) = 0$ and $J(0) = i \in S_1$. As before, denote by τ the length of the first busy period, and let ι denote the duration of the idle period immediately following the first busy period. Now $\zeta = \tau + \iota$ is the length of the first busy cycle, i.e., the return time to the set $\{0\} \times S_1$.

We note that a busy period can start only with phase in S_1, and it always ends with phase in S_2. Similarly, an idle period can start only with phase in S_2 and must end with phase in S_1. Bearing these in mind, for $t \geq 0$, let us define the matrices $\Upsilon(t)$ and $B(t)$ to be the matrix with elements:

$$\begin{aligned} [\Upsilon(t)]_{ik} &= \mathcal{P}_{0i}\left[\iota \leq t,\ J(\iota) = k\right], \quad i \in S_2,\ k \in S_1; \\ [B(t)]_{ik} &= \mathcal{P}_{0i}\left[\zeta \leq t,\ J(\zeta) = k\right], \quad i,k \in S_1. \end{aligned}$$

Denote the respective LSTs of these matrices by $\hat{\Upsilon}(s)$ and $\hat{B}(s)$. These respectively give the distributions of the idle period and the busy cycle of the MMFF taking note of the states at their beginning and end. It is trivial to note that

$$\hat{B}(s) = \Psi(s)\hat{\Upsilon}(s). \qquad (26)$$

The following result gives a formula for $\hat{\Upsilon}(s)$.

Lemma 5 *We have*

$$\hat{\Upsilon}(s) \;=\; (sI-\hat{D}(s))^{-1}\,[Q_{21}+Q_{23}(sI-Q_{33})^{-1}Q_{31}], \tag{27}$$

where

$$\hat{D}(s) = Q_{22}+Q_{23}(sI-Q_{33})^{-1}Q_{32}. \tag{28}$$

Proof: It is elementary to verify that the matrix $\hat{\Upsilon}(s)$ satisfies the equation:

$$\hat{\Upsilon}(s) = (sI-Q_{22})^{-1}[Q_{21}+Q_{23}(sI-Q_{33})^{-1}Q_{31}] \\ + (sI-Q_{22})^{-1}Q_{23}(sI-Q_{33})^{-1}Q_{32}\hat{\Upsilon}(s).$$

The lemma follows by re-arranging the terms of this equation. □

At this time, we introduce the matrix $\hat{R}(s)$ of Laplace transforms, which is of order $|S_2|\times|S_2\cup S_3|$ and whose (i,j)-th element is given by

$$[\hat{R}(s)]_{ij} = \int_0^\infty e^{-st}\,\mathcal{P}_{0i}[F(u)=0,\;\forall 0\le u\le t,\;J(t)=j],\quad i\in S_2,\; j\in S_2\cup S_3, \tag{29}$$

which is the transform of the probability that having started in the state $(0,i)$, $i\in S_2$, the MMFF is in the state $(0,j)$ at time t without the fluid level becoming positive in $[0,t]$. Noting that the event of interest occurs only if the set S_1 is avoided in the interval $[0,t]$, we have the formula

$$\hat{R}(s) = (sI-\hat{D}(s))^{-1}\left[I_{|S_2|}\;\vdots\; Q_{23}(sI-Q_{33})^{-1}\right], \tag{30}$$

where $\hat{D}(s)$ is given by (28).

Armed with these results, we can now prove the following theorem.

Theorem 3

(a) We have,

$$\tilde{z}^0_{11}(s) = \mathbf{0} \tag{31}$$

$$[\tilde{z}^0_{12}(s)\;\vdots\;\tilde{z}^0_{13}(s)] = (I-\hat{B}(s))^{-1}\Psi(s)\hat{R}(s), \tag{32}$$

where $\hat{R}(s)$ is given by Equation (30).

(b) For $x\ge 0$, we have

$$\tilde{w}^0_{11}(s,x) = (I-\hat{B}(s))^{-1}(-K(s))^{-1}e^{K(s)x}C_1^{-1} \tag{33}$$

$$\tilde{w}^0_{12}(s,x) = (I-\hat{B}(s))^{-1}(-K(s))^{-1}e^{K(s)x}\Psi(s)C_2^{-1}, \tag{34}$$

$$\tilde{w}^0_{13}(s,x) = (I-\hat{B}(s))^{-1}(-K(s))^{-1}e^{K(s)x}\Theta(s). \tag{35}$$

Proof: Equation (31) is trivial by noting that for $F(t) = 0$ and $J(t) = j \in S_1$, the epoch t should be the epoch of the end of the n-th busy cycle for some $n \geq 1$ which ends through a visit into state j; the probability that such an event happens preceisely at t is zero. Equation (32) is obtained easily from the interpretation of $\hat{R}(s)$ and by noting that if a point t is such that $F(t) = 0$, then t is in the idle period following some $n \geq 1$ busy periods, and the expression $[B(s)]^{n-1}\Psi(s)$ gives the transform of the joint distribution of the length of time up to the end of the n-th busy period and the phase at that epoch. Part (b) follows similarly by noting that if $F(t) > 0$ holds, then t is an epoch in the n-th busy period for some $n \geq 1$. In the stated formulae, the term $(I - \hat{B}(s))^{-1}$ takes care of the busy cycles, if any, that elapse before time t, and the remainder term characterizes what occurs within the busy period containing t and comes from integrating in (x, ∞), the formulae (8)-(10) of Theorem 1 for the density of the fluid level within a busy period; we omit the details. [Incidentally, the existence of the inverse of $K(s)$ for $Re(s) > 0$ simultaneously follows from the (obvious) finiteness of the transforms that are being evaluated.]

4.2 Case: $\mathbf{F(0) = a > 0}$

We begin by recalling quickly from [5] the distribution of the busy period starting in the state (x, i), $x > 0$, $i \in S_2$; for its proof that is quite elementary and based once again on a level crossing argument, we refer to [5].

Lemma 6 *Let*

$$\begin{aligned} H(s) = {} & C_2^{-1}(Q_{22} - sI) + C_2^{-1}Q_{23}(sI - Q_{33})^{-1}Q_{32} \\ & + C_2^{-1}\{Q_{21} + Q_{23}(sI - Q_{33})^{-1}Q_{31}\}\Psi(s). \end{aligned} \tag{36}$$

For $x \geq 0$ and $i, j \in S_2$,, the matrix $e^{H(s)x}$ is such that its (i,j)-th element gives for the MMFF the joint distribution of the first passage time to fluid level 0 and the phase at the end of such first passage, given that the MMFF starts in state (x, i).

The above result helps us to write the formulae for the transform of the emptiness probability as follows.

Theorem 4 *We have, for the Laplace transform of emptiness probabilities at time t, the following formulae.*

$$\tilde{z}_{11}^{a}(s) = \mathbf{0}. \tag{37}$$

$$[\tilde{z}^a_{12}(s) \vdots \ \tilde{z}^a_{13}(s)] =$$
$$\Psi(s)\, e^{H(s)a}\, \hat{R}(s) + \Psi(s)\, e^{H(s)a}\, \hat{\Upsilon}(s)[\tilde{z}^0_{12}(s) \vdots \ \tilde{z}^0_{13}(s)], \qquad (38)$$

where $[\tilde{z}^0_{12}(s) \vdots \ \tilde{z}^0_{13}(s)]$ *is given by Theorem 3.*

Proof: The argument leading to (37) is similar to that of (31). The proof of (38) follows by noting that starting in $\{a\} \times S_1$, for $F(t) = 0$ and $J(t) \in S_2 \cup S_3$, the epoch t must be in an idle period. Thus, the MMFF must first return to a in the set $\{a\} \times S_2$ (this is characterized by $\Psi(s)$), and then make a first entrance (characterized by $e^{H(s)a}$) into level 0 before time t. At that point, the process is in the set $\{0\} \times S_2$. There are now two cases to consider: either the idle period continues to time t; or it ends before t with a visit into the set S_1. We have noted already that $\hat{\Upsilon}(s)$ governs the distribution of the idle period. The two terms in the right side of (38) correspond to the two respective cases identified, and we have used the interpretation of $\hat{R}(s)$ in equation (29) in the first term and the characterization in Theorem 3 in the second term. □

For $x > 0$, we now recall the matrix of transforms $\hat{U}(s,x)$ of order $|S_2| \times |S_1|$, with elements $\hat{U}(s,x)]_{ij},\ i \in S_2,\ j \in S_1$ given by

$$[\hat{U}(s,x)]_{ij} = \int_0^\infty e^{-st}\ {}_0\mathcal{P}_{xi}[F(u) = x,\ J(u) = j \text{ for some } u \in (t, t+dt)]. \qquad (39)$$

This matrix was introduced by Ramaswami in [16], and $[\hat{U}(s,x)]_{ij}$ is the Laplace transform (with respect to time) of the density that the MMFF crosses fluid level x in the state (x,j), $j \in S_1$ at time $t > 0$ avoiding level 0 in the interval $(0,t]$, given that the MMFF starts in (x,i), $i \in S_2$. It is easy to see that

$$\hat{U}(s,x) = \int_{(0,x)} e^{H(s)y} C_2^{-1}[Q_{21} + Q_{23}(sI - Q_{33})^{-1} Q_{31}] e^{K(s)y}\, dy, \quad x > 0. \qquad (40)$$

The formula in (40) is trivial from the interpretations of $H(s)$ and $K(s)$ and follows by noting that it is based on a conditioning argument that conditions on the lowest level $x - y > 0$ attained by the MMFF in the time interval $(0,t]$. As noted in [16], we can post-multiply both sides of (40) by $K(s)$ and obtain, using integration by parts, the equation

$$\hat{U}(s,x)K(s) + H(s)\hat{U}(s,x) = \hat{A}(s,x),$$

where

$$\hat{A}(s,x) = e^{H(s)x}C_2^{-1}[Q_{21} + Q_{23}(sI - Q_{33})^{-1}Q_{31}]e^{K(s)x} \\ - C_2^{-1}[Q_{21} + Q_{23}(sI - Q_{33})^{-1}Q_{31}].$$

As noted in [16], we can solve the above linear system uniquely and write its solution in a computable form as

$$vec\big(\hat{U}(s,x)\big) = \big[K^t(s) \otimes I_{|S_2|} + I_{|S_1|} \otimes H(s)\big]^{-1} vec\big(\hat{A}(s,x)\big), \tag{41}$$

where, for any matrix A, $vec(A)$ denotes a column vector obtained by writing the successive columns of A one below the other, B^t denotes the transpose of a matrix B, and $\otimes$ denotes the Kronecker product of matrices; in [16], it has been shown that the inverse in (41) does indeed exist.

Now, for $a > 0$, denote by ${}_0\tilde{w}^a(s,x)$ the transform matrix whose (i,j)-th element is the Laplace transform

$$[{}_0\tilde{w}^a(s,x)]_{ij} = \int_0^\infty e^{-st}\, {}_0\mathcal{P}_{ai}[F(t) > x, J(t) = j]\, dt, \quad i \in S_1,\ j \in S;$$

this gives the behavior of the MMFF within a busy period that starts in the state (a,i). We need these quantities as an intermediate step in computing the quantities $\tilde{w}^a(s,x),\ x > 0$ of interest to us.

Lemma 7 *With the convention we have adopted with respect to partitioning matrices, we have,*

(a) For $0 < a \leq x$*,*

$$
\begin{aligned}
{}_0\tilde{w}^a_{11}(s,x) &= (-K(s))^{-1}e^{K(s)(x-a)}C_1^{-1} \\
&\quad + \Psi(s)\hat{U}(s,a)(-K(s))^{-1}e^{K(s)(x-a)}C_1^{-1}, \qquad (42)\\
{}_0\tilde{w}^a_{12}(s,x) &= (-K(s))^{-1}e^{K(s)(x-a)}\Psi(s)C_2^{-1} \\
&\quad + \Psi(s)\hat{U}(s,a)(-K(s))^{-1}e^{K(s)(x-a)}\Psi(s)C_2^{-1}, \qquad (43)\\
{}_0\tilde{w}^a_{13}(s,x) &= (-K(s))^{-1}e^{K(s)(x-a)}\Theta(s) \\
&\quad + \Psi(s)\hat{U}(s,a)(-K(s))^{-1}e^{K(s)(x-a)}\Theta(s). \qquad (44)
\end{aligned}
$$

(b) For $0 < x < a$,

$$
\begin{aligned}
&{}_0\tilde{w}^a_{11}(s,x) \\
&\qquad = (-K(s))^{-1}C_1^{-1} + \Psi(s)(-H(s))^{-1}\{I - e^{H(s)(a-x)}\} \\
&\qquad\quad \times C_2^{-1}[Q_{21} + Q_{23}(sI - Q_{33})^{-1}Q_{31}](-K(s))^{-1}C_1^{-1} \\
&\qquad\quad + \Psi(s)e^{H(s)(a-x)}\hat{U}(s,x)(-K(s))^{-1}C_1^{-1}, \qquad (45)
\end{aligned}
$$

$$
\begin{aligned}
&{}_0\tilde{w}^a_{12}(s,x) \\
&\qquad = (-K(s))^{-1}\Psi(s)C_2^{-1} + \Psi(s)(-H(s))^{-1}\{I - e^{H(s)(a-x)}\} \\
&\qquad\quad \times C_2^{-1}[Q_{21} + Q_{23}(sI - Q_{33})^{-1}Q_{31}](-K(s))^{-1}\Psi(s)C_2^{-1} \\
&\qquad\quad + \Psi(s)e^{H(s)(a-x)}\hat{U}(s,x)(-K(s))^{-1}\Psi(s)C_2^{-1}, \qquad (46)
\end{aligned}
$$

$$
\begin{aligned}
&{}_0\tilde{w}^a_{13}(s,x) \\
&\qquad = (-K(s))^{-1}\Theta(s) + \Psi(s)(-H(s))^{-1}\{I - e^{H(s)(a-x)}\} \\
&\qquad\quad \times C_2^{-1}[Q_{21} + Q_{23}(sI - Q_{33})^{-1}Q_{31}](-K(s))^{-1}\Theta(s) \\
&\qquad\quad + \Psi(s)e^{H(s)(a-x)}\hat{U}(s,x)(-K(s))^{-1}\Theta(s). \qquad (47)
\end{aligned}
$$

Proof:
Let $y > a$. Starting in $\{a\} \times S_1$ at time 0, there are two ways in which the MMFF can be at $\{y\} \times S_1$ at time t avoiding the level 0 throughout: (i) the process avoids fluid level a in the interval $(0, t]$ altogether; (ii) the process returns to level a, then, avoiding fluid level 0, visits the set $\{a\} \times S_1$ at some later epoch, and then finally moves on to $\{y\} \times S_1$ avoiding fluid level a. These lead to the expression

$$e^{K(s)(y-a)}C_1^{-1} + \Psi(s)\hat{U}(s,a)e^{K(s)(y-a)}C_1^{-1}$$

for the transform of the density of the associated event; here, in addition to the interpretation of the matrices $K(s)$ and $\hat{U}(s,a)$, we have also used the spatial homogeneity of the MMFF. For $x \geq a$, integrating this expression with respect to y over the interval (x, ∞) immediately gives equation (42). Equations (43) and (44) are proved analogously, and we omit the details.

Now, we prove (45). For $0 < x < a$, starting in $\{a\} \times S_1$ at time 0, there are three ways in which the MMFF can be in $(x, \infty) \times S_1$ at time t avoiding the level 0 throughout: (i) at time t, it is at some (y, j) with $y > a$ and $j \in S_1$ and furthermore it avoids the fluid level a in $(0, t]$; this contributes the term $e^{K(s)(y-a)}C_1^{-1}$ which integrated over $y > a$ gives the first term on the right side of (45); (ii) at time t, it is at $(x + y, j)$ for some $y > 0$ and

$j \in S_1$ and during such a visit, the process does indeed visit the fluid level x; in this instance, the MMFF must first return to fluid level a, then make a first passage to fluid level $x < a$, then make a visit to $\{x\} \times S_1$ avoiding level 0, and then finally move onto $(x + y, j)$ avoiding fluid level x; these lead to the transform

$$\Psi(s)e^{H(s)(a-x)}\hat{U}(s,x)e^{K(s)y}C_1^{-1};$$

this,when integrated over y in the interval $(0, \infty)$, leads to the third term in the right of (45); (iii) in this final case to be considered, the MMFF does drop below a but does not hit x; then we can condition on the lowest level $x + z$ with $0 < z < a - x$ it does indeed hit and the epoch of exit from that level z, and write a transform for the density of being at $x + y$ as

$$\Psi(s)e^{H(s)(a-x-z)}C_2^{-1}[Q_{21} + Q_{23}(sI - Q_{33})^{-1}Q_{31}]e^{K(s)(y-z)}C_1^{-1};$$

integrating this expression with respect to (z, y) over the interval $(0, a-x) \times (z, \infty)$ gives us the second term in the right of (45). Thus, (45) is proved. The proof of (46) and (47) are similar and omitted. □

It is now trivial to write down a formula for the submatrices of $\tilde{w}^a(s,x)$ of interest to us.

Theorem 5 *For $a > 0$, and $j = 1, 2, 3$, we have,*

$$\tilde{w}^a_{1j}(s,x) = {}_0\tilde{w}^a_{1j}(s,x) + e^{H(s)a}\hat{\Upsilon}(s)\tilde{w}^0_{1j}(s,x), \;\; x \geq 0, \tag{48}$$

where the the transforms on the right side of the above equation have been characterized in Lemma 7 and Theorem 3.

Proof: The result is obtained by noting that the visit to a fluid level in (x, ∞) could occur either avoiding fluid level 0 or after a first passage to fluid level 0. □

5 Concluding Remarks

We have derived the time dependent distribution of the MMFF $(F(t), J(t))$ for an arbitrary initial state in the form of a set of transforms. All the terms appearing in our formulae can be expressed explicitly in terms of the busy period transform matrix $\Psi(s)$ for which we have developed a powerful algorithm in [5]. In [3], we have demonstrated that (even with a naive algorithm compared to that given in [5] for $\Psi(s)$), these transforms lead to accurate numerical results compared with the time domain results of Sericola [21], often surpassing the latter in accuracy and computational time.

The elementary presentation given here should make our work more accessible to practitioners. However, as the discussion shows, the approach via elementary methods based on differential equations is quite limited even in the case of a finite dimensional phase space and requires one to make many technical assumptions. Our own interest in the matrix-geometric approach to fluid models in our previous work was motivated by the need to identify a systematic approach that could possibly be generalized to the case of a general modulating Markov process (i.e., one not necessarily with a finite state space.) Thus, the more mathematically oriented reader should view this paper as one that facilitates the reading and placement in a proper context of our earlier work in [3], [4], [16] so that attempts at initiating operator based extensions of our work could be made possible.

Acknowledgment: Soohan Ahn was supported by Korea Research Foundation Grant (KRF-2004-003-C00044).

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