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Spatial Point Patterns of Phase Type

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Motivated by applications to the performance analysis of wireless communication systems, we develop a constructive procedure to generate random spatial point patterns that are natural generalizations of the Poisson process. The special case of models that are multi-dimensional generalizations of the Markovian Arrival Processes (MAP) of Neuts are discussed in some detail with examples. Like their counterparts on the nonnegative half line, these spatial MAPs offer the versatility to model a wide variety of spatial dependencies and burstiness characteristics while maintaining computational tractability.

Keywords: Spatial point patterns, phase type models.

1. INTRODUCTION

Spatial point patterns - a random set of points in a multi-dimensional space - are of interest in many application areas ranging from epidemiology and geosciences to newly emerging ones like the performance analysis of wireless, personal, and mobile communication systems. The "points" in such a process correspond to the location of certain events such as the occurrence of an infection or the initiation of a wireless call. The work reported here does not include a temporal component, and what is modeled may be viewed as a snapshot of a spatial-temporal process; for example, it could represent the spatial location of arrivals in an observation interval of time. Some models that incorporate both space and time variations will be presented by us in other papers, and for them the material presented here shall form the basic building block.

Although, it is widely recognized [2,3] that spatial dependencies are important to applications, because of the complexity of general models and the difficulties in specifying models incorporating specific types of dependencies of interest, the most commonly used ones still remain to be Poisson, which unfortunately entail the restrictive assumption of spatial independence in the counts. Our goal here is to develop a class of spatial models that can incorporate spatial dependencies while maintaining simplicity, tractability, and modeling versatility.

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For random point patterns on the line, for example the arrival epochs to a queue, the Markovian Arrival Process (MAP) introduced by Neuts [10] has been found to be a convenient generalization of the Poisson process on the line, as one that can incorporate a wide variety of qualitative features without losing computational tractability. Such models are now used extensively in telecommunications performance analysis, particularly in the high speed ATM context [1,4,11,13] where the process of arrivals of cells is known to be "bursty" and correlated. In Neuts' construction, qualitative characteristics like "burstiness" and "rush hours" are modeled through an environment state, called the phase, and instantaneous "arrival rates" depend on the phase of the system. With Markovian assumptions on the phase process, one enables the modeling of complex dependencies in the process of counts in a tractable manner.

Although the construction made by us can be applied much more generally, we restrict ourselves to generalizing the MAPs to patterns in the plane. We note here that a similar construction, based on renewal processes, is defined and analyzed by V. Isham [5,6]

2. THE POISSON PATTERN IN THE PLANE

The homogeneous Poisson point pattern with rate λ on $\mathbb{R}^2$ is characterized by the following axioms, where the symbol $||A||$ is used to denote the Lebesgue measure of a set A, and N_A denotes the number of point falling in the set A:

a. Given non-overlapping subsets of $\mathbb{R}^2$, i.e., given Borel sets A_i, $1 \leq i \leq n$ such that $||A_i \cap A_j|| = \phi$ for $i \neq j$, the counts $N_{A_i}, 1 \leq i \leq n$ are independent.

b. The probability that two or more points lie in a set A is $o(||A||)$ as $||A|| \to 0$.

c. The probability that A contains a point is $\lambda||A|| + o(||A||)$ as $||A|| \to 0$.

The following theorem is an immediate consequence of the above axioms.

Theorem 2.1 *Consider the homogeneous Poisson point pattern of rate λ and let N_A denote the number of points falling in the Borel set A. Then, N_A has a Poisson distribution with parameter $\lambda||A||$. That is,*

$$P[N_A = n] = e^{-\lambda||A||}(\lambda||A||)^n/n!, \qquad n \geq 0.$$

The intensity of the measure is λ at all $(x, y) \in \mathbb{R}^2$; that is $\lambda\, dx\, dy$ is the elementary probability that a small rectangular neighborhood of (x, y) with sides of lengths dx and dy contains a point of the pattern. ∎

Indeed, the spatial independence of the counts over non-overlapping sets (the axiom (b) above) and the Poisson distribution with parameter $\lambda||A||$ for the count N_A for all Borel sets A together characterize the homogeneous Poisson point pattern, and some authors use these two as the axioms for defining the homogeneous Poisson point pattern [3].

The following elementary property of the homogeneous Poisson point pattern which is useful for simulating such a process is also indicative of the restrictiveness of the model: we omit the proof of this well-known result.

Theorem 2.2 *Conditional on $A \subset \mathbb{R}^2$ having n points of a homogeneous Poisson point pattern, the conditional distribution of those points is the same as that of a random sample of size n from the uniform distribution on the set A.* ■

The homogeneous Poisson pattern has many simplifying features and spatial symmetries that make it almost trivial to analyze. Of those, the independence property characterized by Axiom (b) is perhaps the most restrictive. Spatial processes seldom exhibit such independence. Examples of practical situations of interesting correlation structures in the plane are easy to conceive of. For example, in wireless communications applications, strong correlations can exist between counts in different regions due to a wide variety of causes such as demography, customer mobility, radio quality and interference. Naturally, there is a need to consider a more general framework.

We conclude our brief discussion of the homogeneous Poisson model by presenting the following result which suggests a construction yielding the types of generalizations we propose later in this paper.

Theorem 2.3 *Consider the random points of the homogeneous Poisson point pattern of rate λ in the plane, and define their polar co-ordinates $\{(R_n, \Theta_n)\}$. The set of points $\{R_n\}$ on the half line $[0, \infty)$ (after ordering them in increasing order) form an inhomogeneous Poisson process with rate $2\pi\lambda r$ at r. Further, the set of points $\{T_n\}$ where $T_n = \pi R_n^2$, form a homogeneous Poisson process of rate λ. Finally, $\{\Theta_n\}$ form a set of iid random variables uniformly distributed in the interval $[0, 2\pi]$.*

Proof It is trivial to prove that the number of points $\{R_n\}$ falling in non-overlapping intervals (in the $\mathbb{R}$-axis) are independent and that the distribution of the number of points in any interval $(0, r]$ has the Poisson distribution with parameter $\pi r^2 \lambda$. Therefore [3], R_n is a Poisson process on the half line $[0, \infty)$ with rate $2\pi\lambda r$, and T_n is a homogeneous Poisson process of rate λ. The statement concerning Θ_n is a trivial consequence of the property of isometry (invariance under rotations) of the homogeneous Poisson point pattern. ■

The above theorem shows that using the planar Poisson point pattern, we can construct a homogeneous Poisson process on the half line by sweeping points of the planar process onto the half line radially along circles centered at the origin and then performing a simple time change. This motivates the following theorem which is an inverse result of the above and gives a method of obtaining a homogeneous Poisson planar pattern starting from a homogeneous Poisson process on the line.

Theorem 2.4 *Suppose $\{T_n\}$ denote the points of a homogeneous Poisson process of rate λ on $[0, \infty)$ and that $\{\Theta_n\}$ is a sequence of iid random variables on $[0, 2\pi]$. Let $R_n = (T_n/\pi)^{1/2}$. Then, the set of points (R_n, Θ_n) form a homogeneous Poisson pattern of rate λ in the plane.*

Proof From the construction and from the decomposition property of the Poisson distribution, it is easy to see that the number of points falling in non-overlapping subsets of the plane are independently distributed, and that the number of points falling in any set A is a Poisson random variable with parameter $\lambda ||A||$. Hence the result. ■

3. THE MARKOVIAN ARRIVAL PROCESS

Turning back to the construction of the Poisson process on the plane embodied in Theorem 2.4, note that a wide class of generalizations of that process can be obtained using the degrees of freedom available to us: firstly, instead of a Poisson process, we might start from any point process on the half line; secondly, we might use a different time scale change than the specific one used in the construction; finally, the scattering of the points onto the plane might be made along any other family of curves and using more complex mechanisms than that implied by the uniform distribution.

The variety of point patterns that can be generated thus is limitless, but the ability to analyze them will depend on imposing additional structure. With this as the concern, we concentrate in this paper on planar point patterns generated from a Markovian Arrival Process (MAP) on the half line and the derivation of some elementary results for such point patterns. We shall demonstrate in the next section that if we choose the point process on the line to be a stationary MAP and scatter points uniformly on the circle centered at the origin on which they reside, then we get a point pattern on the plane, stationary in the mean. We can view that process as a random walk in the plane driven by a phase process; the points ordered according to their radial distances from the origin give the successive points visited. That process will be called a stationary Spatial MAP (SMAP) on the plane. In the next section, we demonstrate some elementary properties of that process primarily to illustrate its tractability. In Section 5, several other constructions are presented, which demonstrate the modeling versatility and tractability of our scheme. We begin our technical results with a brief review of MAPs on the line.

MAPs on the Line

Our starting point is a construction of M.F. Neuts [10]. To fix ideas, consider the number of arrivals in the interval $[0, t]$ for some arrival process. We assume that the counting process $N_t, t \geq 0$ together with an auxiliary process $J_t, t \geq 0$ to be called the "phase process" yields a two-dimensional Markov process on a set of the form $\mathbb{N} \times \mathbb{N}_m$, where $\mathbb{N}_m = \{1, \ldots, m\}$; the restriction of the phases to a finite number is for computational ease only. Partitioning the states of the process into subsets $\ell(i) = \{(i, 1), \ldots, (i, m)\}$ and making a corresponding partitioning into blocks of the infinitesimal generator of the Markov process, it is easy to see that such a generator has the form

$$Q = \begin{bmatrix} D_0 & D_1 & 0 & 0 & \cdot & \cdot \\ 0 & D_0 & D_1 & 0 & \cdot & \cdot \\ 0 & 0 & D_0 & D_1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & & \end{bmatrix},$$

where D_0 and D_1 are $m \times m$ matrices such that $D_0(i,i) < 0, D_0(i,j) \geq 0$ for $i \neq j, D_1(i,j) \geq 0$ for all i, j and $(D_0 + D_1)\mathbf{1} = \mathbf{0}$; here, $\mathbf{1}$ is a column vector of one's and $\mathbf{0}$ is a vector of zeros. The elements of Q have the following interpretation: with probability $D_0(i,j)dt, i \neq j$, there occurs in time interval $(t, t + dt)$ a change of phase from i to j without an arrival, given that the phase at time t is i, while with probability $D_1(i,j)dt$ there occurs an arrival with or without a change in phase depending on whether $j \neq i$ or $j = i$. Furthermore, given that the phase at time t is i, the distribution of the time to the next event is exponentially distributed with parameter $-D_0(i,i)$. The Poisson process corresponds to the very special case $m=1, D_0=-\lambda$ and $D_1=\lambda$.

The joint process (N_t, J_t) has been studied in great detail by D.M. Lucantoni [8,9], M.F. Neuts [10], and Ramaswami [12]. For the purposes here, it is more convenient to take into account not only the pair (N_t, J_t) but also the number M_t of phase changes without any arrival in the interval $[0,t]$. Obviously, $\{(M_t, N_t, J_t) : t \geq 0\}$ is also a time homogeneous Markov process, and its state space is $\mathbb{N} \times \mathbb{N} \times \mathbb{N}_m$. Its possible transitions are from (m,n,i) to $(m+1,n,j)$, for $j \neq i$, at the rate $D_0(i,j)$, and from (m,n,i) to $(m,n+1,j)$, at the rate $D_1(i,j)$.

Definition 3.1 *We call the process (M_t, N_t, J_t) the Markovian Arrival Process (MAP) generated by the pair (D_0, D_1).*

Without loss of generality [12], we assume that the Markov process of phases governed by the infinitesimal generator $D = D_0 + D_1$ is irreducible, and for avoiding triviality assume that $D_1 \neq 0$. Let $\boldsymbol{\delta}$ denote the stationary probability vector of D, i.e. $\boldsymbol{\delta} D = \mathbf{0}, \boldsymbol{\delta}\mathbf{1} = 1$. The MAP with the initial phase at the time origin having distribution $\boldsymbol{\delta}$ is stationary as seen from the trivial fact that $\boldsymbol{\delta}$ is also the distribution of the phase at any time point t.

It is elementary to show that the matrices $P(m,n,t)$ defined by the elements

$$P_{ij}(m,n,t) = P[M_t = m, N_t = n, J_t = j | J_0 = i] \tag{1}$$

have generating function

$$P^*(z_0, z_1, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} z_0^m z_1^n P(m,n,t) = \exp[(z_0 D_0 + z_1 D_1)t] \tag{2}$$

From this it follows that the joint probability generating function of M_t and N_t for the stationary MAP is given by

$$\boldsymbol{\delta} P^*(z_0, z_1, t)\mathbf{1} = \boldsymbol{\delta} \exp[(z_0 D_0 + z_1 D_1)]\mathbf{1}; \tag{3}$$

the premultiplication by $\boldsymbol{\delta}$ takes account of the distribution of the "initial phase" of the process at 0 and the post-multiplication by $\mathbf{1}$ reflects the fact that we do not care for the specific value of the phase at t while computing this marginal probability generating function.

Note that in the Poisson case, this reduces to the familiar expression $\exp[-\lambda(1-z)t]$ for the probability generating function of the Poisson random variable with mean λt. Since there is only one underlying phase, all quantities reduce to scalars and all points generated are of one kind only, namely, arrival epochs.

The Markovian structure of the tri-variate process of the counts and the phase shall enable the computation of many quantities of interest through routine techniques used for Markov chains. Although our construction and results can be trivially modified to address the batch case (called BMAP by D.M. Lucantoni [8]) as well as discrete analogues based on discrete time Markov chains (called DMAP by some authors), to keep the exposition simple, in this paper we consider only the MAP.

Before we proceed further, let us note a simple but useful fact concerning MAPs in the following theorem. Its proof is straightforward.

Let us consider a time homogeneous MAP $\{(M_t, N_t, J_t) : t \geq 0\}$ generated by (D_0, D_1), and suppose a time change is effected through the transformation $t = \pi r^2$. To be specific, given a realization of event epochs $T_1 \leq T_2 \leq \cdots$ the associated phases $J_1, J_2 \cdots$ immediately after each of these epochs, and the indicators of whether an epoch corresponds to an arrival or not, we create on the transformed time scale, event points at $R_i, i \geq 1$, where $R_i = \sqrt{T_i/\pi}$, set the phase of the system in $[R_i, R_{i+1})$ to be J_i, and finally we also treat R_i as an arrival epoch if T_i is. For the transformed process, let $(\bar{M}_r, \bar{N}_r, \bar{J}_r)$ denote respectively the number of phase changes without arrivals in $[0, r]$, the number of arrivals in $[0, r]$ and the phase of the system at r_+ respectively.

Theorem 3.2 *The process $\{(\bar{M}_r, \bar{N}_r, \bar{J}_r) : r \geq 0\}$ is a time inhomogeneous Markov chain on $\mathbb{N} \times \mathbb{N} \times \mathbb{N}_m$ with feasible transitions as follows:*

- *from (m, n, i) to $(m+1, n, j)$, for $j \neq i$, at the rate $2\pi r D_0(i, j)$;*
- *from (m, n, i) to $(m, n+1, j)$, at the rate $2\pi r D_1(i, j)$.*

In particular,

$$\bar{P}^*(z_0, z_1, r) = \exp[(z_0 D_0 + z_1 D_1)\pi r^2]. \tag{4}$$

where the (i, j)-th entry of the above matrix is given by

$$\sum_{m-0}^{\infty} \sum_{n=0}^{\infty} z_0^m z_1^n P[\bar{M}_r = m, \bar{N}_r = n, \bar{J}_r = j | \bar{J}_0 = i].$$

■

4. THE STATIONARY SPATIAL MAP ON THE PLANE

We begin with a construction akin to the one discussed for generating a homogeneous Poisson pattern on the plane. The steps of our construction are given below.

a) Generate a stationary MAP on $t \geq 0$ defined by a pair of matrices D_0 and D_1.

b) Given $\{T_n\}$, the successive epochs at which events (phase changes and/or arrivals) occur, make a time change using the transformation $T = \pi R^2$ to yield a sequence of points $R_n = [T_n/\pi]^{1/2}$.

c) Now, consider the points $(R_n, 0)$ and rotate them on the circle centered at the origin and with radius R_n by a random angle Θ_n uniformly distributed in $(0, 2\pi)$.

d) With each point (R_n, Θ_n) thus obtained attach a "phase" which is the same as the phase at the point T_n+ for the process in (a), and also mark that point as a point of arrival in the plane if T_n is a point of arrival for the process in (a).

We shall call the resulting set of points a stationary spatial Markovian arrival process or stationary SMAP. The nomenclature SMAP is justified by the following result.

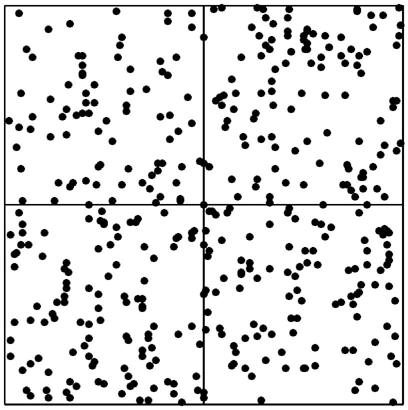

Figure 1. Poisson Process

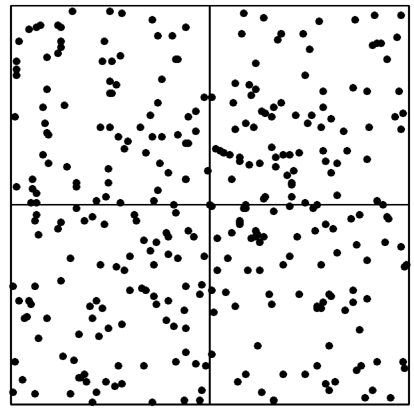

Figure 2. Stationary PH renewal process

Theorem 4.1 *Let S be a connected compact set in $\mathbb{R}^2$. Let M_S and N_S be the number of points without and with arrivals falling in the set S. For the planar point pattern constructed above, the expected values of M_S and N_S are given by*

$$E(M_S) = (\boldsymbol{\delta} D_0 \mathbf{1})||S||, \quad E(N_S) = (\boldsymbol{\delta} D_1 \mathbf{1})||S||, \tag{5}$$

In particular, these mean values depend only on the size of the set S and not on its specific location in $\mathbb{R}^2$; that is, the processes of event counts M_S and N_S are mean stationary.

Proof The proof of the formulae for a set of the form

$S = \{(r, \theta) : r_0 \leq r \leq r_1, \theta_0 \leq \theta \leq \theta_1\}$

is obtained by a direct computation. The measures on the Borel sets S of $\mathbb{R}^2$ defined by $E(M_S)$ and $E(N_S)$ being the extensions of these set functions defined over (r, θ) intervals, the result follows from the unique extension theorem for σ-finite measures from interval sets to the set of all Borel sets [14]. ■

Theorem 4.2 *The bivariate counting process (M_S, N_S) is isotropic; that is, it is invariant under rotations.*

Proof Let $\bar{S}$ be the set obtained by rotating the set S by an angle θ. The result is immediate by noting that each point $(R_n, 0)$ is thrown into $S \setminus \bar{S}$, $\bar{S} \setminus S$, $S \cap \bar{S}$ or outside $S \cup \bar{S}$ according to a multinomial trial and that the first two sets have the same probability because the sets are obtained by a simple rotation. ■

Although the SMAP we constructed is stationary in the mean and isotropic, unlike the homogeneous Poisson process, it is *not*, in general, strictly stationary; the distribution of the counts over a set S is *not* determined by the Lebesgue measure of S alone. Also, the counts over non-overlapping sets are not necessarily independent.

We give in Figures 1 – 3 the sample realizations of three stationary SMAPs, each with the same intensity of 100 points per unit area. These examples are respectively generated starting from the stationary versions of:

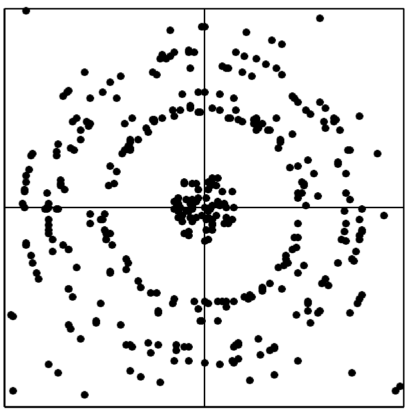

Figure 3. Stationary interrupted Poisson process

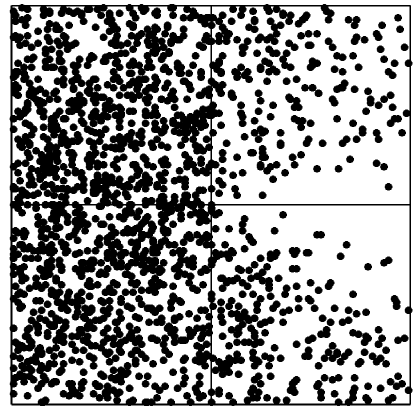

Figure 4. Triangular distribution for Θ

1 the homogeneous Poisson process;

2 a PH-renewal process with inter-arrival time distribution that is a mixture of two exponentials: $F(x) = 0.83(1 - \exp(-1500)) + 0.17(1 - \exp(-18))$;

3 an interrupted Poisson process which has rate 750 when it is on, and on and off times are respectively exponentially distributed with mean 0.0088 and 0.57.

The plots show the area from -1 to 1 in both coordinates; we draw the two axis for easy visual reference.

At first glance, it might appear that there is no difference between the two processes of Figures 1 and 2. This is an appearance only since the planar processes have different distributions for the number of points in any area of the plane.

The process used for Figure 3 is highly bursty, and this gives rise to a pattern such that the points are concentrated on circular portions of the plane. The differences between this plot and the first two is remarkable given that they all come from isotropic, mean stationary processes with the same rate.

5. NON-STATIONARY MAPS — SOME EXAMPLES

In this Section, we provide some examples to illustrate how one may exploit the freedom available in the construction to model different types of spatial structures. For brevity, we only describe the structures in terms of the plots given below of their realizations indicating briefly how they were generated and what they may model.

In Figure 4, we give the plot that we obtain if we take the construction of the homogeneous Poisson process on the plane and modify it so that points of the line process are thrown along concentric circles by a triangular distribution for Θ on $(0, 2\pi)$, instead of a uniform distribution. Note that there is a heavy concentration of points around the negative half line $x < 0$, and that the points become less densely packed as we move away from that half line. This could for example model a scenario where the negative half line

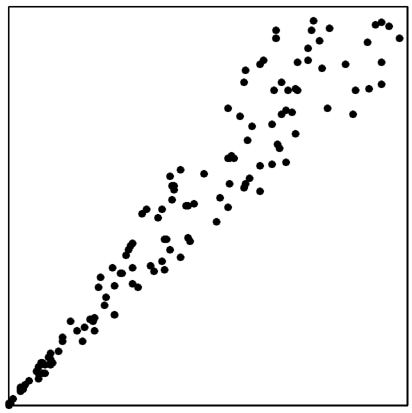

Figure 5. Θ restricted to an interval

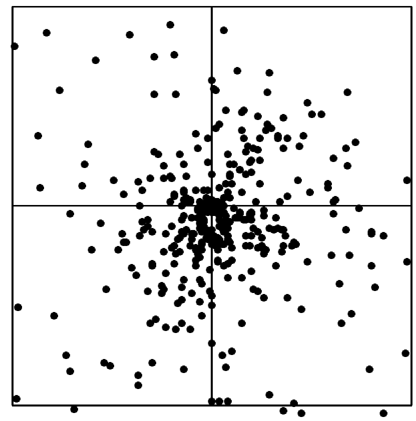

Figure 6. $R(T)$ slowly increasing at 0

represents a road with a large number of office towers and the points corresponds to the location of vehicles just prior to start of work or immediately after the end of work; in this case we would expect a much larger intensity of cars near the work locations relative to areas further away.

The example on Figure 5 differs in two respects from the construction of SMAPs. Starting from the Poisson process, we set $R_n = T_n$, so that the radial distance to the origin increases at a constant rate. Secondly, the points are rotated in a cone $\theta_0 \leq \Theta \leq \theta_1$. We only show the positive quadrant in this case since the remainder of the plane is empty of point. The resulting plot has then the following properties: the intensity of the points decays like $1/\sqrt{R}$ radially as R gets large, and the points lie within the cone of interest. Such a model could be used for a vehicular traffic merge at the origin of some highways from the North East direction ignoring the structures of local tributaries or for the scatter of particles from a particle gun fired at the origin in the North East direction.

The process in Figure 6 is obtained by starting from the Poisson process with rate one, and choosing the time transformation $R = 2^{-4}(2^{T/100} - 1)$; the angles are uniform on $(0, 2\pi)$. The specific time transformation used here is not important; what matters is that we use a function which is slowly increasing at first and rapidly increasing away from 0. The result is a heavy concentration of points near the origin and low concentrations away from it. The resulting point pattern can be used to model burst of calls that occur right after a ball game of after the arrival of a train at a station.

6. CONCLUDING REMARKS

In this paper, we have specified and illustrated a procedure for constructing spatial point patterns. In particular, we have provided a generalization to spatial dimension of Markovian point processes that appears natural and versatile from a modeling perspective. Because of their close connection to Markovian point processes on the line, these processes remain tractable, and many interesting quantities can be computed. Our current work on the topic falls under the following major heads:

(a) obtaining formulae and algorithms for various descriptors of the spatial point patterns generated by such procedures;

(b) examination of simple wireless communication scenarios to understand the impact of the spatial components in the problem (location of antennae, demand patterns, interference, etc.);

(c) simple techniques of "fitting" parameters by extending the methods commonly used for line processes. These will be discussed elsewhere.

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