Front propagation in geometric and phase field models of stratified media

A. Cesaroni * C. B. Muratov † M. Novaga ‡

September 18, 2014

Abstract

We study front propagation problems for forced mean curvature flows and their phase field variants that take place in stratified media, i.e., heterogeneous media whose characteristics do not vary in one direction. We consider phase change fronts in infinite cylinders whose axis coincides with the symmetry axis of the medium. Using the recently developed variational approaches, we provide a convergence result relating asymptotic in time front propagation in the diffuse interface case to that in the sharp interface case, for suitably balanced nonlinearities of Allen-Cahn type. The result is established by using arguments in the spirit of Γ-convergence, to obtain a correspondence between the minimizers of an exponentially weighted Ginzburg-Landau type functional and the minimizers of an exponentially weighted area type functional. These minimizers yield the fastest traveling waves invading a given stable equilibrium in the respective models and determine the asymptotic propagation speeds for front-like initial data. We further show that generically these fronts are the exponentially stable global attractors for this kind of initial data and give sufficient conditions under which complete phase change occurs via the formation of the considered fronts.

*Dipartimento di Matematica, Università di Padova, Via Trieste 63, 35121 Padova, Italy, email: acesar@math.unipd.it

†Department of Mathematical Sciences, New Jersey Institute of Technology, Newark, NJ 07102, USA, email: muratov@njit.edu

‡Dipartimento di Matematica, Università di Pisa, Largo Bruno Pontecorvo 5, 56127 Pisa, Italy, email: novaga@dm.unipi.it

Contents

1 Introduction **2**

2 Assumptions **8**

3 Traveling waves in the diffuse interface case **10**
3.1 Variational principle 11
3.2 Existence of traveling waves 13
3.3 Uniform bounds 14

4 Traveling waves in the sharp interface case **17**
4.1 Variational principle 18
4.2 Existence, uniqueness and stability of generalized traveling waves 24

5 Asymptotic behavior as $\varepsilon \to 0$ **27**

A Density estimates **35**

1 Introduction

Front propagation is a phenomenon ubiquitous to nonlinear systems governed by reaction-diffusion mechanisms and their analogs, and arises in many applications, including phase transitions, combustion, chemical reactions, population dynamics, developmental biology, etc. There is now a huge literature on the subject dealing with various aspects of front propagation, from existence of traveling wave solutions, long time asymptotic behavior, various singular limits in the presence of small parameters to generalized notions of fronts and the effects of advection, randomness or stochasticity (see, e.g., the review in [44] and references therein). By a *front*, one usually understands a narrow transition region (interface) in which the solution of the underlying reaction-diffusion equation changes abruptly between two equilibria. At the core of the phenomenon of propagation is the fact that such fronts may exhibit *wave-like* long-time behavior, whereby the level sets of the solution advance in space with some positive average velocity. This geometric aspect of the problem also leads to an alternative modeling viewpoint, whereby fronts are regarded as infinitesimally thin, i.e., as hypersurfaces whose motion is governed by a geometric evolution law. In the context of phase field models considered in this paper (see below) the connection between the *diffuse* and *sharp* fronts in the respective diffuse interface and sharp interface models has been the subject of many studies [1, 2, 8, 9, 19, 20, 22, 25, 29, 34], starting with the early works [3, 14, 15, 24, 26, 41] (these lists of references are, of course, far from being exhaustive).

As a prototypical model, consider the following version of the Allen-Cahn equation in the presence of a heterogeneous forcing term:

$$\phi_t = \Delta\phi + f(\phi) + \varepsilon g(\varepsilon x). \tag{1.1}$$

Here $\phi = \phi(x,t) \in \mathbb{R}$ is a variable that depends on the spatial coordinate $x \in \mathbb{R}^n$ and time $t \geq 0$, $f(\phi) = \phi(1-\phi)(\phi - \frac{1}{2})$ is a balanced bistable nonlinearity with $\phi = 0$ and $\phi = 1$ being stable equilibria and $\phi = \frac{1}{2}$ an unstable equilibrium, $g(x)$ is some sufficiently regular periodic function and $\varepsilon > 0$ is a parameter. Such an equation may arise, e.g., in modeling the dynamics of two co-existing phases in a phase transition with non-conserved order parameter in a medium with periodically varying properties. When $\varepsilon \ll 1$, the variations of the properties are weak and slowly changing in space. It is then easy to show that in this regime there exist two uniquely defined equilibrium states (periodic with the same period as g), v_0 and v_1, with the properties:

$$v_0(x) = 2\varepsilon g(\varepsilon x) + O(\varepsilon^2) \qquad v_1 = 1 + 2\varepsilon g(\varepsilon x) + O(\varepsilon^2). \tag{1.2}$$

These correspond to the perturbations of the two coexisting phases $\phi = 0$ and $\phi = 1$, respectively, in the homogeneous Allen-Cahn equation. Now define $u = \phi - v_0$. It solves

$$u_t = \Delta u + f(u) + \varepsilon a(\varepsilon x, u), \tag{1.3}$$

where $a(\varepsilon x, u) = 6g(\varepsilon x)(u - u^2) + O(\varepsilon)$. This type of equation for $\varepsilon \ll 1$ and its solutions that invade the $u = 0$ equilibrium will be the main subject of this paper.

On formal asymptotic grounds [1, 2, 8, 14, 25, 41], the dynamics governed by (1.3) with some fixed initial condition is expected to converge as $\varepsilon \to 0$, after rescaling space and time as

$$x \to \frac{x}{\varepsilon}, \qquad t \to \frac{t}{\varepsilon^2}, \tag{1.4}$$

to a forced mean curvature flow. More precisely, for each (x,t) fixed the function $u^\varepsilon(x,t) = u(\varepsilon^{-1}x, \varepsilon^{-2}t)$, where $u(x,t)$ solves (1.3), is expected to converge to either 0 or 1 everywhere except for an $(n-1)$-dimensional evolving hypersurface $\Gamma(t) \subset \mathbb{R}^n$ separating the regions where $u = 0$ and $u = 1$ in the limit and whose equation of motion reads

$$V(x) = \frac{g(x)}{c_W} - \kappa(x), \tag{1.5}$$

where we used the fact that $g(x) = \lim_{\varepsilon\to 0} \int_0^1 a(x,u)du$. Here $V(x)$ is the velocity in the direction of the outward normal (i.e., pointing into the region where $u = 0$ in the limit) at a given point $x \in \Gamma(t)$, κ is the sum of the principal curvatures (positive if the limit set where $u = 1$ is convex), and

$$c_W := \int_0^1 \sqrt{2W(u)}\, du, \qquad W(u) := -\int_0^u f(s)\, ds, \tag{1.6}$$

where we defined the double-well potential W, associated with f, which is nonnegative and whose only zeros are $u = 0$ and $u = 1$. In view of (1.2), the same result would then hold for $\phi^\varepsilon(x,t) = \phi(\varepsilon^{-1}x, \varepsilon^{-2}t)$, where $\phi(x,t)$ solves (1.1). For well-prepared initial data such a result was rigorously established by Barles, Soner and Souganidis, interpreting (1.5) in the viscosity sense [8] (for related results on unforced mean curvature flows see [19, 20, 22, 29]). The case of more general initial conditions was also treated by Barles and Souganidis in [9]. More recently, the problem above was treated within the varifold formalism by Mugnai and Röger under weaker assumptions on the forcing term and in dimensions two and three [34]. Rigorous leading order asymptotic formulas for solutions of (1.1) in terms of solutions of (1.5) were also recently provided by Alfaro and Matano [2].

Note that, since the above mentioned results are local in space and time, they are not suitable for drawing conclusions about the behavior as $t \to +\infty$ of solutions of (1.1) for $\varepsilon \ll 1$, via the analysis of (1.5). Nevertheless, it is widely believed that (1.5) should be able to provide information about the long-time behavior of solutions of (1.1) for $\varepsilon \ll 1$. For example, in the context of the Allen-Cahn equation it is interesting to know how fast the energetically more favored phase invades the energetically less favored phase following a nucleation event. In the homogenous setting (i.e., with $g(x) = \bar{g} > 0$) this would occur via a radial front moving asymptotically with constant normal velocity, consistent with (1.5) [6, 12, 30]. In this paper we provide results for this type of questions for a particular class of heterogeneities in (1.1).

We focus on reaction-diffusion equations and mean curvature flows in infinite cylinders that describe the so-called *stratified media*. These are media that are fibered along the cylinder, i.e., those whose properties do not change along the cylinder axis, and this property can be characterized by the dependence of the nonlinearity for reaction-diffusion equations and of the forcing term for mean curvature flow only on the transverse coordinate of the cylinder. By a cylinder (in the original, unscaled variables), we mean a set $\Sigma_\varepsilon = \Omega_\varepsilon \times \mathbb{R} \subset \mathbb{R}^n$, where $\Omega_\varepsilon = \varepsilon^{-1}\Omega$ and $\Omega \subset \mathbb{R}^{n-1}$ is either a bounded domain with sufficiently smooth boundary or an $(n-1)$-dimensional parallelogram with periodic boundary conditions, covering the case discussed earlier in the presence of an axis of translational symmetry. In the case of a general bounded domain Ω, we also supplement the problem with homogeneous Neumann boundary conditions. Our main interest is to provide a convergence result to relate the asymptotic characteristics of front propagation in the diffuse interface case with those in the sharp interface case as $t \to \infty$ when $\varepsilon \ll 1$. More precisely, we wish to characterize the asymptotic propagation speeds for fronts in Σ_ε invading the $u = 0$ equilibrium in the case of (1.3) or, equivalently, the $\phi = v_0$ equilibrium in the case of (1.1), in terms of uniformly translating graphs solving (1.5). We also wish to characterize the shape of the long time limit of the fronts for (1.3) and their relation to those for (1.5) in the spirit of Γ-convergence (as is done for stationary fronts in [33]).

Variational formulation. Our methods are essentially variational. This stems from the basic observation [35] (in the one-dimensional setting the idea goes back to [26]) that, when the nonlinearity (1.3) is translationally invariant along the cylinder axis, the solution of this equation in the reference frame moving with speed $c > 0$ along the cylinder Σ_ε may be viewed as a gradient flow in $L^2_c(\Sigma_\varepsilon) := L^2(\Sigma_\varepsilon; e^{cz}dx)$ generated by the exponentially weighted Ginzburg-Landau type functional

$$\Phi_c(u) = \int_{\Sigma_\varepsilon} e^{cz}\left(\frac{1}{2}|\nabla u|^2 + V(u,y)\right)dx. \tag{1.7}$$

Here $x = (y,z) \in \Sigma_\varepsilon$, with $y \in \Omega_\varepsilon$ being the transverse coordinate in the cylinder cross-section and $z \in \mathbb{R}$ the coordinate along the cylinder axis in the direction of propagation, and $V(u,y)$ is a suitably chosen potential (see Section 3). In particular, traveling wave solutions of (1.3) with speed c that belong to the exponentially weighted Sobolev space $H^1_c(\Sigma_\varepsilon)$, i.e., the space consisting of all functions in $L^2_c(\Sigma_\varepsilon)$ with first derivatives in $L^2_c(\Sigma_\varepsilon)$, are fixed points of this gradient flow (see [31, 36]). In the simplest case of (1.1) with the considered cubic nonlinearity and spatially homogeneous forcing $g(x) = \bar{g} > 0$, it follows from [36] via analysis of (1.3) and an explicit computation that for all $\bar{g} < \sqrt{3}/(36\varepsilon)$ there exists a unique value of $c^\dagger > 0$ satisfying

$$c^\dagger - \frac{8}{9}(c^\dagger)^3 = 6\sqrt{2}\varepsilon\bar{g}, \tag{1.8}$$

and a profile $\bar{u} \in H^1_{c^\dagger}(\Sigma_\varepsilon)$ depending only on z such that $\bar{u}$ is the unique (up to translations) minimizer of $\Phi_{c^\dagger}$ over its natural domain (see below). Furthermore, by the results of [38] the solution of the initial value problem for (1.1) with the initial datum in the form of a sharp front: $\phi(x,0) = v_0$ for $z > h(y)$ and $\phi(x,0) = v_1$ for $z \le h(y)$, with $h \in C(\overline{\Omega}_\varepsilon)$, converges as $t \to \infty$ exponentially fast to $v_0 + \bar{u}$ after a translation by $R_\infty - c^\dagger t$ for some $R_\infty \in \mathbb{R}$. Thus, for every $\varepsilon > 0$ sufficiently small the solution approaches a flat front perpendicular to the cylinder axis, which, after the rescaling in (1.4), moves with the normal velocity

$$V = 6\sqrt{2}\bar{g} + O(\varepsilon^2). \tag{1.9}$$

This is consistent with the plane wave solution of (1.5), in view of the fact that $c_W = 1/(6\sqrt{2})$ according to (1.6). A flat front with speed $c_0^\dagger = 6\sqrt{2}\bar{g}$ is also the asymptotic solution of (1.5) with $g(x) = \bar{g}$ and an initial condition in the form of a graph on Ω [18]. Furthermore, the corresponding function $\psi(y) = const$ minimizes, for $c = c_0^\dagger$, the following exponentially weighted area type functional

$$F_c(\psi) = \int_\Omega e^{c\psi(y)}\left(c_W\sqrt{1 + |\nabla\psi(y)|^2} - \frac{g(y)}{c}\right)dy, \tag{1.10}$$

among all $\psi \in C^1(\overline{\Omega})$. Here ψ defines the graph $z = \psi(y)$ that represents the sharp interface front. Note that the functional F_c has a well-known geometric characterization, which, however, requires some care [28]. Let us introduce the following exponentially weighted perimeter of a set $S \subseteq \Sigma$:

$$\mathrm{Per}_c(S,\Sigma) := \sup \left\{ \int_S e^{cz} \left(\nabla \cdot \phi + c\hat{z} \cdot \phi \right) dx : \phi \in C_c^1(\Sigma;\mathbb{R}^n),\ |\phi| \leq 1 \right\}, \tag{1.11}$$

where $\hat{z}$ denotes the unit vector pointing in the z-direction. Notice that, if the set S has locally finite perimeter, we can write

$$\mathrm{Per}_c(S,\Sigma) = \int_{\partial^* S \cap \Sigma} e^{cz}\, d\mathcal{H}^{n-1}(x), \tag{1.12}$$

where $\partial^* S$ denotes the reduced boundary of S [5,18].

We then define the following geometric functional on measurable sets $S \subset \Sigma$ with weighted volume $\int_S e^{cz} dx < +\infty$:

$$\mathcal{F}_c(S) := c_W \,\mathrm{Per}_c(S,\Sigma) - \int_S e^{cz}\, g(y)\, dx. \tag{1.13}$$

By our assumptions, this functional is indeed well defined for all such sets. Since the functionals in (1.10) and (1.13) agree whenever $S = \{z < \psi(y)\}$ and ψ is sufficiently smooth [18], in the example in which $g(x) = \bar{g} > 0$ the sets $\{z < \mathit{const}\}$ minimize $\mathcal{F}_c$ for $c = c_0^\dagger$ over all such sets. In fact, it is easy to see that they also minimize $\mathcal{F}_c$ over its entire domain (see Sec. 4).

Problem formulation and main results. The purpose of this paper is to study the long-time behavior of solutions of (1.1) or (1.3) for $\varepsilon \ll 1$ via the analysis of traveling wave solutions to (1.5). In particular we characterize the asymptotic propagation speed and the shape of the long time limit of fronts invading the $u = 0$ equilibrium in the case of (1.3) or, equivalently, the $\phi = v_0$ equilibrium in the case of (1.1), in terms of uniformly translating graphs solving (1.5).

Throughout the paper we always assume that Ω is a bounded domain with a sufficiently smooth boundary (for precise assumptions see Section 2). All the results obtained in this paper remain valid in the periodic setting, so we do not treat this case separately. We set $\Sigma := \Omega \times \mathbb{R}$, and in Σ we consider the family of singularly perturbed reaction-diffusion equations for $u = u(x,t) \in \mathbb{R}$, with parameter $\varepsilon > 0$ and the space and time rescaled according to (1.4):

$$\varepsilon u_t = \varepsilon \Delta u + \frac{1}{\varepsilon} f(u) + a(y,u) \qquad (x,t) \in \Sigma \times (0,+\infty), \tag{1.14}$$

with initial datum $u(x,0) = u_0(x) \geq 0$ and Neumann boundary conditions on $\partial\Sigma$. Here $f(u)$ is a balanced bistable nonlinearity with $f(0) = f(1) = 0$, and $|a(y,u)| \leq Cu$ for some

$C > 0$. For simplicity, we also assume that $a(x,u)$ does not depend on ε. Once again, the obtained results remain valid after perturbing a with terms that can be controlled by $C\varepsilon u$ for some $C > 0$ independent of ε.

As was already mentioned, the singular limit of (1.14) as $\varepsilon \to 0$ was considered in [8], where convergence, in a suitable sense, of positive solutions to the level-set formulation of the mean curvature flow with a suitable forcing term g was proved. Consider a family of measurable sets $S(t) \subseteq \Sigma$ with regular boundary, such that $\Gamma(t) = \partial S(t)$ evolves according to (1.5) with

$$g(y) := \int_0^1 a(y,s)ds. \tag{1.15}$$

We associate to this flow the following quasilinear parabolic problem for $h = h(y,t) \in \mathbb{R}$ in Ω, which corresponds to (1.5):

$$h_t = \sqrt{1 + |\nabla h|^2} \left[\nabla \cdot \left(\frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right) + \frac{g}{c_W} \right] \quad \text{in } \Omega \times (0, +\infty), \tag{1.16}$$

with initial datum $h(y,0) = h_0(y)$, and Neumann boundary conditions on $\partial\Omega$. Note that the subgraph $S(t) = \{(y,z) \in \Sigma : z < h(y,t)\}$ of the solution of (1.16) coincides with the family of sets evolving according to (1.5), with initial datum $S_0 = \{(y,z) \in \Sigma : z < h_0(y)\}$.

In Section 3, we extend the results of [36] on existence of traveling waves solutions to (1.14) of maximal propagation speed $c_\varepsilon^\dagger$ to the considered problem for every ε sufficiently small, under an assumption on the forcing term g for the limit problem, which is Assumption 3 (see Theorem 3.5). Moreover, we show that under a stronger condition on the forcing term g, which is Assumption 4, the traveling wave with maximal speed of propagation is connecting two nondegenerate stable equilibria (see Proposition 5.6). The nondegeneracy of the equilibria is an important property for proving that these waves are long-time attractors for solutions to (1.14).

As for the forced mean curvature flow, we study in Section 4 the existence of generalized traveling wave solutions, according to Definition 4.2, appropriately adapting to the present case the arguments developed for the periodic case in [18]. The main result is Theorem 4.8 which states, under Assumption 3, the existence of a maximal speed of propagation of generalized traveling waves and provides an accurate description of the waves traveling at maximal speed. Moreover, in Theorem 4.11 and Theorem 4.12 it is proved that, under the stronger Assumption 4, the traveling waves moving with the maximal speed are unique and are attractors for the forced mean curvature flow (1.16).

Section 5 contains the main results of the paper. The first one is Theorem 5.3, which provides a convergence result relating the propagation of diffuse and sharp interfaces. In particular, we prove that as $\varepsilon \to 0$ the maximal propagation speed of the traveling waves of (1.14) converges to the maximal speed of propagation for (generalized) traveling waves of (1.16) (for some previous related results see [37, Proposition 4.3 and Theorem 5.3]). By Corollary 5.4, the latter is then the average speed of the leading edge for general front like initial data in the limit $\varepsilon \to 0$. We also show that as $\varepsilon \to 0$ the traveling waves of (1.14) moving with the maximal speed converge, up to translations, to the characteristic function of a set whose boundary is a traveling wave of (1.16), moving with maximal speed. The convergence is along subsequences and holds under Assumption 3. Under the stronger Assumption 4, we can show that the limit is independent of the subsequence. The result is Theorem 5.5, which states that under Assumption 4, the long time limit of solutions to (1.14) converges, as $\varepsilon \to 0$, to a traveling wave solution to (1.16), translating with maximal speed $c^\dagger$. In addition, in our proofs we employed some new uniform estimates for minimizers of Ginzburg-Landau functionals with respect to compactly supported perturbations, which extend those of [13, 23, 39] and are of independent interest. These are presented in the Appendix.

Notations. Throughout the paper H^1, BV, L^p, C^k, C^k_c, $C^{k,\alpha}$ denote the usual spaces of Sobolev functions, functions of bounded variation, Lebesgue functions, continuous functions with k continuous derivatives, k-times continuously differentiable functions with compact support, continuously differentiable functions with Hölder-continuous derivatives of order k for $\alpha \in (0,1)$ (or Lipschitz-continuous when $\alpha = 1$), respectively. For a point $x \in \Sigma$ in the cylinder $\Sigma = \Omega \times \mathbb{R}$, where $\Omega \subset \mathbb{R}^{n-1}$, we always write $x = (y,z)$, where $y \in \Omega$ is the transverse coordinate and $z \in \mathbb{R}$ is the coordinate along the cylinder axis. Depending on the context, the symbol ∇ is understood to denote differential operators acting on functions defined on either the whole cylinder Σ, or on its cross-section Ω. The symbol $B(x,r)$ stands for the open ball in $\mathbb{R}^n$ with radius r centered at x, and for a set A the symbols $\overline{A}$, $|A|$ and χ_A always denote the closure of A, the Lebesgue measure of A and the characteristic function of A, respectively. We also use the notation $⨍_A u^2 dx = \frac{1}{|A|}\int_A u^2 dx$, and the convention that $\ln 0 = -\infty$ and $e^{-\infty} = 0$.

2 Assumptions

We start by listing the assumptions we shall make on the nonlinearities f and a, and the corresponding forcing g appearing in the evolution problems. We associate to f and a the potentials

$$W(u) := -\int_0^u f(s)\,ds, \qquad G(y,u) := \int_0^u a(y,s)\,ds. \tag{2.1}$$

Recall the definition of the forcing term g in (1.15):

$$g(y) := G(y,1). \tag{2.2}$$

We now state our assumptions on the functions a and f. Let $\alpha \in (0,1]$ be such that $\partial\Omega$ is of class $C^{2,\alpha}$.

Assumption 1. $a \in C^\alpha_{loc}(\overline{\Omega} \times \mathbb{R})$, $a_u \in C^\alpha_{loc}(\overline{\Omega} \times \mathbb{R})$, $a(\cdot,0) = 0$.

Assumption 2. $f \in C^{1,\alpha}_{loc}(\mathbb{R})$, $f(0) = f(1) = 0$, $f'(0) < 0$, $f'(1) < 0$, $W(1) = W(0) = 0$, $W(u) > 0$ for all $u \neq 0, 1$, and $\liminf_{|u|\to\infty} W(u) > 0$.

Assumption 2 implies that $W(u)$ is a balanced non-degenerate double-well potential (as a model function one could think of $W(u) = \frac{1}{4}u^2(1-u)^2$ corresponding to the example considered in the introduction). However, we do not require that f has only one other zero, which is located in $(0,1)$, as is usually done in the literature. Instead, we only assume that $u = 0$ and $u = 1$ have the same value of W, and that W is greater for all other values of u, including at infinity. Note that by Assumptions 1 and 2 there exists $C, \delta_0 > 0$, depending only on f and a, such that for every $\varepsilon \leq C^{-1}\delta_0$

$$\varepsilon^{-1}W(u) - G(y,u) \geq 0 \qquad \forall (y,u) \in \overline{\Omega} \times \left(\mathbb{R}\backslash(1 - C\sqrt{\varepsilon}, 1 + C\sqrt{\varepsilon})\right), \tag{2.3}$$

and

$$\varepsilon^{-1}W(\cdot) - G(y,\cdot) \text{ is increasing on } [1 + C\varepsilon, 1 + \delta_0] \qquad \forall y \in \overline{\Omega}. \tag{2.4}$$

Remark 2.1. Observe that, if the initial datum u_0 satisfies $0 \leq u_0(x) \leq 1 + \delta$ for some $\delta \in (0, \delta_0)$ and all $x \in \Sigma$, then by the maximum principle and (2.4) we have $0 \leq u(x,t) \leq 1 + \delta$ for all $(x,t) \in \Sigma \times [0, +\infty)$ and all $\varepsilon \leq C^{-1}\delta$.

We recall the standard definition of the perimeter of a measurable set $A \subseteq \Omega$ relative to Ω [5, 28]:

$$\mathrm{Per}(A,\Omega) := \sup\left\{ \int_A \nabla \cdot \phi(y)dy : \ \phi \in C^1_c(\Omega; \mathbb{R}^{n-1}), \ |\phi| \leq 1 \right\}. \tag{2.5}$$

With the help of (2.5), the standing assumption to study front propagation problem for (1.14) and (1.16) will be the following condition on the forcing term g:

Assumption 3. Let $g \in C^\alpha(\overline{\Omega})$. Then there exists $A \subseteq \Omega$ such that

$$\int_A g(y)dy \ > c_W \,\mathrm{Per}(A,\Omega). \tag{2.6}$$

This assumption basically ensures that the trivial state $u = 0$ is energetically less favorable for ε sufficiently small, resulting in the existence of the invasion fronts.

Remark 2.2. Notice that (2.6) implies, in particular, that $\sup_\Omega g > 0$, and is automatically satisfied if

$$\int_\Omega g(y)dy > 0. \tag{2.7}$$

Finally, we list an additional assumption under which stronger conclusions about the convergence of fronts can be made.

Assumption 4. Let $g \in C^{\alpha}(\overline{\Omega})$ and assume that (2.7) holds. Then $\Omega \times \mathbb{R}$ is the unique minimizer of $\mathcal{F}_{c^\dagger}$ under compact perturbations among sets $S = \omega \times \mathbb{R}$ with $\omega \subseteq \Omega$ and $c^\dagger := \inf\{c > 0 : \inf \mathcal{F}_c \geq 0\} \in (0, \infty)$.

Clearly, Assumption 4 is quite implicit. In Proposition 4.10 we give some sufficient conditions for it to hold, first in the two-dimensional case and then in every dimension.

Throughout the rest of the paper Assumptions 1–3 are always taken to be satisfied, with g defined by (2.2). The consequences of Assumption 4 will be explored in Section 4.2.

3 Traveling waves in the diffuse interface case

In this section we consider the front propagation problem in the cylinder Σ for the reaction-diffusion equation in (1.14) with $\varepsilon > 0$. We are particularly interested in the special solutions of the reaction-diffusion equation (1.14) in the form of *traveling waves*, i.e., solutions of (1.14) of the form $u(x,t) = \bar{u}(y, z - ct)$, for some $c \in \mathbb{R}$ and $\bar{u} \in C^2(\Sigma) \cap C^1(\overline{\Sigma}) \cap L^\infty(\Sigma)$. The constant c is referred to as the wave speed and the function $\bar{u}$ as the traveling wave profile. In particular, the profile of the traveling wave solves the equation

$$\varepsilon \Delta \bar{u} + c\varepsilon \bar{u}_z + \frac{1}{\varepsilon} f(\bar{u}) + a(y, \bar{u}) = 0 \qquad (y, z) \in \Sigma, \tag{3.1}$$

with Neumann boundary conditions $\nu \cdot \nabla \bar{u} = 0$ on $\partial\Sigma$.

More specifically, we are interested in the traveling wave solutions in the form of fronts invading the equilibrium $v = 0$ from above. By an equilibrium for (1.14), we mean a function $v : \overline{\Omega} \to \mathbb{R}$ which solves

$$\varepsilon \Delta v + \frac{1}{\varepsilon} f(v) + a(y, v) = 0 \qquad y \in \Omega, \tag{3.2}$$

with $\nu \cdot \nabla v = 0$ on $\partial\Omega$. Note that by our assumptions $v = 0$ is always an equilibrium. In terms of the propagation speed and the traveling wave profile, front solutions invading zero from above are bounded solutions of (3.1) that satisfy

$$c > 0, \qquad \bar{u} > 0, \qquad \text{and } \bar{u}(\cdot, z) \to 0 \text{ uniformly as } z \to +\infty. \tag{3.3}$$

Note that existence and qualitative properties of traveling fronts in a variety of settings have been extensively studied, starting with the classical work of Berestycki and Nirenberg [10], who analyzed (in our setting and using our notation) traveling fronts connecting zero with some equilibrium $v > 0$, i.e., those positive front solutions of (3.1) that also satisfy $\bar{u}(\cdot, z) \to v$ uniformly as $z \to -\infty$. We point out that existence of the considered solutions is not guaranteed in general. In particular, we have to impose some condition on g assuring the existence of non-trivial positive equilibria. The existence of such non-trivial equilibria for $\varepsilon \ll 1$ will be a consequence of Assumption 3 (see Proposition 3.4). Similarly, although for a fixed equilibrium $v > 0$ as the limit at $z = -\infty$ there is at most one (modulo translations) front solution (see [10], under some technical assumptions, and [38] for a general result in the class of the so-called variational traveling waves), in general front solutions of (3.1) may not be unique. There is, however, at most one front solution of (3.1) which governs the propagation behavior of solutions of (1.14) with front-like initial data. These solutions can be characterized variationally (see [36] and the following section) and are the main subject of our study.

3.1 Variational principle

Following the variational approach to front propagation problems [36] (see also [31, 35, 37, 38]), for every $c > 0$ we associate to the reaction-diffusion equation in (1.14) the energy functional (for fixed $\varepsilon > 0$)

$$\Phi_c^\varepsilon(u) = \int_\Sigma e^{cz} \left(\frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) - G(y, u) \right) dx. \tag{3.4}$$

This functional is naturally defined on $H_c^1(\Sigma) \cap L^\infty(\Sigma)$, where $H_c^1(\Sigma)$ is an exponentially weighted Sobolev space with the norm

$$\|u\|^2_{H_c^1(\Sigma)} = \int_\Sigma e^{cz} (|\nabla u|^2 + |u|^2)\, dx. \tag{3.5}$$

Furthermore, the functional Φ_c^ε is differentiable in $H_c^1(\Sigma) \cap L^\infty(\Sigma)$, and its critical points satisfy the traveling wave equation (3.1) [31, 36].

Remark 3.1. Following [10, Section 4] (see also [43, Theorem 4.1] and [36, Theorem 3.3(iii)]) one can show that every traveling wave solution (c, u) of (1.14), with $c > 0$ and satisfying (3.3), belongs to $H_c^1(\Sigma)$ and is a critical point of Φ_c^ε. In particular, non-trivial minimizers of Φ_c^ε are the fastest traveling wave solutions of (1.14) invading the equilibrium $u = 0$ [31, 36]. We observe that non-trivial critical points of Φ_c^ε which are not minimizers may also exist and correspond to traveling waves with lower speeds (see the discussion in [36, Section 6]).

Critical points of Φ_c^ε and, in particular, minimizers of Φ_c^ε play an important role for the long-time behavior of the solutions of the initial value problem associated with (1.14) in the case of front-like initial data. Indeed, in [38] it is proved under generic assumptions on the nonlinearity (see also [35, 36]) that the non-trivial minimizers of Φ_c^ε over $H_c^1(\Sigma)$ are selected as long-time attractors for the initial value problem associated to (1.14) with front-like initial data. Also, in [36] it was proved under minimal assumptions on the nonlinearity that the speed of the leading edge of the solution is determined by the unique value of $c_\varepsilon^\dagger > 0$ for which $\Phi^\varepsilon_{c_\varepsilon^\dagger}$ has a non-trivial minimizer. Appropriate assumptions to guarantee existence of minimizers of Φ_c^ε were given in [36]. Here we show that in our case these conditions are verified for every ε sufficiently small (see Theorem 3.5).

Let us introduce an auxiliary functional

$$E^\varepsilon(v) = \int_\Omega \left(\frac{\varepsilon}{2}|\nabla v|^2 + \frac{W(v)}{\varepsilon} - G(y,v) \right) dy \qquad v \in H^1(\Omega) \cap L^\infty(\Omega). \tag{3.6}$$

Note that $v = 0$ is always a critical point of this functional for every ε. Moreover, every critical point of E^ε is an equilibrium for the reaction-diffusion equation (1.14).

Remark 3.2. By [33] (see also [11]) we have that

$$\Gamma - \lim_{\varepsilon \to 0} E^\varepsilon(u) = \begin{cases} E^0(A) := c_W \mathrm{Per}(A,\Omega) - \int_A g\, dy & \text{if } u = \chi_A, \\ +\infty & \text{otherwise,} \end{cases} \tag{3.7}$$

where c_W is defined in (1.6), A is Lebesgue measurable and the convergence is understood in the sense of Γ-convergence in $L^1(\Omega)$ [11].

Definition 3.3. *A function $v \in H^1(\Omega) \cap L^\infty(\Omega)$ is a stable critical point of E^ε if it is a critical point of the functional and the second variation of E^ε is nonnegative, i.e.*

$$\int_\Omega \Big(\varepsilon |\nabla \phi|^2 + \big(\varepsilon^{-1} W''(v) - G_{uu}(y,v)\big)\, \phi^2 \Big) dy \geq 0 \qquad \forall \phi \in H^1(\Omega)\,. \tag{3.8}$$

Moreover, v is a nondegenerate stable critical point of E^ε if strict inequality holds in (3.8).

Proposition 3.4. *Under Assumptions 1, 2 and 3, there exist positive constants ε_0 and C such that for all $\varepsilon < \varepsilon_0$ there exists $v_\varepsilon^0 \in H^1(\Omega)$ such that $0 \leq v_\varepsilon^0 \leq 1$ and $E^\varepsilon(v_\varepsilon^0) < 0$.*

Proof. Without loss of generality we may assume that the set $A \subseteq \Omega$ in Assumption 3 has a smooth boundary. Then, by Remark 3.2 we have

$$\Gamma - \lim_{\varepsilon \to 0} E^\varepsilon(\chi_A) = E^0(A)$$

where E^0 is defines in (3.7). Recalling that A has smooth boundary, by the Γ-limsup construction in [33] (see also Theorem 5.3 below) there exists a family of functions $v_\varepsilon^0 \in H^1(\Omega)$, with $0 \leq v_\varepsilon^0 \leq 1$ such that $v_\varepsilon^0 \to \chi_A$ in $L^1(\Omega)$ as $\varepsilon \to 0$, and

$$\lim_{\varepsilon \to 0} E^\varepsilon(v_\varepsilon^0) = E^0(A) < 0.$$

In particular, $E^\varepsilon(v_\varepsilon^0) < 0$ for all ε sufficiently small. $\square$

By Assumption 2, $v = 0$ is a non-degenerate stable critical point of the functional E^ε for every ε sufficiently small. Indeed, defining

$$\nu_0^\varepsilon := \min_{\int_\Omega \phi^2 = 1} \int_\Omega \Big(\varepsilon |\nabla \phi|^2 + \big(\varepsilon^{-1} W''(0) - G_{uu}(y,0)\big)\, \phi^2 \Big) dy, \tag{3.9}$$

observe that there exists $\varepsilon_0 > 0$, depending on $W''(0)$ and $\|a_u(\cdot,0)\|_\infty$, such that

$$\nu_0^\varepsilon > 0 \qquad \text{for all } \varepsilon < \varepsilon_0. \tag{3.10}$$

3.2 Existence of traveling waves

We now state an existence result for the diffuse interface problem. This result is a minor modification of the one in [36, Theorem 3.3].

Theorem 3.5. *Under Assumptions 1, 2 and 3, there exist positive constants ε_0 and C, depending on f, a and Ω, such that for all $0 < \varepsilon < \varepsilon_0$ there exists a unique $c_\varepsilon^\dagger > 0$ such that*

i) $\Phi^\varepsilon_{c_\varepsilon^\dagger}$ *admits a non-trivial minimizer* $\bar{u}_\varepsilon \in H^1_{c_\varepsilon^\dagger} \cap L^\infty(\Sigma)$ *which satisfies*

$$\sup\left\{z \in \mathbb{R} \mid \sup_{y\in\Omega} \bar{u}_\varepsilon(y,z) > \frac{1}{2}\right\} = 0. \tag{3.11}$$

ii) $\bar{u}_\varepsilon \in C^2(\overline{\Sigma}) \cap W^{1,\infty}(\Sigma)$, *and* $(c_\varepsilon^\dagger, \bar{u}_\varepsilon)$ *is a traveling wave solution to* (1.14).

iii) $0 < \bar{u}_\varepsilon \leq 1 + C\varepsilon$, $(\bar{u}_\varepsilon)_z < 0$ *in* $\overline{\Sigma}$, *and*

$$\lim_{z\to+\infty} \bar{u}_\varepsilon(\cdot, z) = 0 \qquad \lim_{z\to-\infty} \bar{u}_\varepsilon(\cdot, z) = v_\varepsilon \qquad \text{in } C^1(\overline{\Omega}),$$

where v_ε is a stable critical point of E^ε in (3.6) *with $E^\varepsilon(v_\varepsilon) < 0$.*

iv) $\Phi^\varepsilon_{c_\varepsilon^\dagger}(\bar{u}_\varepsilon) = 0$, *and all non-trivial minimizers of* $\Phi^\varepsilon_{c_\varepsilon^\dagger}$ *are translates of* $\bar{u}_\varepsilon$ *along* z.

Proof. By (2.3), (2.4) and Assumption 2 it follows that, for ε sufficiently small, there holds

$$\varepsilon^{-1}W(u) - G(\cdot, u) > 0$$

for all $u < 0$, and

$$\varepsilon^{-1}W(u) - G(\cdot, u) > \varepsilon^{-1}W(1 + C\varepsilon) - G(\cdot, 1 + C\varepsilon)$$

for all $u > 1 + C\varepsilon$. By a cutting argument (as in [36, Theorem 3.3(i)]), we then get, for any $c > 0$ and $u \in H^1_c(\Sigma) \cap L^\infty(\Sigma)$, that

$$\Phi^\varepsilon_c(\tilde{u}) \leq \Phi^\varepsilon_c(u),$$

where $\tilde{u}(x) := \max(0, \min(u(x), 1 + C\varepsilon))$, with strict inequality if $\operatorname{ess\,sup}_\Sigma u > 1 + C\varepsilon$ or $\operatorname{ess\,inf}_\Sigma u < 0$. Therefore, the result follows from [36, Theorem 3.9] by minimizing Φ^ε_c over functions with values in $[0, 1+C\varepsilon]$. Notice that the assumptions in [36] are satisfied thanks to Proposition 3.4 and (3.10). The estimate (3.11) is due to the fact that, since $\Phi^\varepsilon_{c_\varepsilon^\dagger}(\bar{u}_\varepsilon) = 0$, by (2.3) we have $\|u_\varepsilon\|_{L^\infty(\Sigma)} > \frac{1}{2}$. Note that in view of our regularity assumption on $\partial\Omega$, we get a slightly higher regularity of $\bar{u}_\varepsilon$ up to the boundary of Σ. The fact that $(\bar{u}_\varepsilon)_z < 0$ up to the boundary of Σ follows by the strong maximum principle applied to the elliptic equation satisfied by $(\bar{u}_\varepsilon)_z$, with Neumann boundary conditions, obtained by differentiating (3.1) in z. ∎

Note that the choice of the value $\frac{1}{2}$ in (3.11) is arbitrary, and every other value in $(0,1)$ could be used equivalently.

Remark 3.6. Observe that $\Phi_c^\varepsilon(u(y,z-a)) = e^{ca}\Phi_c^\varepsilon(u(y,z))$ for all $c>0$, $a\in\mathbb{R}$ and $u\in H_c^1(\Sigma)\cap L^\infty(\Sigma)$. In particular, if $\bar{u}_\varepsilon$ is a non-trivial minimizer of $\Phi^\varepsilon_{c_\varepsilon^\dagger}$, then $\Phi^\varepsilon_{c_\varepsilon^\dagger}(\bar{u}_\varepsilon(y,z)) = \Phi^\varepsilon_{c_\varepsilon^\dagger}(\bar{u}_\varepsilon(y,z-a)) = 0$. Moreover, from [36, Theorem 3.9] we have that $\Phi_c^\varepsilon(u)>0$ for every non-zero $u\in H_c^1(\Sigma)\cap L^\infty(\Sigma)$ and $c>c_\varepsilon^\dagger$, while $\inf\Phi_c^\varepsilon(u)=-\infty$ for $c<c_\varepsilon^\dagger$.

3.3 Uniform bounds

We next establish several properties of the traveling wave solutions $(c_\varepsilon^\dagger,\bar{u}_\varepsilon)$ in Theorem 3.5 that will allow us to pass to the limit as $\varepsilon\to 0$ in Section 5.

We begin by proving an isoperimetric type inequality for the weighted perimeter Per_c.

Proposition 3.7. *Let $c>0$ and let $S\subset\Sigma$ be a measurable set with $\int_S e^{cz}dx<\infty$. Then*

$$\mathrm{Per}_c(S,\Sigma)\geq c\int_S e^{cz}dx. \tag{3.12}$$

Proof. By the definition of the weighted perimeter in (1.11), we have

$$\mathrm{Per}_c(S,\Sigma)\geq \int_S e^{cz}\left(\nabla\cdot\phi + c\hat{z}\cdot\phi\right)dx, \tag{3.13}$$

for any admissible test function ϕ. Choosing $\phi(y,z)=\eta_\delta(y,z)\hat{z}$, where the cutoff function $\eta_\delta(y,z):=\eta(\delta^{-1}\mathrm{dist}(y,\partial\Omega))(1-\eta(\delta|z|))$ and $\eta\in C^1(\mathbb{R})$ with $0\leq\eta'(x)\leq 2$ for all $x\in\mathbb{R}$, $\eta(x)=0$ for all $x\leq 1$ and $\eta(x)=1$ for all $x\geq 2$, for $\delta>0$ sufficiently small we find that

$$\mathrm{Per}_c(S,\Sigma)\geq c\int_S e^{cz}\,\eta_\delta(y,z)\,dx - \delta\int_S e^{cz}\eta'(\delta|z|)dx\,. \tag{3.14}$$

Then, passing to the limit as $\delta\to 0$, we conclude by the monotone convergence theorem and the fact that $\int_S e^{cz}\eta'(\delta|z|)dx\leq 2\int_S e^{cz}dx<\infty$. □

We now prove a uniform upper bound for the speeds $c_\varepsilon^\dagger$ as $\varepsilon\to 0$.

Proposition 3.8. *Let ε and $c_\varepsilon^\dagger$ be as in Theorem 3.5. Then there exist constants $\varepsilon_1>0$ and $M>0$ depending only on W and G such that $0<\bar{u}_\varepsilon<\frac{3}{2}$ and $c_\varepsilon^\dagger\leq M$ for all $0<\varepsilon<\varepsilon_1$.*

Proof. By Theorem 3.5 and Assumptions 1 and 2 we may take $\varepsilon_1\in(0,\varepsilon_0)$ so small that $0<\bar{u}_\varepsilon<\frac{3}{2}$ and that $\varepsilon^{-1}W(u)-2G(y,u)\geq 0$ whenever $0\leq u\leq\frac{3}{4}$. In particular, the set $\{\bar{u}_\varepsilon > \frac{3}{4}\}$ has positive measure. Furthermore, we have $W(u) \geq K(1-u)^2$ whenever $u \in [\frac{1}{2}, \frac{3}{2}]$, for some $K \in (0,1]$. Hence

$$\begin{aligned}
\Phi^\varepsilon_{c^\dagger_\varepsilon}(\bar{u}_\varepsilon) &\geq \frac{K}{2}\int_{\{\bar{u}_\varepsilon > \frac{1}{2}\}} e^{c^\dagger_\varepsilon z}\Big(\varepsilon|\nabla \bar{u}_\varepsilon|^2 + \varepsilon^{-1}(1-\bar{u}_\varepsilon)^2\Big)dx \\
&\quad - \int_{\{\bar{u}_\varepsilon > \frac{3}{4}\}} e^{c^\dagger_\varepsilon z} G(y,\bar{u}_\varepsilon)\,dx \\
&\geq \frac{K}{2}\int_{\{\bar{u}_\varepsilon > \frac{1}{2}\}} e^{c^\dagger_\varepsilon z}|2(1-\bar{u}_\varepsilon)\nabla \bar{u}_\varepsilon|dx - \int_{\{\bar{u}_\varepsilon > \frac{3}{4}\}} e^{c^\dagger_\varepsilon z} G(y,\bar{u}_\varepsilon)\,dx \\
&= \frac{K}{2}\int_{\{\bar{u}_\varepsilon > \frac{1}{2}\}} e^{c^\dagger_\varepsilon z}|\nabla h(\bar{u}_\varepsilon)|dx - \int_{\{\bar{u}_\varepsilon > \frac{3}{4}\}} e^{c^\dagger_\varepsilon z} G(y,\bar{u}_\varepsilon)\,dx\,,
\end{aligned}$$

where $h(u) := \frac{1}{4} + (u-1)|u-1|$ is an increasing function of u with $h(\frac{1}{2}) = 0$. In turn, by the co-area formula [5] we get

$$\Phi^\varepsilon_{c^\dagger_\varepsilon}(\bar{u}_\varepsilon) \geq \frac{K}{2}\int_0^{\frac{1}{2}} \mathrm{Per}_{c^\dagger_\varepsilon}(\{h(\bar{u}_\varepsilon) > t\})dt - \int_{\{\bar{u}_\varepsilon > \frac{3}{4}\}} e^{c^\dagger_\varepsilon z} G(y,\bar{u}_\varepsilon)\,dx. \tag{3.15}$$

On the other hand, by Proposition 3.7 we have

$$\mathrm{Per}_{c^\dagger_\varepsilon}(\{h(\bar{u}_\varepsilon) > t\}) \geq c^\dagger_\varepsilon \int_{\{h(\bar{u}_\varepsilon) > t\}} e^{c^\dagger_\varepsilon z}dx,$$

where we noted that by Theorem 3.5(iii) the sets $\{(y,z) : h(\bar{u}_\varepsilon(y,z)) > t\}$ are bounded above in z uniformly in y for each $t > 0$ and, hence, Proposition 3.7 applies. Therefore, substituting this inequality into (3.15) and using the layer cake theorem yields

$$\begin{aligned}
\Phi^\varepsilon_{c^\dagger_\varepsilon}(\bar{u}_\varepsilon) &\geq \frac{Kc^\dagger_\varepsilon}{2}\int_0^{\frac{1}{2}}\int_{\{h(\bar{u}_\varepsilon) > t\}} e^{c^\dagger_\varepsilon z}dx\,dt - \int_{\{\bar{u}_\varepsilon > \frac{3}{4}\}} e^{c^\dagger_\varepsilon z} G(y,\bar{u}_\varepsilon)dx \\
&\geq \int_{\{\bar{u}_\varepsilon > \frac{3}{4}\}} e^{c^\dagger_\varepsilon z}\left(\frac{1}{2}Kc^\dagger_\varepsilon h(\bar{u}_\varepsilon) - G(y,\bar{u}_\varepsilon)\right)dx \\
&\geq \int_{\{\bar{u}_\varepsilon > \frac{3}{4}\}} e^{c^\dagger_\varepsilon z}\left(\frac{3}{32}Kc^\dagger_\varepsilon - G(y,\bar{u}_\varepsilon)\right)dx.
\end{aligned}$$

We then conclude that $c^\dagger_\varepsilon \leq M$, where

$$M := \frac{32}{3K}\max_{(y,u)\in\overline{\Omega}\times[\frac{3}{4},\frac{3}{2}]} G(y,u) < +\infty,$$

for if not, then from above we have $\Phi^\varepsilon_{c^\dagger_\varepsilon}(\bar{u}_\varepsilon) > 0$, contradicting the conclusion of Theorem 3.5(iv). $\square$

We now state an ε-independent density estimate for minimizers of Φ^ε_c.

Proposition 3.9. *Let ε, $c^\dagger_\varepsilon$ and $\bar u_\varepsilon$ be as in Theorem 3.5. Given $\delta \in (0,1)$ and $\bar x \in \overline{\Sigma}$, there exist $C, \bar r, \bar\varepsilon > 0$ depending only on W, G, Ω and δ such that, for every $\varepsilon \in (0, \bar\varepsilon)$ and $r \in (\varepsilon, \bar r)$, there holds*

$$\bar u_\varepsilon(\bar x) \geq \delta \quad \Rightarrow \quad \int_{B(\bar x, r)\cap\Sigma} \bar u_\varepsilon^2 \, dx \geq C r^n, \tag{3.16}$$

$$\bar u_\varepsilon(\bar x) \leq 1-\delta \quad \Rightarrow \quad \int_{B(\bar x, r)\cap\Sigma} (1-\bar u_\varepsilon)^2 \, dx \geq C r^n. \tag{3.17}$$

Proof. We only prove (3.16), since (3.17) follows analogously. For $\varepsilon \in (0, \varepsilon_0)$ we let $\tilde u_\varepsilon(x) := \bar u_\varepsilon(\varepsilon x) \in C^1(\overline{\Sigma}_\varepsilon)$, with $\Sigma_\varepsilon = \varepsilon^{-1}\Sigma$. Then $\tilde u_\varepsilon$ is a minimizer of the functional

$$\Phi_{c^\dagger_\varepsilon}(u; B(\bar x, \rho)) := \int_{\Sigma_\varepsilon \cap B(\bar x,\rho)} e^{\varepsilon c^\dagger_\varepsilon z} \left(\frac{1}{2}|\nabla u|^2 + W(u) - \varepsilon G(\varepsilon y, u) \right) dx \tag{3.18}$$

for any $\rho > 0$, among functions $u \in C^1\big(\overline{\Sigma_\varepsilon \cap B(\bar x, \rho)}\big)$ with fixed boundary data on $\partial B(\bar x, \rho) \cap \Sigma_\varepsilon$.

For $\rho > 2$, let us first consider the case in which $B(\bar x, \rho) \subset \Sigma_\varepsilon$. Without loss of generality we may assume that $\bar x = 0$. By our assumptions and standard regularity theory [27], there exists a constant $M \geq 1$ independent of ε and ρ such that

$$\|\nabla \tilde u_\varepsilon\|_{L^\infty(B(0,\rho))} \leq M. \tag{3.19}$$

Recalling that $\tilde u_\varepsilon(0) \geq \delta$, (3.19) implies that

$$⨍_{B(0,R)} \tilde u_\varepsilon^2 \, dx \geq \frac{1}{|B(0,R)|} \int_{B\left(0, \frac{\delta}{2M}\right)} \frac{\delta^2}{4} \, dx = \frac{\delta^{n+2}}{2^{n+2} R^n M^n} \qquad \forall R \in [1, \rho]. \tag{3.20}$$

We now note that in the considered case the functional in (3.18) satisfies the assumptions of Theorem A.1, with all the constants independent of ε and ρ, as long as $\rho \leq \varepsilon^{-1}$ and $\varepsilon < \varepsilon_1$, where ε_1 is given by Proposition 3.8. Therefore, if $r_0 \geq 1$ is the integer independent of ε and ρ defined in Theorem A.1, and

$$\alpha := \frac{\delta^{n+2}}{2^{n+2} r_0^n M^n}, \tag{3.21}$$

we have $0 < \alpha < r_0^{-n} \leq r_0^{1-n}$. Then by Theorem A.1 we obtain

$$⨍_{B(0,R)} \tilde u_\varepsilon^2 \, dx \geq \alpha \qquad \text{for all } R \in \mathbb{N} \cap [r_0, R_0] \text{ and } \varepsilon < \varepsilon_1, \tag{3.22}$$

provided that $R_0 = \lfloor \varepsilon^{-1} r_1 \rfloor$ satisfies $r_0 + 1 \le R_0 < \rho$ and

$$r_1 < \frac{\alpha}{1 + \|G\|_{L^\infty(\Omega\times(0,\frac{3}{2}))}}. \tag{3.23}$$

Note that since $\alpha < 1$, this statement is non-empty for all $\varepsilon < \varepsilon_2(r_1)$, where $\varepsilon_2(r_1) := \min(\varepsilon_1, r_1(2+r_0)^{-1})$, and all ρ satisfying $r_1 < \varepsilon\rho \le 1$. Furthermore, since $R \ge 1$ in (3.22), extending that estimate to an interval yields

$$⨍_{B(0,R)} \tilde{u}_\varepsilon^2 \, dx \ge 2^{-n}\alpha \qquad \forall R \in [1, \varepsilon^{-1} r_1], \tag{3.24}$$

where we also observed that by the definition of α and (3.20) the estimate in (3.22) holds for all $R \in [1, r_0]$ as well. By a rescaling and a translation, this then proves (3.16) for all $\bar{x} \in \Sigma$ such that $\mathrm{dist}(\bar{x}, \partial\Sigma) > r_1$, for every $r_1 > 0$ satisfying (3.23) and every $\varepsilon \in (0, \varepsilon_2(r_1))$.

We now consider the case of $\bar{x} \in \overline{\Sigma}$ such that $\mathrm{dist}(\bar{x}, \partial\Sigma) \le r_1$, for $r_1 > 0$ to be fixed momentarily. Once again, assume without loss of generality that $\bar{x} = 0$, which implies that $\mathrm{dist}(0, \partial\Sigma_\varepsilon) \le \varepsilon^{-1} r_1$. Observe that since $\partial\Omega$ is bounded and of class C^2, there exists $r_1 > 0$ satisfying (3.23) and a diffeomorphism $\phi_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ of class C^2 such that $\phi_\varepsilon(B(0,\rho) \cap \Sigma_\varepsilon) = B_\varepsilon^+$, where B_ε^+ denotes the intersection of the ball $B(0,\rho)$ with the half-space $\{x \in \mathbb{R}^n : x_n > -\mathrm{dist}(0, \partial\Sigma_\varepsilon)\}$, and $\rho = 2\varepsilon^{-1} r_1$. In the new coordinate system the functional in (3.18) becomes

$$\begin{aligned} \Phi_{c_\varepsilon^\dagger}(u; B(0,\rho)) = \int_{B_\varepsilon^+} e^{\varepsilon c_\varepsilon^\dagger \hat{z}\cdot\phi_\varepsilon^{-1}(x')} \Big(\frac{1}{2} |D\phi_\varepsilon(x')\nabla\tilde{u}|^2 + W(\tilde{u}) \\ - \varepsilon G(\varepsilon\phi_\varepsilon^{-1}(x'), \tilde{u}) \Big) |\det D\phi_\varepsilon(x')|^{-1} \, dx', \end{aligned} \tag{3.25}$$

where $\tilde{u}(x') := u(\phi_\varepsilon^{-1}(x'))$. By a standard reflection argument, the minimizer $\tilde{u}_\varepsilon \circ \phi_\varepsilon^{-1}$ of (3.25) can be extended from B_ε^+ to a minimizer of an energy functional as in (3.25), but defined on the whole of $B(0,\rho)$. Then, since this energy functional still satisfies the assumptions of Theorem A.1, we can repeat the arguments from the preceding part of the proof and, possibly reducing the value of r_1 to some $\bar{r} > 0$, establish the estimate in (3.24) in this case as well, provided that $\varepsilon < \bar{\varepsilon}$ for some $\bar{\varepsilon} \in (0, \varepsilon_2(\bar{r}))$. □

4 Traveling waves in the sharp interface case

In this section we consider the front propagation problem in the cylinder Σ, for the forced mean curvature flow (1.5). Also in this case, we are interested in traveling wave solutions with positive speed, which are special solutions to the forced mean curvature flow.

Definition 4.1 (Traveling waves)**.** *A traveling wave for the forced mean curvature flow is a pair (c, ψ), where $c > 0$ is the speed of the wave and the graph of the function $\psi \in C^2(\overline{\Omega})$ is the profile of the wave, such that $h(y,t) = \psi(y) + ct$ solves* (1.16).

Observe that to prove existence of a traveling wave solution it is sufficient to determine $c > 0$ such that the equation

$$-\nabla \cdot \left(\frac{\nabla \psi}{\sqrt{1 + |\nabla \psi|^2}} \right) = \frac{1}{c_W} g(y) - \frac{c}{\sqrt{1 + |\nabla \psi|^2}}, \qquad y \in \Omega, \tag{4.1}$$

with Neumann boundary condition $\nu \cdot \nabla \psi = 0$ on $\partial \Omega$, admits a classical solution. The graph of this solution will be the profile of the traveling wave.

4.1 Variational principle

Following the variational approach proposed in [37, Section 4] and developed in [18] (see also [7]) for the forced mean curvature flow, for $c > 0$ we consider the family of functionals F_c defined in (1.10). Note that if ψ is bounded and is a critical point the functional F_c then it is a solution to (4.1).

After the change of variable $\zeta(y) := \frac{e^{c\psi(y)}}{c} \geq 0$, the functional F_c is equivalent to

$$G_c(\zeta) = \int_\Omega \Big(c_W \sqrt{c^2 \zeta^2(y) + |\nabla \zeta(y)|^2} - g(y)\zeta(y) \Big) dy, \tag{4.2}$$

in the sense that $F_c(\psi) = G_c(\zeta)$ for all $\zeta \in C^1(\overline{\Omega})$ [18]. Since the functional G_c is naturally defined on $BV(\Omega)$ as the lower-semicontinuous relaxation (see, e.g., [4] and references therein), we introduce the following generalization to the notion of a traveling wave for (1.16). We use the convention that $\ln 0 = -\infty$.

Definition 4.2 (Generalized traveling waves)**.** *A generalized traveling wave for the forced mean curvature flow* (1.16) *is a pair* (c, ψ)*, where* $c > 0$ *is the speed of the wave,* $\psi = \frac{1}{c} \ln c\zeta$ *is the profile of the wave, and* $\zeta \in BV(\Omega)$ *is a non-negative critical point of* G_c*, not identically equal to zero.*

This definition is consistent with the earlier definition in the following sense. Defining $\omega \subseteq \Omega$ to be the interior of the support of ζ, again, by standard regularity of minimizers of perimeter type functionals [28, 32] we have that ψ solves (4.1) classically in ω with $\nu \cdot \nabla \psi = 0$ on $\partial\Omega \cap \partial\omega$ and, therefore, we have that $h(y,t) = \psi(y) + ct$ solves (1.16) in ω with Neumann boundary conditions on $\partial\Omega \cap \partial\omega$. In particular, if $\omega = \Omega$, then the above definition implies that (c, ψ) is a traveling wave in the sense of Definition 4.1. In general, however, ω may differ from Ω by a set of positive measure, in which case the traveling wave profile ψ obeys the following kind of boundary condition:

$$\lim_{y \to \bar{y}} \psi(y) = -\infty \qquad \forall \, \bar{y} \in \partial\omega \cap \Omega. \tag{4.3}$$

In this situation a generalized traveling wave may have the form of one or several "fingers" invading the cylinder from left to right with speed c.

The next proposition explains the relation between Assumption 3 and the minimization problems associated with functionals G_c and, hence, F_c.

Proposition 4.3. *Let Assumption 3 hold. Then there exists a unique $c^\dagger > 0$ such that*

i) $\frac{1}{c_W|\Omega|}\int_\Omega g\,dy \le c^\dagger \le \frac{1}{c_W}\sup_\Omega g$.

ii) If $0 < c < c^\dagger$, *then* $\inf\{G_c(\zeta)\ :\ \zeta \in BV(\Omega),\ \zeta \ge 0\} = -\infty$.

iii) If $c > c^\dagger$, *then* $\inf\{G_c(\zeta)\ :\ \zeta \in BV(\Omega),\ \zeta \ge 0\} = 0$, *and* $G_c(\zeta) > 0$ *for every non-trivial* $\zeta \ge 0$.

iv) If $c = c^\dagger$, *then there exists a non-trivial* $\bar\zeta \ge 0$, *with* $\bar\zeta \in BV(\Omega)$, *such that* $G_c(\bar\zeta) = \inf\{G_c(\zeta)\ :\ \zeta \in BV(\Omega),\ \zeta \ge 0\} = 0$.

Proof. The result follows from [18, Proposition 3.1 and Corollary 3.2] (see also [37, Proposition 4.1]). □

Note that the same argument as in [18, Proposition 3.4, Lemma 3.5] gives that for all $\zeta \ge 0$ such that $\zeta \in BV(\Omega)$ we have

$$G_c(\zeta) = \mathcal{F}_c(S_\psi) \tag{4.4}$$

where $S_\psi = \{(y,z) \in \Omega \times \mathbb{R}\ :\ z < \psi(y)\}$ is the subgraph of $\psi = \frac{1}{c}\ln c\zeta$. Moreover if $\zeta \ge 0$ is a non trivial minimizer of G_c, then the subgraph S_ψ of ψ is a minimizer, under compact perturbations, of the functional $\mathcal{F}_c$ defined in (1.13). This can be proved as in [28, Theorem 14.9], for more details see [18, Lemma 3.5].

We now prove a density estimate for minimizers under compact perturbations of the functional $\mathcal{F}_c$, which will be useful in the sequel. Note that related density estimates for the level sets of minimizers of Allen-Cahn type functionals will be proved in Appendix A. Throughout the rest of this section, a set of locally finite perimeter is identified with its measure theoretic interior (see [5]).

Lemma 4.4. *Given $\bar c > 0$, there exist $r_0 > 0$ and $\lambda > 0$ such that for all minimizers S of $\mathcal{F}_c$ under compact perturbations, with $c \in (0, \bar c]$, and for all $\bar x \in \overline{S}$, all $\bar x' \in \overline{\Sigma\setminus S}$ and all $r \in (0, r_0)$ the following density estimates hold:*

$$\begin{aligned} |S \cap B(\bar x, r)| &\ \ge\ \lambda\, r^n, &(4.5)\\ |(\Sigma \setminus S) \cap B(\bar x', r)| &\ \ge\ \lambda\, r^n. &(4.6)\end{aligned}$$

Furthermore, we have $S \subset \Omega \times (-\infty, M]$ for some $M \in \mathbb{R}$.

Proof. Let S be a minimizer of $\mathcal{F}_c$ under compact perturbations, $\bar x \in \overline{S}$ and $r > 0$. Up to a translation in the z-direction, we may assume $\bar x = (\overline{y}, 0)$ for some $\overline{y} \in \overline{\Omega}$. By minimality of S we have

$$\mathrm{Per}_c(S,\Sigma) - \frac{1}{c_W}\int_S e^{cz} g(y)dx \le \mathrm{Per}_c(S \setminus B(\bar x, r), \Sigma) - \frac{1}{c_W}\int_{S\setminus B(\bar x, r)} e^{cz} g(y)dx.$$

Therefore, letting $S_r = S \cap B(\bar{x}, r)$, we obtain

$$\mathrm{Per}_c(S, B(\bar{x}, r) \cap \Sigma) \le \int_{\partial B(\bar{x},r)\cap S} e^{cz} d\mathcal{H}^{n-1}(x) + \frac{1}{c_W} \int_{S_r} e^{cz} g(y) dx.$$

On the other hand, we have

$$\mathrm{Per}_c(S_r, \Sigma) = \mathrm{Per}_c(S, B(\bar{x}, r) \cap \Sigma) + \int_{\partial B(\bar{x},r)\cap S} e^{cz} d\mathcal{H}^{n-1}(x),$$

and hence

$$\begin{aligned} \mathrm{Per}_c(S_r, \Sigma) &\le 2 \int_{\partial B(\bar{x},r)\cap S} e^{cz} d\mathcal{H}^{n-1}(x) + \frac{1}{c_W} \int_{S_r} e^{cz} g(y) dx \\ &\le 2 \frac{\partial}{\partial r} \left(\int_{S_r} e^{cz} dx \right) + \frac{\|g\|_\infty}{c_W} \int_{S_r} e^{cz} dx. \end{aligned} \tag{4.7}$$

Recalling that $\bar{x} = (\bar{y}, 0)$, by the relative isoperimetric inequality in $\Sigma_0 := \Omega \times (-1, 1)$ [5], there exists $C > 0$ such that for all $r_0 \le 1$ and all $r \in (0, r_0)$ we have

$$\begin{aligned} \mathrm{Per}_c(S_r, \Sigma) &= \mathrm{Per}_c(S_r, \Sigma_0) \\ &\ge e^{-\bar{c}r_0} \mathrm{Per}(S_r, \Sigma_0) \\ &\ge e^{-\bar{c}r_0} C_{\Sigma_0} |S_r|^{\frac{n-1}{n}} \\ &\ge C \left(\int_{S_r} e^{cz} dx \right)^{\frac{n-1}{n}} \end{aligned} \tag{4.8}$$

for some $C > 0$ depending only on $\bar{c}$ and Ω. Let now $U(r) := \int_{S_r} e^{cz} dx$, and notice that $\lim_{r\to 0} U(r) = 0$ and that $0 < U(r) \le e^{\bar{c}r_0} |S_r|$ for all $r \in (0, r_0)$ by our choice of $\bar{x}$. From (4.7) and (4.8), we then get

$$\frac{dU}{dr}(r) \ge C U(r)^{\frac{n-1}{n}} \qquad \text{for a.e. } r \in (0, r_0), \tag{4.9}$$

for some $r_0 > 0$ and $C > 0$ depending only on $\bar{c}$, $\|g\|_\infty$, c_W and Ω.

Estimate (4.5) follows from (4.9) by integration. Estimate (4.6) follows by the same argument, working with $\Sigma \backslash S$ instead of S. Finally, the fact that $S \subset \Omega \times (-\infty, M]$ for some $M \in \mathbb{R}$ follows directly from (4.5) and the volume bound $\int_S e^{cz}\, dx < +\infty$. □

Remark 4.5. As a consequence of Lemma 4.4 and the regularity theory for almost minimal surfaces in bounded domains (see, [21, Corollary 1.4], [28,32]), $\overline{\partial S \cap \Sigma}$ is a hypersurface of class C^2 out of a closed singular set $\Xi_0 \subset \overline{\Sigma}$ of Hausdorff dimension at most $n - 4$ (in fact $\Xi_0 \cap \Sigma$ has Hausdorff dimension at most $n - 8$). In particular, for the physically relevant case $n \le 3$ the hypersurface $\partial S \cap \Sigma$ is of class C^2 uniformly in Σ (see also [42]).

In the sequel we need the following lemma, based on the rearrangement argument in the proof of [18, Lemma 3.5].

Lemma 4.6. *Let $c > 0$ and let $S \subset \Sigma$ with $\int_S e^{cz} dx \in (0, \infty)$. Then there exists a set $S_\psi = \{(y,z) \in \Sigma : z < \psi(y)\}$ such that $e^{c\psi} \in BV(\Omega)$ and*

$$\mathcal{F}_c(S_\psi) \leq \mathcal{F}_c(S), \tag{4.10}$$

with strict inequality if $S \not\equiv S_\psi$.

Proof. We begin by defining $\psi : \Omega \to [-\infty, \infty)$ as

$$\psi(y) := \frac{1}{c} \ln \left(c \int_{S^y} e^{cz} dz \right) \qquad \text{for a.e. } y \in \Omega, \tag{4.11}$$

where $S^y := \{z \in \mathbb{R} : (y,z) \in S\}$ and, as usual, we use the convention that $\ln 0 = -\infty$. Notice that if $S_\psi := \{(y,z) \in \Sigma : z < \psi(y)\}$, then by construction $\int_S e^{cz} dx = \int_{S_\psi} e^{cz} dx$ and

$$\int_S e^{cz} g(y)\, dx = \int_{S_\psi} e^{cz} g(y)\, dx = \frac{1}{c} \int_\Omega e^{c\psi(y)} g(y)\, dy. \tag{4.12}$$

Now, testing (1.11) with $\phi(y,z) := (\tilde{\phi}(y) + \chi(y)\hat{z})\eta_\delta(y,z)$, where $\tilde{\phi} \in C_c^1(\Omega; \mathbb{R}^{n-1})$, $\chi \in C_c^1(\Omega)$, $\delta > 0$ and η_δ is as in Proposition 3.7, we have for small enough δ depending on $\tilde{\phi}$ and χ:

$$\begin{aligned}
\int_S e^{cz}(\nabla \cdot \phi + c\hat{z} \cdot \phi) dx \geq & \int_\Omega \int_{S^y \cap (-\delta^{-1}, \delta^{-1})} e^{cz}(\nabla \cdot \tilde{\phi} + c\chi) dz\, dy \\
& - \int_{S \setminus (\Omega \times (-\delta^{-1}, \delta^{-1}))} e^{cz}(|\nabla \cdot \tilde{\phi}| + \delta\, \eta'(\delta|z|) + c|\chi|) dx \\
\geq & \int_\Omega \int_{S^y} e^{cz}(\nabla \cdot \tilde{\phi} + c\chi) dz\, dy - C \int_{S \setminus (\Omega \times (-\delta^{-1}, \delta^{-1}))} e^{cz} dx \\
= & \int_{S_\psi} e^{cz}(\nabla \cdot \tilde{\phi} + c\chi) dx - C \int_{S \setminus (\Omega \times (-\delta^{-1}, \delta^{-1}))} e^{cz} dx,
\end{aligned} \tag{4.13}$$

for some $C > 0$ independent of δ. Observing that the last term in the last line of (4.13) vanishes as $\delta \to 0$, we obtain

$$\mathrm{Per}_c(S, \Sigma) \geq \limsup_{\delta \to 0} \int_S e^{cz}(\nabla \cdot \phi + c\hat{z} \cdot \phi) dx \geq \int_{S_\psi} e^{cz}(\nabla \cdot \tilde{\phi} + c\chi) dx. \tag{4.14}$$

In particular, since

$$\int_{S_\psi} e^{cz}(\nabla \cdot \tilde{\phi} + c\chi) dx = \frac{1}{c} \int_\Omega e^{c\psi}(\nabla \cdot \tilde{\phi} + c\chi) dy\,, \tag{4.15}$$

this implies that $e^{c\psi} \in BV(\Omega)$.

We claim that taking the supremum in (4.14) over all $\tilde{\phi}$ and χ satisfying $|\tilde{\phi}|^2 + \chi^2 \leq 1$ yields $\mathrm{Per}_c(S_\psi, \Sigma)$ (for similar arguments, see [28, Theorem 14.6]). Indeed, from (4.13) with S replaced by S_ψ we obtain, after sending $\delta \to 0$ and then taking the supremum over all $\tilde{\phi}$, that

$$\begin{aligned} \mathrm{Per}_c(S_\psi, \Sigma) &\geq \sup_{|\tilde{\phi}|^2+\chi^2\leq 1} \int_{S_\psi} e^{cz}(\nabla \cdot \tilde{\phi} + c\chi)dx \\ &=: \int_\Omega e^{c\psi}\sqrt{1+|\nabla\psi|^2}\, dy\,. \end{aligned} \tag{4.16}$$

We now approximate ψ by smooth functions ψ_ε such that

$$\lim_{\varepsilon\to 0} \int_\Omega e^{c\psi_\varepsilon}\, dy = \int_\Omega e^{c\psi}\, dy$$

and

$$\lim_{\varepsilon\to 0} \int_\Omega e^{c\psi_\varepsilon}\sqrt{1+|\nabla\psi_\varepsilon|^2}\, dy = \int_\Omega e^{c\psi}\sqrt{1+|\nabla\psi|^2}\, dy\,.$$

By the lower semicontinuity of the perimeter functional Per_c, we obtain

$$\begin{aligned} \mathrm{Per}_c(S_\psi, \Sigma) &\leq \liminf_{\varepsilon\to 0}\ \mathrm{Per}_c(S_{\psi_\varepsilon}, \Sigma) \\ &= \liminf_{\varepsilon\to 0} \sup_{\substack{\phi\in C^1_c(\Sigma;\mathbb{R}^n)\\ |\phi|\leq 1}} \int_{S_{\psi_\varepsilon}} e^{cz}(\nabla\cdot\phi + c\hat{z}\cdot\phi)\, dx \\ &= \liminf_{\varepsilon\to 0} \sup_{\substack{\phi\in C^1_c(\Sigma;\mathbb{R}^n)\\ |\phi|\leq 1}} \int_{\partial S_{\psi_\varepsilon}} e^{cz}\, \phi\cdot\nu\ d\mathcal{H}^{n-1}(x), \end{aligned} \tag{4.17}$$

where ν is the normal to $\partial S_{\psi_\varepsilon}$ pointing out of S_{ψ_ε}, and we used Gauss-Green theorem to arrive at the last line. From this we obtain

$$\begin{aligned} \mathrm{Per}_c(S_\psi, \Sigma) &\leq \liminf_{\varepsilon\to 0} \int_{\partial S_{\psi_\varepsilon}} e^{cz} d\mathcal{H}^{n-1}(x) \\ &= \lim_{\varepsilon\to 0}\int_\Omega e^{c\psi_\varepsilon}\sqrt{1+|\nabla\psi_\varepsilon|^2}\, dy = \int_\Omega e^{c\psi}\sqrt{1+|\nabla\psi|^2}\, dy\,. \end{aligned} \tag{4.18}$$

Therefore, we have, in fact, an equality in (4.16), and (4.10) then follows by combining (4.14) with (4.12).

Finally we show that the inequality in (4.10) is strict if $S \not\equiv S_\psi$. By Gauss-Green theorem we have

$$\begin{aligned} \int_S e^{cz}(\nabla\cdot\phi + c\hat{z}\cdot\phi)dx &= \int_{\partial^* S} e^{cz}\phi\cdot\nu\, d\mathcal{H}^{n-1}(x) \\ &\leq \int_{\partial^* S\cap\{\nu\cdot\hat{z}\geq -\varepsilon\}} e^{cz}\, d\mathcal{H}^{n-1}(x) + \int_{\partial^* S\cap\{\nu\cdot\hat{z}< -\varepsilon\}} e^{cz}\phi\cdot\nu\, d\mathcal{H}^{n-1}(x), \end{aligned} \tag{4.19}$$

where ϕ is as before, ν is the unit normal vector to $\partial^* S$ pointing out of S and $\varepsilon > 0$ is arbitrary. Therefore, for $\chi \geq 0$ we can write

$$\begin{aligned}
\int_{S_\psi} e^{cz}(\nabla \cdot \tilde{\phi} + c\chi)dx &\leq \limsup_{\delta \to 0} \int_S e^{cz}(\nabla \cdot \phi + c\hat{z} \cdot \phi)dx \\
&\leq \int_{\partial^* S \cap \{\nu \cdot \hat{z} \geq -\varepsilon\}} e^{cz}\, d\mathcal{H}^{n-1}(x) + \sqrt{1-\varepsilon^2} \int_{\partial^* S \cap \{\nu \cdot \hat{z} < -\varepsilon\}} e^{cz}\, d\mathcal{H}^{n-1}(x) \\
&\leq \int_{\partial^* S} e^{cz}\, d\mathcal{H}^{n-1}(x) - \frac{\varepsilon^2}{2} \int_{\partial^* S \cap \{\nu \cdot \hat{z} < -\varepsilon\}} e^{cz}\, d\mathcal{H}^{n-1}(x), \quad (4.20)
\end{aligned}$$

where we also used (4.14). We now take the supremum of the left-hand side in (4.20) over all $\tilde{\phi}$ and χ, noting that we can restrict χ to non-negative functions without affecting the value of the supremum. Then, reasoning as in the first part of the proof, we conclude that the left-hand side of (4.20) converges to $\mathrm{Per}_c(S_\psi, \Sigma)$. On the other hand, if $S_\psi \not\equiv S$ there exists $\varepsilon > 0$ such that the last integral in (4.20) is strictly positive, implying that

$$\mathrm{Per}_c(S_\psi, \Sigma) < \int_{\partial^* S} e^{cz}\, d\mathcal{H}^{n-1}(x).$$

Indeed, it is enough to choose $x \in \partial^* S \cap \{\nu \cdot \hat{z} < 0\}$ and let $\varepsilon = -(\nu(x) \cdot \hat{z})/2$. By the properties of the reduced boundary [32], it then follows that the set $x \in \partial^* S \cap \{\nu \cdot \hat{z} < -\varepsilon\}$ has positive $\mathcal{H}^{n-1}$ measure. Combining this with (4.12) yields the desired result. $\square$

We summarize all the conclusions above into the following proposition connecting the non-trivial minimizers of G_c with those of $\mathcal{F}_c$ on its natural domain, i.e., among all measurable sets $S \subset \Sigma$ with $\int_S e^{cz} dx < \infty$.

Proposition 4.7. *Let Assumption 3 hold, and let $c^\dagger$ be as in Proposition 4.3. Then*

i) If $0 < c < c^\dagger$, then $\inf \mathcal{F}_c = -\infty$.

ii) If $c > c^\dagger$, then $\mathcal{F}_c(S) > 0$ for all $S \subset \Sigma$ with $\int_S e^{cz} dx > 0$.

iii) There exists a non-trivial minimizer of $\mathcal{F}_{c^\dagger}$, and $\mathcal{F}_{c^\dagger}(S) = 0$. Furthermore, S is a non-trivial minimizer of $\mathcal{F}_{c^\dagger}$ if and only if $S = \{(y,z) \in \Sigma : z < \psi(y)\}$, where $\psi = \frac{1}{c^\dagger} \ln c^\dagger \zeta$ and $\zeta \geq 0$ is a non-trivial minimizer of $G_{c^\dagger}$.

Proof. i) follows from (4.4) and Proposition 4.3 (ii). To prove ii), we note that in view Proposition 4.3 (iii) and (4.4) we have $\mathcal{F}_c(S_\psi) > 0$ for all S_ψ as in Lemma 4.6 and apply (4.10). Finally, iii) follows from (4.4), Proposition 4.3 (iv) and Lemma 4.6. $\square$

4.2 Existence, uniqueness and stability of generalized traveling waves

The content of this section is a slight extension of [18, Section 3]. We provide proofs of the results for the reader's convenience.

The characterization of minimizers of the geometric functional $\mathcal{F}_c$ in Proposition 4.7 yields the following existence result for generalized traveling waves.

Theorem 4.8 (Existence of generalized traveling waves)**.** *Let Assumption 3 hold. Then there exists a unique $c^\dagger > 0$, which coincides with the one in Proposition 4.3, such that:*

i) There exist a function $\psi : \Omega \to [-\infty, \infty)$ such that $(c^\dagger, \psi)$ is a generalized traveling wave for the forced mean curvature flow and the set $S_\psi := \{(y, z) \in \Sigma \mid z < \psi(y)\}$ is a minimizer of $\mathcal{F}_{c^\dagger}$.

ii) The set $\omega := \{\psi > -\infty\}$ is open and satisfies $E^0(\omega) < 0$, where E^0 is defined in (3.7)*. Moreover, $\omega \times \mathbb{R}$ is a minimizer of $\mathcal{F}_{c^\dagger}$ under compact perturbations, and $\psi \in C^2(\omega)$.*

iii) ψ is unique up to additive constants on every connected component of ω, in the following sense: there exists a number $k \in \mathbb{N}$ and functions $\psi_i : \Omega \to [-\infty, \infty)$ for each $i = 1, \ldots, k$ such that $\omega_i := \{\psi_i > -\infty\} \neq \varnothing$ are open, connected and disjoint, $\psi_i \in C^2(\omega_i)$ and $\psi = \ln\left(\sum_{i=1}^k e^{\psi_i + k_i}\right)$, for some $k_i \in [-\infty, \infty)$.

iv) There exists a closed set $\sigma \subset \overline{\partial\omega \cap \Omega}$ of Hausdorff dimension at most $n - 5$ (and $\sigma \cap \Omega$ has Hausdorff dimension at most $n - 9$) such that $\partial\omega \setminus \sigma$ is a C^2 solution to the prescribed curvature problem

$$c_W \kappa = g \qquad \textit{on } (\partial\omega \cap \Omega) \setminus \sigma, \tag{4.21}$$

where κ is the sum of the principal curvatures of $(\partial\omega \cap \Omega) \setminus \sigma$, with Neumann boundary conditions $\nu_{\partial\omega} \cdot \nu_{\partial\Omega} = 0$ at $(\overline{\partial\omega \cap \Omega} \cap \partial\Omega) \setminus \sigma$.

Proof. We recall that by Proposition 4.7 (iii) there exists a set S_ψ which is a non-trivial minimizer of $\mathcal{F}_{c^\dagger}$, and $\mathcal{F}_{c^\dagger}(S_\psi) = 0$. Furthermore, by Remark 4.5 S_ψ is of class C^2 out of a singular set Ξ_0.

Observe also that the class of minimizers $\mathcal{F}_{c^\dagger}$ is invariant with respect to shifts along z and is closed with respect to the L^1_{loc} convergence of their characteristic functions [28]. Therefore, translating the minimizer towards $z = +\infty$ and passing to the limit, we get that $\omega \times \mathbb{R}$ is a minimizer under compact perturbations of $\mathcal{F}_{c^\dagger}$. The regularity of $\partial(\omega \times \mathbb{R}) = \partial\omega \times \mathbb{R}$ is then a consequence of the classical regularity theory for minimal surfaces with prescribed mean curvature in bounded domains (see [21, 28]). In particular (4.21), with Neumann boundary conditions, follows from the Euler-Lagrange equation for $\mathcal{F}_{c^\dagger}$, observing that $\nu_{\partial\omega\times\mathbb{R}} \cdot \hat{z} = 0$. The inequality $E^0(\omega) < 0$ follows from [18, Remark 3.12].

From the density estimate (4.5), reasoning as in [28, Theorem 14.10], we derive that ψ is bounded above in ω. Moreover, reasoning as in [28, Theorem 14.13] (see also [18, Proposition 3.7]), we obtain that ψ is regular on ω, in particular $\psi \in C^2(\omega)$. From this, we get that ψ is a solution of (4.1) in ω with $c = c^\dagger$. Moreover ψ satisfies the Neumann boundary conditions on $\partial\omega \cap \partial\Omega$ [21,28]. The fact that ψ is uniquely defined up to translations on every connected component of ω follows from Proposition 4.7(iii) and the convexity of $G_{c^\dagger}$ (see [18, Propositions 3.7 and 3.10]). □

Remark 4.9. We observe that if ψ is a regular solution to (4.1) in some set $\omega \subseteq \Omega$, such that ψ is bounded from above, $\psi(y) \to -\infty$ as $y \to \partial\omega \cap \Omega$, and $\psi = -\infty$ on $\Omega \backslash \omega$, then $\zeta = \frac{e^{c\psi}}{c} \in BV(\Omega)$ and $G_c(\zeta) = 0$. This, in particular, implies that $c \leq c^\dagger$. Indeed, if $c > c^\dagger$, $\zeta = \frac{e^{c\psi}}{c}$ would be a non trivial minimizer of G_c, contradicting Proposition 4.3(iii). This means that the variational method selects the *fastest* generalized traveling waves which are bounded from above.

Under Assumption 4, which is considerably stronger than Assumption 3, we can prove uniqueness and stability of traveling waves for the mean curvature flow. We begin by giving several sufficient conditions that lead to Assumption 4.

Proposition 4.10. *Let* (2.7) *hold, and let* C_Ω *be the relative isoperimetric constant of* Ω. *Then Assumption 4 holds if one of the following conditions is verified:*

i) there is no embedded hypersurface $\partial\omega \subseteq \Omega$ *which solves the prescribed curvature problem*

$$c_W \kappa = g \qquad \partial\omega \cap \Omega\,, \tag{4.22}$$

with Neumann boundary conditions $\nu_{\partial\omega} \cdot \nu_{\partial\Omega} = 0$ *on* $\partial\omega \cap \partial\Omega$,

ii) $n = 2$ *and* $g > 0$ *on* $\overline{\Omega}$,

iii) $\min_{\overline{\Omega}} g \leq 0$ *and* $\max_{\overline{\Omega}} g - \min_{\overline{\Omega}} g < C_\Omega c_W 2^{\frac{1}{n-1}} |\Omega|^{-\frac{1}{n-1}}$,

iv) $n > 2$, $g > 0$ *on* $\overline{\Omega}$ *and* $\max_{\overline{\Omega}} g < C_\Omega c_W 2^{\frac{1}{n-1}} |\Omega|^{-\frac{1}{n-1}}$,

v) $n > 1$, $g \in C^1(\overline{\Omega})$, $g > 0$ *on* $\overline{\Omega}$, *and* $\min_{\overline{\Omega}} (g^2 - (n-1)|\nabla g|) > 0$.

Proof. (i) follows from Theorem 4.8. Indeed, if ω is as in Theorem 4.8, then $\partial\omega$ is a solution of the prescribed curvature problem (4.22). (ii) comes from (i), observing that if ω is a solution of the prescribed curvature problem in $\mathbb{R}$, then $g = 0$ on $\partial\omega$. The proof of (iii) and (iv) is given in [18, Proposition 3.16] (see also [17, Proposition 4.6]), and (v) is proved in [16]. □

We now state an existence and uniqueness result. Note that, in view of Proposition 4.7, the value of $c^\dagger$ in Assumption 4 coincides with that in Proposition 4.3.

Theorem 4.11 (Existence and uniqueness of traveling waves). *Under Assumption 4, there exist a unique $c^\dagger > 0$ and a unique $\psi \in C^2(\overline{\Omega})$ such that $\max_{y\in\overline{\Omega}} \psi(y) = 0$, and $(c^\dagger, \psi)$ is a traveling wave for the forced mean curvature flow* (1.16). *Moreover, ψ is the unique minimizer of the functional $F_{c^\dagger}$ over $C^1(\overline{\Omega})$, up to additive constants, and $S = \{(y, z) \in \Sigma : z < \psi(y)\}$ is the unique minimizer of $\mathcal{F}_{c^\dagger}$ up to translations in z.*

Proof. From Theorem 4.8 we get the existence of a generalized variational traveling wave $(c^\dagger, \psi)$ with $\omega \subseteq \Omega$, and $\omega \times \mathbb{R}$ is a minimizer of $\mathcal{F}_{c^\dagger}$ under compact perturbations. Then by Assumption 4 necessarily $\omega = \Omega$ and, hence, $\psi > -\infty$ in Ω.

We now claim that $\psi \geq M$ in Ω for some $M \in \mathbb{R}$. Assume by contradiction that there exists $x_n \to x \in \partial\Omega$ such that $\psi(x_n) \to -\infty$. By construction we have that the subgraph S_ψ of ψ is a minimizer of $\mathcal{F}_{c^\dagger}$. So, we can apply the density estimate (4.6) to $\Sigma \setminus S_\psi$ at x_n and obtain a contradiction, if n is sufficiently large.

Since ψ is a bounded regular minimizer of $F_{c^\dagger}$, it satisfies the Neumann boundary conditions on $\partial\Omega$. Moreover, by standard elliptic regularity theory $\psi \in C^2(\overline{\Omega})$. Finally, uniqueness of the pair $(c^\dagger, \psi)$ is a consequence of the strong maximum principle. Indeed, if there are two smooth solutions $(c_1^\dagger, \psi_1)$ and $(c_2^\dagger, \psi_2)$ to (4.1) with $c_2^\dagger > c_1^\dagger$, then by a suitable translation we may assume that $\psi_2 < \psi_1$. Then, using those functions as initial data for (1.16), we find that the solutions of (1.16) touch at some $t > 0$, contradicting the comparison principle for (1.16) [40]. If, on the other hand, $c_1^\dagger = c_2^\dagger$, again, by a suitable translation the two solutions can be made to touch at a point, while $\psi_2 \leq \psi_1$. Then by strong maximum principle for (4.1) we have $\psi_1 = \psi_2$ [40]. $\square$

Moreover, we get the following stability result.

Theorem 4.12. *Let Assumption 4 hold, let $(c^\dagger, \psi)$ be as in Theorem 4.11, and let $h(y, t)$ be the unique solution to* (1.16) *with Neumann boundary conditions and initial datum $h(y, 0) = h_0(y) \in C(\overline{\Omega})$. Then there exists a constant $k \in \mathbb{R}$ such that*

$$h(\cdot, t) - c^\dagger t - k \longrightarrow \psi \qquad \text{in } C^{1,\alpha}(\overline{\Omega}), \text{ as } t \to +\infty.$$

Proof. The proof can be obtained by a straightforward adaptation of the argument in [18, Corollary 4.9]. $\square$

Remark 4.13. If we assume a weaker assumption than Assumption 4, i.e. that there is at most one set $\omega' \subseteq \Omega$ such that $\omega' \times \mathbb{R}$ is a minimizer under compact perturbations of the geometric functional $\mathcal{F}_{c^\dagger}$, then we can prove an analogue of the previous stability result. Indeed under this assumption there exists a unique (up to additive constants) generalized traveling wave $(c^\dagger, \psi)$, and ψ is supported on ω, where ω can be either ω' or the whole Ω.

Moreover, there exists a constant $k \in \mathbb{R}$ such that, as $t \to +\infty$,

$$h(\cdot, t) - c^\dagger t - k \longrightarrow \begin{cases} \psi & \text{in } C^{1,\alpha}_{loc}(\omega), \\ -\infty & \text{locally uniformly in } \Omega \setminus \omega, \end{cases}$$

where $h(y,t)$ is the unique solution to (1.16) with Neumann boundary conditions, and initial datum $h(\cdot,0) = h_0 \in C(\overline{\Omega})$. For the proof, see [18, Theorem 4.7 and Remark 4.8].

5 Asymptotic behavior as $\varepsilon \to 0$

In this section we prove the main result of this paper, namely, the convergence result, as $\varepsilon \to 0$, of the traveling waves of (1.14) to the generalized traveling waves for (1.16). For $M > 0$ and $k > 0$, let us introduce the notations that will be used throughout the proofs in this section:

$$\Sigma_M := \Omega \times (-M, M), \qquad \|G\|_{k,\infty} = \max_{u\in[0,k], y\in\overline{\Omega}} |G(y,u)|. \tag{5.1}$$

We begin with the following basic compactness result.

Lemma 5.1. *Let $c > 0$ and $c_\varepsilon \to c$ as $\varepsilon \to 0$. Let $u_\varepsilon \in H^1_{c_\varepsilon}(\Sigma)$ be such that $0 \le u_\varepsilon \le k$ and $\Phi^\varepsilon_{c_\varepsilon}(u_\varepsilon) \le K$, for some $k, K > 0$ independent of ε. Assume that:*

$$\exists \delta \in (0,1) \text{ such that } u_\varepsilon(y,z) \le \delta \text{ for all } y \in \Omega \text{ and } z > 0. \tag{5.2}$$

Then there exists $u \in BV_{loc}(\Sigma, \{0,1\})$ such that $u(\cdot, z) = 0$ for all $z > 0$ and

$$u_\varepsilon \to u \qquad \text{in } L^1_{loc}(\Sigma),$$

upon extraction of a sequence.

Proof. Recall that by (2.3) there exists $\varepsilon_\delta > 0$ such that for every $\varepsilon < \varepsilon_\delta$ the integrand in (3.4) is positive for all $z \ge 0$. Then, by our assumptions, for every $M > 0$ we get

$$\begin{aligned} K &\ge \Phi^\varepsilon_{c_\varepsilon}(u_\varepsilon) \ge \int_{\Sigma_M} e^{c_\varepsilon z} \frac{W(u_\varepsilon)}{\varepsilon} dx - \int_{-\infty}^{M} \int_\Omega e^{c_\varepsilon z} G(y, u_\varepsilon) dy\, dz \\ &\ge \int_{\Sigma_M} e^{c_\varepsilon z} \frac{W(u_\varepsilon)}{\varepsilon} dx - \frac{|\Omega|}{c_\varepsilon} \|G\|_{k,\infty} e^{c_\varepsilon M}. \end{aligned}$$

Therefore, for every $M > 0$ and ε small enough we have

$$\int_{\Sigma_M} e^{c_\varepsilon z} W(u_\varepsilon) dx \le \varepsilon K \left(1 + \frac{2|\Omega|}{cK} \|G\|_{k,\infty} e^{cM}\right). \tag{5.3}$$

On the other hand, by our assumptions and the Modica-Mortola trick [33]

$$\begin{aligned}
&\int_{\Sigma_M} e^{c_\varepsilon z}|\nabla u_\varepsilon|\sqrt{2W(u_\varepsilon)}dx \\
\leq\ & \int_{\Sigma_M} e^{c_\varepsilon z}\left[\left(\sqrt{\frac{\varepsilon}{2}}|\nabla u_\varepsilon| - \sqrt{\frac{W(u_\varepsilon)}{\varepsilon}}\right)^2 + |\nabla u_\varepsilon|\sqrt{2W(u_\varepsilon)}\right]dx \\
=\ & \int_{\Sigma_M} e^{c_\varepsilon z}\left(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{W(u_\varepsilon)}{\varepsilon}\right)dx \\
\leq\ & \Phi^\varepsilon_{c_\varepsilon}(u_\varepsilon) + \frac{|\Omega|}{c_\varepsilon}\|G\|_{k,\infty}e^{c_\varepsilon M} \leq K\left(1 + \frac{2|\Omega|}{cK}\|G\|_{k,\infty}\, e^{cM}\right).
\end{aligned}$$

We define

$$\phi(u) := \int_0^u \sqrt{2W(s)}ds, \tag{5.4}$$

and rewrite the previous inequality as

$$\int_{\Sigma_M} e^{c_\varepsilon z}|\nabla\phi(u_\varepsilon)|dx \leq K\left(1 + \frac{2|\Omega|}{cK}\|G\|_{k,\infty}\, e^{cM}\right). \tag{5.5}$$

This implies that $\phi(u_\varepsilon)$ are uniformly bounded in $BV(\Sigma_M)$ for every $M > 0$. By compactness theorem in BV (see [5]), we then get that upon extraction of a sequence $\phi(u_\varepsilon)$ converges in $L^1_{loc}(\Sigma)$ to a function $w \in BV_{loc}(\Sigma)$. Therefore, since $u \mapsto \phi(u)$ is a continuous one-to-one map, this implies that up to a subsequence u_ε converges to $u = \phi^{-1}(w)$ almost everywhere and in $L^1_{loc}(\Sigma)$. Furthermore, by (5.3) u takes values in $\{0,1\}$ and, hence, $u = c_W^{-1}w$ almost everywhere. Therefore, $u \in BV_{loc}(\Sigma;\{0,1\})$.

Eventually, by assumption (5.2) it follows that $u = 0$ in $\Omega \times (0,+\infty)$. $\square$

We will need the following technical result from the proof of [36, Theorem 3.3].

Lemma 5.2. *Let $\varepsilon > 0$ and $c^\dagger_\varepsilon$ be as in Theorem 3.5. Then for every $c > 0$ and every $u \in H^1_c(\Sigma)$ we have*

$$\Phi^\varepsilon_c(u) \geq \frac{c^2 - (c^\dagger_\varepsilon)^2}{c^2}\int_\Sigma e^{cz}\frac{\varepsilon}{2}|u_z|^2dx. \tag{5.6}$$

Proof. We define

$$\tilde{u}(y,z) := u\left(y, \frac{c^\dagger_\varepsilon}{c}z\right). \tag{5.7}$$

Note that $\tilde{u} \in H^1_{c^\dagger_\varepsilon}(\Sigma)$. By a simple change of variables we then get

$$\begin{aligned}
\Phi^\varepsilon_c(u) &= \frac{c^\dagger_\varepsilon}{c}\int_\Sigma e^{c^\dagger_\varepsilon z}\left[\frac{\varepsilon}{2}|\nabla\tilde{u}|^2 + \frac{\varepsilon}{2}\left(\frac{c}{c^\dagger_\varepsilon}\right)^2|\tilde{u}_z|^2 + \frac{W(\tilde{u})}{\varepsilon} - G(y,\tilde{u})\right]dx \\
&= \frac{c^\dagger_\varepsilon}{c}\Phi^\varepsilon_{c^\dagger_\varepsilon}(\tilde{u}) + \frac{c^2 - (c^\dagger_\varepsilon)^2}{cc^\dagger_\varepsilon}\int_\Sigma e^{c^\dagger_\varepsilon z}\frac{\varepsilon}{2}|\tilde{u}_z|^2dx,
\end{aligned}$$

which gives the result, since $\Phi^{\varepsilon}_{c^\dagger_\varepsilon}(\tilde u) \geq 0$. $\square$

We now state our main result.

Theorem 5.3. *Let Assumptions 1, 2 and 3 hold. Let $c^\dagger_\varepsilon$, $\bar u_\varepsilon$ and v_ε be as in Theorem 3.5, and let $c^\dagger$ be as in Theorem 4.8.*

i) There holds
$$\lim_{\varepsilon\to 0} c^\dagger_\varepsilon = c^\dagger. \tag{5.8}$$

ii) For every sequence $\varepsilon_n \to 0$ there exist a subsequence (not relabeled) and an open set $S \subset \Sigma$ such that
$$\bar u_{\varepsilon_n} \to \chi_S \qquad \text{in } L^1_{loc}(\Sigma),$$
where S is a non-trivial minimizer of $\mathcal{F}_{c^\dagger}$ satisfying $S \subseteq \Omega \times (-\infty, 0)$ and $\partial S \cap (\overline{\Omega} \times \{0\}) \neq \varnothing$. Moreover,
$$\bar u_{\varepsilon_n} \to \chi_S \qquad \text{locally uniformly on } \overline{\Sigma} \setminus \partial S,$$
and for every $\theta \in (0,1)$ the level sets $\{\bar u_{\varepsilon_n} = \theta\}$ converge to ∂S locally uniformly in the Hausdorff sense.

iii) If also Assumption 4 holds, then S is the unique minimizer of $\mathcal{F}_{c^\dagger}$ from Theorem 4.11 satisfying $S \subseteq \Omega \times (-\infty, 0)$ and $\partial S \cap (\overline{\Omega} \times \{0\}) \neq \varnothing$. Moreover
$$v_\varepsilon \to 1 \qquad \text{uniformly in } \overline{\Omega}.$$

Proof. We divide the proof into four steps.

Step 1: we shall prove that
$$\liminf_{\varepsilon\to 0} c^\dagger_\varepsilon \geq c^\dagger.$$

The proof follows by the standard Modica-Mortola construction of a recovery sequence [33]. Let S_ψ be as in Theorem 4.8. Then the hypersurface $\partial S_\psi \cap \Sigma$ is of class C^2 uniformly on compact subsets of $\overline{\Sigma}$, and by Proposition 4.7(iii) S_ψ satisfies
$$c_W \mathrm{Per}_{c^\dagger}(S_\psi, \Sigma) = \int_{S_\psi} e^{c^\dagger z} g(y)\,dx. \tag{5.9}$$

We consider d_{S_ψ} to be the signed distance function from ∂S_ψ, i.e.,
$$d_{S_\psi}(x) := \mathrm{dist}(x, \Sigma \setminus S_\psi) - \mathrm{dist}(x, S_\psi)$$

and $\gamma : \mathbb{R} \to \mathbb{R}$ to be the unique solution to $\gamma' = \sqrt{2W(\gamma)}$ with $\gamma(0) = \frac{1}{2}$. Note that the map $t \mapsto \gamma(t)$ is monotone increasing and, by Assumption 2, converges exponentially to 0 for $t \to -\infty$ and to 1 for $t \to +\infty$. For $\varepsilon, M > 0$, we let

$$u_{\varepsilon,M} := \gamma\left(\frac{d_{S_\psi}}{\varepsilon}\right)\eta\left(\frac{z+M}{\varepsilon}\right),$$

where $\eta : \mathbb{R} \to [0,1]$ is a smooth increasing function such that $\eta(z) = 1$ for all $z \geq 1$ and $\eta(z) = 0$ for all $z \leq 0$. Note that, since ψ is bounded from above by Theorem 4.8, we have that $u_{\varepsilon,M} \in H^1_{c^\dagger}(\Sigma)$ for all M and ε small enough. Note also that, for $M > \sup_\Omega \psi$, we have $u_{\varepsilon,M} \to \chi_{S_\psi \cap \Sigma_M}$ in $L^1(\Sigma)$, as $\varepsilon \to 0$. For $M > \sup_\Omega \psi$, we compute

$$\begin{aligned}
\Phi^\varepsilon_{c^\dagger}(u_{\varepsilon,M}) &\leq \int_\Sigma e^{c^\dagger z}\sqrt{2W(u_{\varepsilon,M})}|\nabla u_{\varepsilon,M}|dx \\
&\quad - \int_\Sigma e^{c^\dagger z} G(y, u_{\varepsilon,M})dx + Ce^{-c^\dagger M} \\
&= \int_\Sigma e^{c^\dagger z}|\nabla \phi(u_{\varepsilon,M})|dx - \int_\Sigma e^{c^\dagger z} G(y, u_{\varepsilon,M})dx + Ce^{-c^\dagger M},
\end{aligned}$$

where ϕ is as in (5.4), and the constant $C > 0$ is independent of ε and M. Notice that, by the C^2-regularity of $\partial S_\psi \cap \Sigma$, we have

$$\mathrm{Per}_{c^\dagger}(\{\phi(u_{\varepsilon,M}) > t\}, \Sigma) \to \mathrm{Per}_{c^\dagger}(S_\psi \cap \Sigma_M, \Sigma),$$

for any $t \in (0, c_W)$, as $\varepsilon \to 0$. Recalling the definition of g in (2.2), as $\varepsilon \to 0$ we also have that $\phi(u_{\varepsilon,M}) \to c_W \chi_{S_\psi \cap \Sigma_M}$ in $L^1(\Sigma)$. Therefore, we can apply the co-area formula (see [5]) and, possibly increasing the value of C, we obtain that

$$\begin{aligned}
\lim_{\varepsilon\to 0} \Phi^\varepsilon_{c^\dagger}(u_{\varepsilon,M}) &\leq \lim_{\varepsilon\to 0}\Big(\int_0^{c_W} \mathrm{Per}_{c^\dagger}(\{\phi(u_{\varepsilon,M}) > t\}, \Sigma)dt \\
&\quad - \int_\Sigma e^{c^\dagger z} G(y, u_{\varepsilon,M})dx + Ce^{-c^\dagger M}\Big) \qquad (5.10)\\
&= c_W \mathrm{Per}_{c^\dagger}(S_\psi \cap \Sigma_M, \Sigma) - \int_{S_\psi} e^{c^\dagger z} g(y)dx + Ce^{-c^\dagger M} \\
&\leq c_W \mathrm{Per}_{c^\dagger}(S_\psi, \Sigma) - \int_{S_\psi} e^{c^\dagger z} g(y)dx + Ce^{-c^\dagger M} \\
&= Ce^{-c^\dagger M},
\end{aligned}$$

where the last equality follows from (5.9).

Assume now by contradiction that there exists a sequence of $c^\dagger_\varepsilon$ converging to a constant $c < c^\dagger$. By (5.6) we have

$$\Phi^\varepsilon_{c^\dagger}(u_{\varepsilon,M}) \geq \frac{(c^\dagger)^2 - (c^\dagger_\varepsilon)^2}{(c^\dagger)^2} \int_\Sigma e^{c^\dagger z} \frac{\varepsilon}{2} \left|(u_{\varepsilon,M})_z\right|^2 dx, \qquad (5.11)$$

and observe that by the definition of $u_{\varepsilon,M}$ and the regularity of ∂S_ψ, for M large enough and ε small enough independently of M, we get that

$$|\nabla\phi(u_{\varepsilon,M})| = \varepsilon|\nabla u_{\varepsilon,M}|^2 \leq 2\varepsilon|(u_{\varepsilon,M})_z|^2,$$

in a ball $B(x,r)$ for some $r>0$, where $x=(y,z)\in\partial S_\psi$ and $y\in\overline{\Omega}$ is a point at which ψ attains its maximum. Combining these two facts yields

$$\begin{aligned}\Phi^\varepsilon_{c^\dagger}(u_\varepsilon) &\geq \frac{(c^\dagger)^2-(c^\dagger_\varepsilon)^2}{4(c^\dagger)^2}\int_{\Sigma\cap B(x,r)} e^{c^\dagger z}|\nabla\phi(u_\varepsilon)|dx \\ &\to \frac{(c^\dagger)^2-c^2}{4(c^\dagger)^2}c_W \mathrm{Per}_{c^\dagger}(S_\psi,\Sigma\cap B(x,r)) =: L>0,\end{aligned}$$

as $\varepsilon\to 0$. This contradicts (5.10), by taking M such that $Ce^{-c^\dagger M}<L$.

Step 2: let us now prove (i). By Proposition 3.8, $c^\dagger_\varepsilon$ is bounded from above by a constant independent of ε. In particular, there exists $c\in[0,+\infty)$ such that $c^\dagger_\varepsilon\to c$ as $\varepsilon\to 0$, along a sequence. By Step 1 we have $c\geq c^\dagger$, so that it is enough to prove that $c\leq c^\dagger$ for every sequence $\varepsilon\to 0$.

Recall that, for ε sufficiently small, we have $0\leq\bar u_\varepsilon\leq 2$ and $\bar u_\varepsilon(y,z)\leq\frac{1}{2}$ for every $y\in\Omega$ and $z>0$. By (2.3) and Theorem 3.5(iv), this implies that for $M>0$ and ε sufficiently small we have

$$\begin{aligned}0 &= \Phi^\varepsilon_{c^\dagger_\varepsilon}(\bar u_\varepsilon) \\ &\geq \int_{\Sigma_M} e^{c^\dagger_\varepsilon z}\left(\frac{\varepsilon}{2}|\nabla\bar u_\varepsilon|^2+\frac{W(\bar u_\varepsilon)}{\varepsilon}-G(y,\bar u_\varepsilon)\right)dx-\frac{|\Omega|}{c^\dagger_\varepsilon}\|G\|_{2,\infty}e^{-c^\dagger_\varepsilon M} \\ &\geq \int_{\Sigma_M} e^{c^\dagger_\varepsilon z}\left(\sqrt{2W(\bar u_\varepsilon)}|\nabla\bar u_\varepsilon|-G(y,\bar u_\varepsilon)\right)dx-\frac{2|\Omega|}{c}\|G\|_{2,\infty}e^{-cM} \\ &= \int_{\Sigma_M} e^{c^\dagger_\varepsilon z}\left(|\nabla\phi(\bar u_\varepsilon)|-G(y,\bar u_\varepsilon)\right)dx-\frac{2|\Omega|}{c}\|G\|_{2,\infty}e^{-cM},\end{aligned} \tag{5.12}$$

where ϕ is as in (5.4). By Lemma 5.1 we get, up to a subsequence, that

$$\bar u_\varepsilon\to\chi_S \qquad \text{in } L^1_{loc}(\Sigma), \tag{5.13}$$

where $\chi_S\in BV_{loc}(\Sigma)$ and $S\subseteq\Omega\times(-\infty,0)$. Moreover, by (3.11) and the density estimate in Proposition 3.9 we have $\int_S e^{cz}dx>0$.

By the lower semicontinuity in BV (see [5]) of the functional

$$u\mapsto\int_{\Sigma_M} e^{cz}\left(|\nabla\phi(u)|-G(y,u)\right)dx,$$

and by the fact that $\phi(u_\varepsilon) \to \phi(\chi_S) = c_W \chi_S$ in $L^1(\Sigma_M)$, we get

$$\liminf_{\varepsilon\to 0} \int_{\Sigma_M} e^{c_\varepsilon^\dagger z} \left(|\nabla \phi(\bar u_\varepsilon)| - G(y, \bar u_\varepsilon)\right) dx \\ \geq c_W \mathrm{Per}_c(S, \Sigma_M) - \int_{S\cap\Sigma_M} e^{cz} g(y) dx. \tag{5.14}$$

Sending now $M \to \infty$, from (5.12) and (5.14) we conclude that

$$\mathcal{F}_c(S) \leq 0, \tag{5.15}$$

which, by Proposition 4.7(ii), implies that $c \leq c^\dagger$.

Step 3: we now prove (ii). By (5.8) it follows that (5.15) holds with $c = c^\dagger$. Therefore, by Proposition 4.7(iii) the inequality in (5.15) is in fact an equality, and by Remark 4.5 and the density estimate (3.16) the set S is a non-trivial minimizer of $\mathcal{F}_{c^\dagger}$, satisfying all the desired properties. Furthermore, S is the subgraph of a function $\psi : \Omega \to [-\infty, \infty)$ that satisfies all the conclusions of Theorem 4.8.

Let $\theta \in (0,1)$ and assume by contradiction that the level sets $\{\bar u_\varepsilon = \theta\}$ do not converge to ∂S locally uniformly in the Hausdorff distance. This means that there exist $\delta, M > 0$ and points $x_\varepsilon \in \Sigma_M$ such that $\bar u_\varepsilon(x_\varepsilon) = \theta$ and $\mathrm{dist}(x_\varepsilon, \partial S) \geq \delta > 0$. Up to extracting a subsequence we can assume that $x_\varepsilon \in S$ or $x_\varepsilon \in \Sigma \setminus S$, for all ε. Assume $x_\varepsilon \in S$, and let $x \in S$, with $\mathrm{dist}(x, \partial S) \geq \delta$, such that $x_\varepsilon \to x \in S$ as $\varepsilon \to 0$. By (5.13) we have that $\bar u_\varepsilon \to 1$ in $L^1(B(x, \delta/2) \cap \Sigma)$, which contradicts the density estimate (3.17). If $x_\varepsilon \in \Sigma \setminus S$ one can reason analogously, contradicting the density estimate (3.16).

The locally uniform convergence of $\bar u_\varepsilon$ to χ_S outside ∂S is a direct consequence of the convergence of the level sets $\{\bar u_\varepsilon = \theta\}$ to ∂S in the Hausdorff sense.

Step 4: it remains to prove (iii). By Theorem 4.11 there exists a unique minimizer S of $\mathcal{F}_{c^\dagger}$ such that $S \subseteq \Omega \times (-\infty, 0)$ and $\partial S \cap (\overline{\Omega} \times \{0\}) \neq \varnothing$, and so $\bar u_\varepsilon \to \chi_S$ in $L^1(\Sigma_M)$ for every $M > 0$. Moreover, in this case S is the subgraph of a bounded function ψ defined in Ω, so we get that $\chi_S(y, z) \equiv 1$ for all $y \in \Omega$ and $z < \min_{\overline{\Omega}} \psi$. Then from the locally uniform convergence proved in Step 3, the monotonicity of $\bar u_\varepsilon(y, z)$ in z and the fact that $\bar u_\varepsilon \leq 1 + C\varepsilon$ for some $C > 0$ and ε small enough (see Theorem 3.5), we have $\bar u_\varepsilon(y, z) \to 1$ uniformly in $\Omega \times (-\infty, M]$ for every $M < \min_{\overline{\Omega}} \psi$. The conclusion then follows from the fact that again by Theorem 3.5 we have $\bar u_\varepsilon(y, z) \leq v_\varepsilon(y) \leq 1 + C\varepsilon$ for every $(y, z) \in \Sigma$. $\square$

The result in Theorem 5.3 allows us to make an important conclusion about spreading of the level sets of solutions of the initial value problem with general front-like initial data for $\varepsilon \ll 1$. We define the *leading edge*, i.e., the quantity[1]

$$R_\theta^\varepsilon(t) := \sup\{z \in \mathbb{R} : \ u^\varepsilon(y, z, t) > \theta \text{ for some } y \in \Omega\}, \tag{5.16}$$

[1]The definition in (5.16) corrects a typo in [36, Eq. (5.1)].

with $\theta > 0$, for the solution u^ε of (1.14). Then the following result is an immediate consequence of Theorem 5.3 and [36, Theorem 5.8].

Corollary 5.4. *Let u^ε be a solution of* (1.14) *with initial datum $u_0^\varepsilon \in W^{1,\infty}(\Sigma) \cap L_c^2(\Sigma)$ for some $c > c^\dagger$, where $c^\dagger$ is as in Theorem 4.8. Assume that $u_0^\varepsilon \leq 1 + \delta$ in Σ, where δ is as in Remark 2.1, and $u_0^\varepsilon(\cdot, z) \geq 1 + C\varepsilon$ for all $z \leq M$, for some $M \in \mathbb{R}$, where C is as in* (2.4)*. Then under Assumptions 1, 2, 3 we have*

$$\lim_{\varepsilon \to 0} \lim_{t \to \infty} \frac{R_\theta^\varepsilon(t)}{t} = c^\dagger, \tag{5.17}$$

for all $\theta \in (0,1)$, where $R_\theta^\varepsilon(t)$ is given by (5.16)*.*

Thus, $R_\theta^\varepsilon(t)$ propagates, for ε small enough, asymptotically as $t \to \infty$ with the average speed that approaches $c^\dagger$ as $\varepsilon \to 0$. The fact that θ can be chosen arbitrarily from $(0,1)$ follows by inspection of the proof of [36, Theorem 5.8] and the conclusion of Theorem 5.3(ii).

We now investigate the long-time behavior of the solutions of (1.14) in more detail. Under Assumption 4, which is stronger than our standing Assumption 3, we show that the long-time limit of solutions to (1.14) with front-like initial data converges, as $\varepsilon \to 0$, to a traveling wave solution to (1.16) moving with speed $c^\dagger$.

Theorem 5.5. *Let Assumptions 1, 2 and 4 hold. Let $\delta > 0$ be such that*

$$(1-u)f(u) > 0 \qquad \text{for all} \qquad u \in [1-\delta, 1) \cup (1, 1+\delta],$$

let $u_0^\varepsilon \in W^{1,\infty}(\Sigma) \cap L^2_{c_\varepsilon^\dagger}(\Sigma)$ be such that

$$0 \leq u_0^\varepsilon \leq 1 + \delta \qquad \text{and} \qquad \liminf_{z \to -\infty} u_0^\varepsilon(y,z) \geq 1 - \delta \text{ uniformly in } \Omega, \tag{5.18}$$

and let u^ε be the solution of (1.14) *with initial datum u_0^ε. Then there exists $R_\infty \in \mathbb{R}$ such that, for all $M > 0$,*

$$\lim_{\varepsilon \to 0} \lim_{t \to \infty} \| u^\varepsilon(y, z + c_\varepsilon^\dagger t + R_\infty, t) - \chi_{S_\psi}(y,z) \|_{L^1(\Sigma_M)} = 0\,, \tag{5.19}$$

where ψ is given by Theorem 4.11. Moreover, the convergence as $\varepsilon \to 0$ after passing to the limit $t \to \infty$ is locally uniform in $\overline{\Sigma} \backslash \partial S_\psi$.

Proof. The proof follows from Theorem 5.3 and from the stability results in Theorem 1 and Corollary 2.1 of [38], which apply to the solutions of the initial value problem for (1.14) under the additional assumption that $v_\varepsilon(y) = \lim_{z \to -\infty} \bar{u}_\varepsilon(y,z)$ is a nondegenerate stable critical point of E^ε. We note that under our assumptions the value of α in Theorems 1 and 3 of [38] does not depend on the parameter ε. This is due to the fact that $u = 1 - \delta$ is a subsolution for (1.14) for all ε small enough. Thus, to conclude we only need to demonstrate that under the assumptions of the theorem v_ε is indeed non-degenerate. This is proved in Lemma 5.6 below. $\square$

Lemma 5.6. *Let Assumptions 1, 2 and 4 hold and let $(c^\dagger_\varepsilon, \bar u_\varepsilon)$ be as in Theorem 3.5. Then there exists $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$, $v_\varepsilon(y) = \lim_{z\to-\infty} \bar u_\varepsilon(y,z)$ is a nondegenerate stable critical point of E^ε.*

Proof. By Theorem 5.3(iii), we have that $v_\varepsilon \to 1$ uniformly in $\overline{\Omega}$. Fix $\delta > 0$ such that $W''(u) > 0$ for every $u \in [1-\delta, 1+\delta]$. Let ε_0 be such that for all $\varepsilon < \varepsilon_0$ we have $v_\varepsilon(y) \in (1-\delta, 1+\delta)$ for all $y \in \overline{\Omega}$. Moreover, eventually decreasing ε_0, we have that

$$\frac{W''(s)}{\varepsilon} - G_{uu}(y,s) > 0 \qquad \forall y \in \overline{\Omega},\ s \in [1-\delta, 1+\delta],\ \varepsilon < \varepsilon_0.$$

This implies that v_ε is a non degenerate stable critical point of E_ε. □

Remark 5.7. To derive the stability result in Theorem 5.5, it is essential that the local minimizer v_ε of E^ε to which the traveling wave $(c^\dagger_\varepsilon, \bar u_\varepsilon)$ is converging as $z \to -\infty$ is nondegenerate, according to Definition 3.3. In general the assumption that v_ε is nondegenerate is quite difficult to check, even if it is generically satisfied, see the discussion in [38]. In Lemma 5.6, we show that a sufficient condition for it is Assumption 4, together with Assumptions 1 and 2. More generally, we expect that the same nondegeneracy condition on v_ε is generically true, when there is at most one set $\omega \subseteq \Omega$ such that $\omega \times \mathbb{R}$ is a minimizer under compact perturbations of the geometric functional $\mathcal{F}_{c^\dagger}$ and, moreover, this unique local minimizer has positive second variation.

Lastly, we briefly discuss what kinds of counterparts to our propagation results can be obtained, using the methods of [8, 9]. We note that because of the local in time nature of convergence in [8, 9], the order of the limits in (5.19) in such results needs to be reversed. Then the conclusion can be obtained via the analysis of the long time limit of (1.16), as is done, e.g., in [18]. To be specific, if the initial data u_0^ε converge to $\chi_{S_{h_0}}$ locally uniformly out of ∂S_{h_0}, for some $h_0 \in W^{1,\infty}(\Omega)$, then by [8, 9] the solutions u^ε of (1.14) with initial data u_0^ε converge locally uniformly to χ_{S_h}, where h is the solution of (1.16) with initial datum h_0. Since by [18], under Assumption 4, the function $h(y,t) - c^\dagger t - R_\infty$ converges uniformly to $\psi(y)$ as $t \to +\infty$, it follows that

$$\lim_{t\to\infty} \lim_{\varepsilon\to 0} \|u^\varepsilon(y, z + c^\dagger t + R_\infty, t) - \chi_{S_\psi}(y,z)\|_{C(K)} = 0\,, \tag{5.20}$$

for any compact set $K \subset \overline{\Sigma} \setminus \partial S_\psi$. Thus, the expectation about the long time behavior of solutions of (1.14) for $\varepsilon \ll 1$ based on the analysis of the mean curvature flow that follows from (5.20) is justified by our result in Theorem 5.5.

Acknowledgements

The work of AC and MN was partially supported by the Fondazione CaRiPaRo Project "Nonlinear Partial Differential Equations: models, analysis, and control-theoretic problems." The work of CBM was supported, in part, by NSF via grants DMS-0908279, DMS-1119724 and DMS-1313687.

Appendix

A Density estimates

In this Appendix we establish a general density estimate in the spirit of [13, 23, 39] for minimizers of Allen-Cahn type functionals. Note, however, that our estimates are in terms of the averages of the L^2 norms of the minimizers with respect to compactly supported perturbations, rather than in terms of the densities associated with their superlevel sets. The key ingredient of the proof is still an application of the Gagliardo-Nirenberg-Sobolev inequality, as in [13,23,39]. However, the use of a simpler test function and of L^2 estimates makes the proof considerably more straightforward. In fact, our proof is in some sense more along the lines of the respective density estimates for minimal surfaces and relies in an essential way on the Modica-Mortola trick [33]. Also, we point out that our functionals, as in [39] and in contrast to [13,23], do not necessarily admit minimizers that are constants. Our assumptions are more general than those of [39], however, since they do not require $G(\cdot, u)$ to have zero mean.

Theorem A.1. *For $\rho > 2$ and $u \in H^1(B(0,\rho)) \cap L^\infty(B(0,\rho))$, let*

$$H(u) := \int_{B(0,\rho)} \Big((a(x)\nabla u) \cdot \nabla u + b(x)W(u) + G(x,u) \Big) dx, \tag{A.1}$$

where W is defined by (2.1) with f satisfying Assumption 2, $a(x)$ is a symmetric $n \times n$ matrix, $a \in W^{1,\infty}(B(0,\rho);\mathbb{R}^{n\times n})$, $b \in L^\infty(B(0,\rho))$, $G(x,u)$ is a Carathéodory function, and a and b satisfy

$$\lambda \le b(x) \le \lambda^{-1} \quad \text{and} \quad \lambda|\xi|^2 \le (a(x)\xi)\cdot\xi \le \lambda^{-1}|\xi|^2 \qquad \forall x \in B(0,\rho),\ \forall \xi \in \mathbb{R}^n, \tag{A.2}$$

for some $\lambda > 0$. Then there exists $r_0 \in \mathbb{N}$ depending only on n, W, $\|a\|_{W^{1,\infty}(B(0,\rho);\mathbb{R}^{n\times n})}$ and λ such that if u is a minimizer of H with prescribed boundary data on $\partial B(0,\rho)$, $\|u - \frac{1}{2}\|_{L^\infty(B(0,\rho))} \le 1$, $\alpha \in (0, r_0^{1-n})$, R_0 is an integer such that $r_0 + 1 \le R_0 < \rho$, and $\|G\|_{L^\infty(B(0,\rho)\times(-\frac{1}{2},\frac{3}{2}))} \le \alpha R_0^{-1}$, then

$$\fint_{B(0,r_0)} u^2 dx \ge \alpha \qquad \Rightarrow \qquad \fint_{B(0,R)} u^2 dx \ge \alpha, \tag{A.3}$$

$$\fint_{B(0,r_0)} (1-u)^2 dx \ge \alpha \qquad \Rightarrow \qquad \fint_{B(0,R)} (1-u)^2 dx \ge \alpha, \tag{A.4}$$

for all $R \in [r_0, R_0]$ integer.

Proof. We only prove (A.3), since (A.4) then follows by a change of variable $u \to 1-u$. Let $\theta \in C^\infty(\mathbb{R})$ with $\theta(x) = 0$ for all $x < 0$, $\theta(x) = 1$ for all $x > 1$ and $\theta'(x) \ge 0$ for all $x \in \mathbb{R}$. For $1 \le R < \rho - 1$, let $\eta(x) := \theta(|x| - R)$ be a cutoff function. Since u is a minimizer of

H with respect to perturbations supported in $B(0,\rho)$, we have $H(u) \le H(u\eta)$. Then by positivity of a, b and W and the fact that $\eta = 0$ in $B(0,R)$, we obtain

$$\int_{B(0,R+1)} (1-\eta^2)\Big((a(x)\nabla u)\cdot\nabla u + b(x)W(u)\Big)dx \le 2|B(0,R+1)|G_0$$
$$+\int_{B(0,R+1)\setminus B(0,R)} \left(\frac{1}{2}(a(x)\nabla u^2)\cdot\nabla\eta^2 + u^2(a(x)\nabla\eta)\cdot\nabla\eta + b(x)W(u\eta)\right)dx, \quad \text{(A.5)}$$

where we introduced

$$G_0 := \|G\|_{L^\infty(B(0,\rho)\times(-\frac{1}{2},\frac{3}{2}))}. \quad \text{(A.6)}$$

Integrating by parts the first term in the integral on the right-hand side of (A.5) and using the assumptions on a and b in the left-hand side, we have

$$\lambda\int_{B(0,R+1)} (1-\eta^2)\Big(|\nabla u|^2 + W(u)\Big)dx \le 2|B(0,R+1)|G_0$$
$$+\int_{B(0,R+1)\setminus B(0,R)} \Big(b(x)W(u\eta) - u^2\eta\nabla\cdot(a\nabla\eta)\Big)dx. \quad \text{(A.7)}$$

Therefore, since our assumptions imply that $W(s) \le C_1 s^2$ for all $|s| \le \frac{3}{2}$ and some $C_1 > 0$ depending only on W, we have

$$\int_{B(0,R+1)} (1-\eta^2)\Big(|\nabla u|^2 + W(u)\Big)dx$$
$$\le C\left(R^n G_0 + \int_{B(0,R+1)\setminus B(0,R)} u^2 dx\right), \quad \text{(A.8)}$$

for some constant $C > 0$ depending only on n, W, $\|a\|_{W^{1,\infty}(\Omega;\mathbb{R}^{n\times n})}$ and λ, which changes from line to line from now on. In particular, since by our assumptions $W(s) \ge C_2 s^2$ for all $|s| \le \frac{1}{2}$ and some $C_2 > 0$ depending only on W, we obtain

$$\int_{B(0,R)\cap\{|u|\le\frac{1}{2}\}} u^2 dx \le C\left(R^n G_0 + \int_{B(0,R+1)\setminus B(0,R)} u^2 dx\right). \quad \text{(A.9)}$$

We now use the Modica-Mortola trick [33] and estimate the left-hand side of (A.8) from below as follows:

$$\int_{B(0,R+1)} |(1-\eta^2)\nabla\phi(u)|dx \le C\left(R^n G_0 + \int_{B(0,R+1)\setminus B(0,R)} u^2 dx\right), \quad \text{(A.10)}$$

where $\phi(u)$ is defined via (5.4). Therefore, with the help of Gagliardo-Nirenberg-Sobolev inequality we get

$$\left(\int_{B(0,R+1)} |(1-\eta^2)\phi(u)|^{\frac{n}{n-1}} dx\right)^{\frac{n-1}{n}} \leq C\left(R^n G_0 + \int_{B(0,R+1)\setminus B(0,R)} \left(u^2 + \phi(u)|\nabla\eta^2|\right) dx\right). \quad (A.11)$$

Moreover, since by our assumptions $C_3 s^2 \leq |\phi(s)| \leq C_4 s^2$ for all $|s| \leq \frac{3}{2}$ and some $C_3, C_4 > 0$ depending only on W, we have

$$\left(\int_{B(0,R)} |u|^{2\frac{n}{n-1}} dx\right)^{\frac{n-1}{n}} \leq C\left(R^n G_0 + \int_{B(0,R+1)\setminus B(0,R)} u^2 dx\right). \quad (A.12)$$

Raising both sides of this inequality to the power $n/(n-1)$, we obtain

$$\int_{B(0,R)\cap\{|u|>\frac{1}{2}\}} u^2 dx \leq C\left(R^n G_0 + \int_{B(0,R+1)\setminus B(0,R)} u^2 dx\right)^{\frac{n}{n-1}}. \quad (A.13)$$

Let us introduce the quantity

$$M_R := \int_{B(0,R)} u^2 dx + |B(0,1)| R^{n+1} G_0. \quad (A.14)$$

Adding (A.9) and (A.13), and expressing the result in terms of M_R yields

$$M_R - |B(0,1)|R^{n+1}G_0 \leq C\left(1 + \left(R^n G_0 + \int_{B(0,R+1)\setminus B(0,R)} u^2 dx\right)^{\frac{1}{n-1}}\right)(M_{R+1} - M_R). \quad (A.15)$$

We can rewrite the inequality in (A.15) in the form

$$M_{R+1} \geq K(u,R) M_R, \quad (A.16)$$

where

$$K(u,R) := 1 + \frac{C\left(1 - \frac{RG_0}{\fint_{B(0,R)} u^2 dx + RG_0}\right)}{1 + R\left(RG_0 + \fint_{B(0,R+1)\setminus B(0,R)} u^2 dx\right)^{\frac{1}{n-1}}} \geq 1. \quad (A.17)$$

Let now $r_0 \in \mathbb{N}$, and let $R_0 \in \mathbb{N}$ and $\alpha \in \mathbb{R}$ be such that $r_0 + 1 \le R_0 < \rho$ and $\alpha \in (0, r_0^{1-n})$. If $⨍_{B(0,R+1)} u^2 dx \ge \alpha$ for all $r_0 \le R \le R_0 - 1$ integer, then there is nothing to prove. So suppose the opposite inequality holds for some integer $r_0 \le R_1 \le R_0 - 1$, and that R_1 is the smallest value of R for which this happens. Then

$$⨍_{B(0,R_1)} u^2 dx \ge \alpha \quad \text{and} \quad ⨍_{B(0,R_1+1)\setminus B(0,R_1)} u^2 dx < \alpha\,, \tag{A.18}$$

and we can estimate $K(u, R_1)$ from below as

$$\begin{aligned} K(u,R_1) &\ge 1 + \frac{C\alpha}{(R_0G_0+\alpha)\left(1 + R_1\left(R_0G_0+\alpha\right)^{\frac{1}{n-1}}\right)} \\ &\ge 1 + \frac{C}{1 + R_1\alpha^{\frac{1}{n-1}}} \ge 1 + \frac{Cr_0}{2R_1}\,. \end{aligned} \tag{A.19}$$

By (A.14) and (A.16), this implies that

$$\int_{B(0,R_1+1)} u^2 dx + 2^n(n+1)|B(0,1)|R_1^n G_0 \ge \left(1 + \frac{Cr_0}{R_1}\right)\int_{B(0,R_1)} u^2 dx, \tag{A.20}$$

and, hence, by our assumptions and (A.18) we obtain

$$⨍_{B(0,R_1+1)} u^2 dx \ge \left(1 + \frac{1}{R_1}\right)^{-n}\left(1 + \frac{Cr_0 - 2^n(n+1)}{R_1}\right)\alpha. \tag{A.21}$$

Since $R_1 \ge 1$, choosing

$$r_0 = \left\lceil \frac{2^n(n+2)-1}{C} \right\rceil, \tag{A.22}$$

where C is the constant appearing in (A.21), we get that the right-hand side of (A.21) is greater or equal than α, contradicting our assumption on R_1. $\square$

Remark A.2. An inspection of the proof of Theorem A.1 shows that $⨍_{B(0,R)} u^2 dx$ is monotonically increasing in $R \in \mathbb{N}$, as long as it is not too big. More precisely, under the assumptions of Theorem A.1, we have that $⨍_{B(0,R)} u^2 dx$ is monotonically increasing for all $R \in [r_0, R_0]$ integer, provided that $⨍_{B(0,R)} u^2 dx < r_0^{1-n}$ and $\|G\|_{L^\infty(B(0,\rho)\times(-\frac{1}{2},\frac{3}{2}))} \le r_0^{1-n} R_0^{-1}$.

We also note that Theorem A.1 yields the kinds of density estimates for the level sets of the minimizers of Ginzburg-Landau functionals with respect to compactly supported perturbations that were previously obtained in [13, 23, 39]. Here we give a result that extends those of [13, 23, 39] to the case of the functional H in (A.1), generalizing the estimates obtained in [13, 23] for the case of functionals that admit constant minimizers, and the estimates of [39] under assumption of periodicity of G.

Corollary A.3. *Under the assumptions of Theorem A.1, let $\beta \in (0,1)$, and for $R > 0$ let*

$$\mu_{\beta,R} := |\{|u| > \beta\} \cap B(0,R)|. \tag{A.23}$$

If $\mu_{\beta,1} > 0$, there exist $C, C' > 0$ depending on n, W, $\|a\|_{W^{1,\infty}(B(0,\rho);\mathbb{R}^{n\times n})}$, λ, β and $\mu_{\beta,1}$, such that

$$\mu_{\beta,R} \geq CR^n, \tag{A.24}$$

for all $R \in [1,\rho]$ satisfying $R \leq C'\|G\|^{-1}_{L^\infty(B(0,\rho)\times(-\frac{1}{2},\frac{3}{2}))}$.

Proof. Throughout the proof, C, C' denote positive constants depending only on n, W, $\|a\|_{W^{1,\infty}(B(0,\rho);\mathbb{R}^{n\times n})}$, λ, β and $\mu_{\beta,1}$ that may change from line to line.

Let $r_0 \geq 1$ be as in Theorem A.1. Then

$$\fint_{B(0,r_0)} u^2 dx \geq \frac{\beta^2 \mu_{\beta,1}}{|B(0,r_0)|} =: \alpha \in (0, r_0^{1-n}). \tag{A.25}$$

Also, clearly $\fint_{B(0,R)} u^2 dx \geq \alpha$ for every $1 \leq R \leq r_0$. Therefore, by (A.25) and Theorem A.1 we have that (A.3) holds for any $R \in [r_0, R_0]$ integer, provided that $r_0 + 1 \leq R_0 < \rho$ is an integer that satisfies $R_0 G_0 \leq \alpha$, where G_0 is defined in (A.6). Extending this estimate to the whole interval then yields

$$\int_{B(0,R)} u^2 dx \geq 2^{-n}\alpha|B(0,R)| \qquad \forall R \in [1, R_0]. \tag{A.26}$$

Moreover, since $\|u\|_{L^\infty(B(0,\rho))} \leq \frac{3}{2}$ by the assumptions in Theorem A.1, we can write

$$\int_{B(0,R)} u^2 dx \leq \int_{B(0,R)\cap\{|u|\leq\beta\}} u^2 dx + \frac{9}{4}\mu_{\beta,R}. \tag{A.27}$$

On the other hand, by (A.9) (with $\frac{1}{2}$ replaced by β) we have

$$\int_{B(0,R)\cap\{|u|\leq\beta\}} u^2 dx \leq C(R^{n-1} + R^n G_0) \qquad \forall R \in [1,\rho]. \tag{A.28}$$

Therefore, by (A.26), (A.27) and (A.28), and recalling that $R_0 G_0 \leq \alpha < 1$, we obtain for all $R \in [1, R_0]$,

$$\frac{9}{4}\mu_{\beta,R} \geq 2^{-n}\alpha|B(0,1)|R^n - CR^{n-1}. \tag{A.29}$$

From this we conclude that $\mu_{\beta,R} \geq CR^n$ whenever $R > C'$. At the same time, by definition $\mu_{\beta,R} \geq \mu_{\beta,1} \geq \mu_{\beta,1}(C')^{-n}R^n$ whenever $R \leq C'$. This concludes the proof. $\square$

References

[1] M. Alfaro, D. Hilhorst, H. Matano. The singular limit of the Allen-Cahn equation and the FitzHugh-Nagumo system. *J. Differential Equations*, 245:505–565, 2008.

[2] M. Alfaro, H. Matano. On the validity of formal asymptotic expansions in Allen-Cahn equation and FitzHugh-Nagumo system with generic initial data. *Discrete Cont. Dyn. Syst. B* 17:1639–1649, 2012.

[3] S. M. Allen, J. W. Cahn. A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening. *Acta Metal. Mater.* 27:1085–1095, 1979.

[4] M. Amar, V. De Cicco, N. Fusco. Lower semicontinuity and relaxation results in BV for integral functionals with BV integrands. *ESAIM Control Opt. Calc. Var.* 14: 456–477, 2008.

[5] L. Ambrosio, N. Fusco, D. Pallara. *Functions of Bounded Variation and Free Discontinuity Problems.* Oxford Mathematical Monographs, 2000.

[6] D. G. Aronson, H. F. Weinberger. Multidimensional diffusion arising in population genetics. *Adv. Math.* 30:33–58, 1978.

[7] G. Barles, A. Cesaroni, M. Novaga. Homogenization of fronts in highly heterogeneous media. *SIAM J. Math. Anal.* 43:212–227, 2011.

[8] G. Barles, H. M. Soner, P. E. Souganidis. Front propagation and phase field theory, *SIAM J. Control Optim.* 31:439–469, 1993.

[9] G. Barles, P. E. Souganidis. A new approach to front propagation problems: Theory and applications. *Arch. Rational Mech. Anal.* 141:237–296, 1998.

[10] H. Berestycki, L. Nirenberg. Travelling fronts in cylinders. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 9:497–572, 1992.

[11] A. Braides. Γ-*convergence for beginners.* Oxford Lecture Series in Mathematics and its Applications, 2002.

[12] L. Bronsard, R. V. Kohn. Motion by mean curvature as the singular limit of Ginzburg-Landau dynamics. *J. Differential Equations* 90:211–237, 1991.

[13] L. A. Caffarelli, A. Córdoba. Uniform convergence of a singular perturbation problem. *Comm. Pure Appl. Math.* 48:1–12, 1995.

[14] G. Caginalp. The role of microscopic anisotropy in the macroscopic behavior of a phase boundary. *Ann. Phys.* 172:136–155, 1986.

[15] G. Caginalp, P. C. Fife. Phase-field methods for interfacial boundaries. *Phys. Rev. B* 33:7792–7794, 1986.

[16] P. Cardaliaguet, P.-L. Lions, P. E. Souganidis. A discussion about the homogenization of moving interfaces. *J. Math. Pures Appl.* 91:339–363, 2009.

[17] A. Cesaroni, C. Muratov, M. Novaga. Asymptotic behavior of attractors for inhomogeneous Allen-Cahn equations. *RIMS Kokyuroku*, to appear.

[18] A. Cesaroni, M. Novaga. Long-time behavior of the mean curvature flow with periodic forcing. *Comm. Partial Differential Equations* 38:780-801, 2013.

[19] X. Chen. Generation and propagation of interfaces in reaction-diffusion equations. *J. Differential Equations* 96:116–141, 1992.

[20] P. De Mottoni, M. Schatzman. Geometrical evolution of developed interfaces. *Trans. Amer. Math. Soc.* 347:1533–1589, 1995.

[21] G. De Philippis, F. Maggi. *Regularity of free boundaries in anisotropic capillarity problems and the validity of Young's law.* Preprint, 2014, available at http://arxiv.org/pdf/1402.0549v1.pdf.

[22] L. C. Evans, H. M. Soner, P. E. Souganidis. Phase transitions and generalized motion by mean curvature. *Comm. Pure Appl. Math.* 45:1097–1123, 1992.

[23] A. Farina, E. Valdinoci. Geometry of quasiminimal phase transitions *Calc. Var. Partial Differential Equations* 33:1–35, 2008.

[24] P. C. Fife. Pattern formation in reacting and diffusing systems. *J. Chem. Phys.* 64:554–563, 1976.

[25] P. C. Fife. *Dynamics of Internal Layers and Diffusive Interfaces.* Society for Industrial and Applied Mathematics, Philadelphia, 1988.

[26] P. C. Fife, J. B. McLeod. The approach of solutions of nonlinear diffusion equations to travelling front solutions. *Arch. Ration. Mech. Anal.*, 65:335–361, 1977.

[27] D. Gilbarg, N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order.* Springer-Verlag, Berlin, 1983.

[28] E. Giusti. *Minimal surfaces and functions of bounded variation.* Monographs in Mathematics, 80. Birkhäuser Verlag, Basel, 1984.

[29] T. Ilmanen. Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature. *J. Differential Geom.* 38:417–461, 1993.

[30] C. K. R. T. Jones. Asymptotic behaviour of a reaction-diffusion equation in higher space dimensions. *Rocky Mountain J. Math.* 13:355–364, 1983.

[31] M. Lucia, C. B. Muratov, M. Novaga. Existence of traveling wave solutions for Ginzburg-Landau-type problems in infinite cylinders. *Arch. Rat. Mech. Anal.* 188:475–508, 2008.

[32] F. Maggi. *Sets of Finite Perimeter and Geometric Variational Problems.* Cambridge Studies in Advanced Mathematics, 2012.

[33] L. Modica, S. Mortola. Un esempio di Γ-convergenza. *Bollettino U.M.I.* 14-B:285–299, 1977.

[34] L. Mugnai, M. Röger. Convergence of perturbed Allen-Cahn equations to forced mean curvature flow. *Indiana Univ. Math. J.* 60:41–75, 2011.

[35] C. B. Muratov. A global variational structure and propagation of disturbances in reaction-diffusion systems of gradient type. *Discrete Cont. Dyn. Syst. B* 4:867–892, 2004.

[36] C. B. Muratov, M. Novaga. Front propagation in infinite cylinders I. A variational approach. *Comm. Math. Sci.* 6:799–826, 2008.

[37] C. B. Muratov, M. Novaga. Front propagation in infinite cylinders. II. The sharp reaction zone limit. *Calc. Var. Partial Differential Equations* 31:521–547, 2008.

[38] C. B. Muratov, M. Novaga. Global exponential convergence to variational traveling waves in cylinders. *SIAM J. Math. Anal.* 44:293–315, 2012.

[39] M. Novaga, E. Valdinoci. The geometry of mesoscopic phase transition interfaces. *Discrete Cont. Dyn. Syst. A* 19:777–798, 2008.

[40] M. H. Protter, H. F. Weinberger, *Maximum principles in differential equations.* Springer-Verlag, New York, 1984.

[41] J. Rubinstein, P. Sternberg, J. B. Keller. Fast reaction, slow diffusion, and curve shortening. *SIAM J. Appl. Math.* 49:116–133, 1989.

[42] J. E. Taylor. Boundary regularity for solutions to various capillarity and free boundary problems. *Comm. Partial Differential Equations* 2:323-357, 1977.

[43] J. M. Vega. The asymptotic behavior of the solutions of some semilinear elliptic equations in cylindrical domains. *J. Differential Equations* 102:119–152, 1993.

[44] J. Xin. Front propagation in heterogeneous media. *SIAM Review* 42:161–230, 2000.