MINIMIZERS FOR NONLOCAL PERIMETERS OF MINKOWSKI TYPE

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Abstract. We study a nonlocal perimeter functional inspired by the Minkowski content, whose main feature is that it interpolates between the classical perimeter and the volume functional.

This nonlocal functionals arise in concrete applications, since the nonlocal character of the problems and the different behaviors of the energy at different scales allow the preservation of details and irregularities of the image in the process of removing white noises, thus improving the quality of the image without losing relevant features.

In this paper, we provide a series of results concerning existence, rigidity and classification of minimizers, compactness results, isoperimetric inequalities, Poincaré-Wirtinger inequalities and density estimates. Furthermore, we provide the construction of planelike minimizers for this generalized perimeter under a small and periodic volume perturbation.

Contents

1. Introduction

The main goal of this paper is to study a class of variational problems which interpolate between the classical perimeter and the volume functionals. Generalized energies of this type have been analyzed in [7, 10, 11, 17], and also in anisotropic contexts and in view of discretizations methods in [8, 9]. In terms of applications, nonlocal

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functionals interpolating between perimeter and volume are often used in image processing to keep fine details and irregularities of the image while denoising additive white noises, see e.g. [2, 13].

These objects are modeled by an energy which, at large scales, resembles the perimeter using an approximation based on the Minkowski content, but on small scales they present predominant volume contributions, giving rise to a sort of nonlocal behavior, which may produce severe losses of regularity and compactness properties. We will call *r-perimeters* these nonlocal perimeter functionals.

We recall that in recent years a lot of attention has been devoted to the analysis of other type of nonlocal perimeter functionals, starting from the seminal work of Caffarelli, Roquejoffre and Savin [6], where it was initiated the study of the Plateau problem for such kind of nonlocal perimeters. A regularity theory for minimizers of such perimeters has been developed in analogy to the regularity theory of classical minimal surfaces, and also the geometric and variational relation with the classical perimeter has been investigated. For a general overview on the subject we refer to [33] and references therein.

In this paper, we develop a preliminary study of the main properties of the r-perimeters. First of all we analyze the main features of sets with finite r-perimeter, in particular compactness properties, to get existence for the Dirichlet problem, and then isoperimetric inequalities. The global isoperimetric inequality is a direct consequence of the Brunn-Minkowski inequality, whereas its local version is valid at the appropriate scale.

We show some rigidity results for minimizers of the r-perimeter, in dimension 2, and we presents some properties of minimizers. In particular we consider their density properties, pointing out an interesting phenomenon not appearing in the classical case. Indeed in the density estimates two scales of growth appear: if the initial density is below a given threshold depending on r, then there is an exponential density growth, then, over the threshold, the growth reduces to the usual one, that is the radius to the power n.

An important feature of these results is that they always need to capture the "local" behavior of the minimizers, which can be rather different than the "global" one, due to nonlocal effects at small scales. In addition, these problems are not scale invariant and they do not possess any associated extended problem of local type, therefore many classical techniques related to scaled iterations and monotonicity formulas are not easily applicable in our setting. In particular, we show with a concrete example (see Theorem 1.19) that compactness and regularity properties can fail, at a small scale, for minimizers of the r-perimeter with the addition of a sufficiently large volume term.

Finally the last section is devoted to the construction of plane-like minimizers for the r-perimeters in a periodic medium. A classical problem in different fields, including geometry, dynamical systems and partial differential equations, consists in the determination of objects that are embedded into a periodic medium and present bounded oscillations with respect to a reference hyperplane. These objects are somehow the natural extension of "flat" objects such as hyperplanes and linear functions and have the important property that, for these solutions, the forcing term produced by the lack of homogeneity of the medium "averages out" at a large scale. We refer to [22, 25] for the first results of this type on geodesics, to [26] for the introduction of this setting in the case of elliptic integrands, to [1, 4, 12] for the case of hypersurfaces with prescribed mean curvature, to [30, 32] for the case of partial differential equations and to [5, 15] for problems related to statistical mechanics. The planelike structures are also useful to construct pinning effects and localized bump solutions, see e.g. [28, 29]. See also [16] for planelike constructions related to nonlocal problems of fractional type and [14] for a general review.

We also address the problem of existence of planelike minimizers for energy functionals in which the r-perimeter in (1.2) is modulated by a volume term which is periodic and with a sufficiently small size. This is a setting not comprised in the existing literature, since, as far as we know, the only nonlocal cases taken into account are the ones arising from fractional minimal surfaces or related to the Ising model.

In the rest of this section, we formalize the mathematical setting in which we work and we present our main results. The nonlocal perimeter functional based on Minkowski content will be introduced in Subsection 1.1. Some rigidity properties of minimizers will be also discussed.

In Subsections 1.2 we present some compactness results at large scales and some Γ-convergence results for this nonlocal perimeter. Then, in Subsection 1.3 we discuss the Dirichlet problem and in Subsection 1.4 we present some rigidity results.

In Subsection 1.5 we introduce global and relative isoperimetric inequalities. Furthermore, we provide density estimates for the nonlocal perimeter, which in turn show that the compactness and regularity properties of the nonlocal perimeter minimizers may be deeply influenced by oscillations at small scales.

Finally, the planelike minimizers for the nonlocal perimeter are discussed in Subsection 1.6.

A detailed organization of the paper is then presented at the end of the Introduction, in Subsection 1.7.

1.1. **The nonlocal perimeter and the corresponding Dirichlet energy.** We start with some preliminary definitions. Given $r > 0$ and $E \subseteq \mathbb{R}^n$, we let

$$\begin{aligned} E \oplus B_r &:= \bigcup_{x\in E} B_r(x) = (\partial E \oplus B_r) \cup E = (\partial E \oplus B_r) \cup (E \ominus B_r), \\ \text{where} \qquad E \ominus B_r &:= E \setminus \left(\bigcup_{x\in\partial E} B_r(x) \right) = E \setminus \big((\partial E) \oplus B_r \big). \end{aligned} \tag{1.1}$$

We shall identify a set $E \subseteq \mathbb{R}^n$ with its points of density one and ∂E with the topological boundary of the set of points of density one.

Given $r > 0$ and a domain $\Omega \subseteq \mathbb{R}^n$, for any measurable set $E \subseteq \mathbb{R}^n$, we use the notation in (1.1) and we consider the functional

$$\mathrm{Per}_r(E, \Omega) := \frac{1}{2r} \mathcal{L}^n \Big(\big((\partial E) \oplus B_r \big) \cap \Omega \Big) = \frac{1}{2r} \mathcal{L}^n \big((\partial E)_r \cap \Omega \big). \tag{1.2}$$

As customary, we denoted here by $\mathcal{L}^n$ the n-dimensional Lebesgue measure. When $\Omega = \mathbb{R}^n$, we simply write $\mathrm{Per}_r(E) := \mathrm{Per}_r(E, \mathbb{R}^n)$. Note that our definition agrees with that in [8, 9], since we are identifying a set with its points of density one, therefore

$$\mathrm{Per}_r(E, \Omega) = \min_{|E' \Delta E|=0} \mathrm{Per}_r(E', \Omega).$$

We observe (see e.g. [10]) that Per_r is weak lower semicontinuous in L^1_{loc} and that, for every A, B measurable sets,

$$\mathrm{Per}_r(A \cap B, \Omega) + \mathrm{Per}_r(A \cup B, \Omega) \leqslant \mathrm{Per}_r(A, \Omega) + \mathrm{Per}_r(B, \Omega).$$

The definition of Per_r is inspired by the classical Minkowski content (which would be recovered in the limit, see e.g. [8, 9, 19]). In particular, for sets with compact and $(n-1)$-rectifiable boundaries, the functional in (1.2) may be seen as a nonlocal approximation of the classical perimeter functional, in the sense that

$$\lim_{r \searrow 0} \mathrm{Per}_r(E) = \mathcal{H}^{n-1}(\partial E).$$

Hence, in some sense, Per_r recovers a perimeter functional for small r and a volume energy for large r.

We observe that recently a great attention has been devoted to the fractional perimeters introduced in [6], which also interpolate the classical perimeter with an area type functional (see e.g. [18] for a review on such topic). The functional in (1.2) is however very different in spirit from that in [6], since the lack of scaling invariance does not allow a classical regularity theory and causes severe lack of compactness at small scales (as we will discuss in details in the sequel).

More generally, given a domain $\Omega \subseteq \mathbb{R}^n$ and a function $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, we define the energy

$$\mathcal{F}_{r,g}(E, \Omega) := \mathrm{Per}_r(E, \Omega) + \int_{E\cap\Omega} g(x)\, dx. \tag{1.3}$$

The functional in (1.2) is related via a coarea formula to a Dirichlet energy which takes into account the local oscillation of a function, which is described as follows. Let $\Omega \subseteq \mathbb{R}^n$ be a domain and $u \in L^1_{\mathrm{loc}}(\Omega \oplus B_r)$. Then for any $x \in \Omega$ we consider the oscillation of u in $B_r(x)$, given by

$$\operatorname*{osc}_{B_r(x)} u := \sup_{B_r(x)} u - \inf_{B_r(x)} u.$$

In this paper, in the sup and inf notation, we mean the "essential supremum and infimum" of the function (i.e., sets of null measure are neglected). It can be checked by using the definition that a triangular inequality holds: for all $v, u \in L^1_{\mathrm{loc}}(\Omega \oplus B_r)$

$$\operatorname*{osc}_{B_r(x)} (u + v) \leqslant \operatorname*{osc}_{B_r(x)} (u) + \operatorname*{osc}_{B_r(x)} (v).$$

We have the following generalized coarea formula (see formulas (4.3) and (5.7) in [11] for similar formulas in very related contexts):

Lemma 1.1. *It holds that*

$$\int_\Omega \operatorname*{osc}_{B_r(x)} u\, dx = 2r \int_{-\infty}^{+\infty} \mathrm{Per}_r(\{u > s\}, \Omega)\, ds. \tag{1.4}$$

In the setting of (1.2) and (1.3), we introduce the definition of local minimizer and Class A minimizer. We are interested in existence, compactness and regularity properties of such minimizers. Moreover we will also provide construction of planelike minimizers for such energies in periodic media.

Definition 1.2 (Local minimizer and Class A minimizer)**.** A set E is a minimizer for Per_r (resp. for $\mathcal{F}_{r,g}$) in a bounded domain Ω if for any measurable set $F \subseteq \mathbb{R}^n$ with $F \setminus (\Omega \ominus B_r) = E \setminus (\Omega \ominus B_r)$ it holds that

$$\mathrm{Per}_r(E,\Omega) \leqslant \mathrm{Per}_r(F,\Omega) \qquad (\text{resp. } \mathcal{F}_{r,g}(E,\Omega) \leqslant \mathcal{F}_{r,g}(F,\Omega)).$$

E is a Class A minimizer if it is a minimizer in this sense in any ball of $\mathbb{R}^n$.

We observe that if E is a Class A minimizer, then

$$\mathrm{Per}_r(E,B_R) \leqslant n\omega_n R^{n-1}, \quad \text{if } R \geqslant 2r.$$

Indeed,

$$\mathrm{Per}_r(E,B_R) \leqslant \mathrm{Per}_r(E\setminus B_{R-r},B_R) \leqslant \mathrm{Per}_r(B_{R-r},B_R) = \frac{\omega_n}{2r}\left(R^n - (R-2r)^n\right).$$

Note that in Definition 1.2 we allow competitors only away from the boundary of the domain, in a way compatible with the natural scale of the problem. Actually, this is the appropriate notion of minimizer, since the following result shows that the problem trivializes if competitors are allowed to produce modifications up to the boundary of the domain.

Proposition 1.3. *Let $E \subseteq \mathbb{R}^n$ be such that for every ball $B \subseteq \mathbb{R}^n$, and for any measurable set $F \subseteq \mathbb{R}^n$ with $F \setminus B = E \setminus B$ it holds that*

$$\mathrm{Per}_r(E,B) \leqslant \mathrm{Per}_r(F,B).$$

Then either $E = \varnothing$ or $E = \mathbb{R}^n$.

1.2. **Γ-convergence results and compactness properties.** We start with some convergence results on Per_{r_k} as $r_k \to r$. We focus also on compactness properties of sets with bounded energy.

Theorem 1.4. *Let Ω be either an open and bounded subset of $\mathbb{R}^n$, or equal to $\mathbb{R}^n$. Let also $r_k \to r \in (0,+\infty)$. Then the following holds.*

(1) $\mathrm{Per}_{r_k}(E,\Omega) \to \mathrm{Per}_r(E,\Omega)$ *and* $\mathcal{F}_{r_k,g}(E,\Omega) \to \mathcal{F}_{r,g}(E,\Omega)$ *for all* $E \subseteq \mathbb{R}^n$.

(2) $\mathrm{Per}_{r_k}(\cdot,\Omega)$ *(resp.* $\mathcal{F}_{r_k,g}(\cdot,\Omega)$*)* Γ*-converges in* $L^1_{\mathrm{loc}}(\Omega)$ *to* $\mathrm{Per}_r(\cdot,\Omega)$ *(resp. to* $\mathcal{F}_{r,g}(\cdot,\Omega)$*).*

(3) *Let* $E_k \subseteq \mathbb{R}^n$ *be such that*

$$\sup_{k\in\mathbb{N}} \mathrm{Per}_{r_k}(E_k,\Omega) < +\infty.$$

Assume that, up to subsequences, $\chi_{E_k} \rightharpoonup u$, *in* $L^1_{\mathrm{loc}}(\Omega)$ *with* $u : \mathbb{R}^n \to [0,1]$. *Let* $\Sigma := \{x \in \Omega : u(x) \in (0,1)\}$ *and*

$$\ell := \liminf_{r_k\to r} \mathrm{Per}_{r_k}(E_k,\Omega). \tag{1.5}$$

Then the following holds true:

$$\mathcal{L}^n(\Sigma \oplus B_r \cap \Omega) \leqslant 2\ell r, \qquad \chi_{E_k} \to u \ \text{ in } \ L^1_{\mathrm{loc}}(\Omega\setminus\Sigma), \qquad \int_\Omega \operatorname*{osc}_{B_r(x)} u\,dx \leqslant 2r\ell. \tag{1.6}$$

Observe that we cannot expect a stronger compactness result, due to the following observation.

Remark 1.5. Families of sets E_k for which $\mathrm{Per}_r(E_k,\Omega) \leqslant 1$ are not necessarily compact in $L^1(\Omega)$ (and, more generally, it is not necessarily true that χ_{E_k} converges pointwise up to a subsequence).

Remark 1.6. When Ω is unbounded and $r_k \searrow r > 0$, some pathological counterexamples to the claim in (1) of Theorem 1.4 may arise. For instance, one may have that

$$\mathrm{Per}_{r_k}(E,\Omega) = +\infty \quad \text{while} \quad \mathrm{Per}_r(E,\Omega) = 0. \tag{1.7}$$

We recall also the following result, which is proved in [9, Theorem 3.1 and Remark 3.4].

Theorem 1.7. *Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded, and $r \to 0$. Then the following holds.*

(1) $\mathrm{Per}_r(\cdot,\Omega)$ Γ*-converges in* $L^1_{\mathrm{loc}}(\Omega)$ *to* $\mathrm{Per}(\cdot,\Omega)$, *where* Per *is the standard perimeter.*

(2) *Let $E_r \subseteq \mathbb{R}^n$ be such that*

$$\sup_r \mathrm{Per}_r(E_r,\Omega) < +\infty.$$

Then there exists $E \subseteq \mathbb{R}^n$ such that

$$\chi_{E_r} \to \chi_E \quad \text{in } L^1_{\mathrm{loc}}(\Omega) \quad \text{up to a subsequence}$$

and

$$\mathrm{Per}(E,\Omega) \leqslant \liminf_{r\to 0} \mathrm{Per}_r(E_r,\Omega).$$

An analogous result holds for the functional $\mathcal{F}_{r,g}$.

Dealing with compactness issues, it is interesting to point out that sequences of minimizers (differently than sequences of sets with bounded r-perimeter) always provide a limit which is also a minimizer, on a smaller set. From the technical point of view, such limit is obtained from the support of the weak limit, once sets of zero measure are neglected. The precise statement goes as follows:

Theorem 1.8. *Let Ω be a bounded open set and E_k be a sequence of local minimizers of $\mathcal{F}_{r,g}$ in Ω such that $\chi_{E_k} \rightharpoonup u$, with $u : \mathbb{R}^n \to [0,1]$, in $L^1(\Omega)$.*

Let E be such that $\{u = 1\} \subseteq E \subseteq \Omega \setminus \{u = 0\}$, and $\Sigma := \{x \in \Omega : u(x) \in (0,1)\}$. Then E is a local minimizer of $\mathcal{F}_{r,g}$ in $\Omega \ominus B_r$ and $g(x) = 0$ for a.e. $x \in \Sigma$.

Moreover, if $\mathcal{L}^n(\{g = 0\}) = 0$ then $\chi_{E_k} \to \chi_E$.

1.3. **The Dirichlet problem.** We now consider the Dirichlet problem for the functional Per_r.

Theorem 1.9. *Let $E_o \subseteq \mathbb{R}^n$ and Ω a bounded open set. Fix $\Omega' \Subset \Omega$. Then, there exists $E \subseteq \mathbb{R}^n$ such that $E \setminus \Omega' = E_o \setminus \Omega'$, and*

$$\mathrm{Per}_r(E,\Omega) \leqslant \mathrm{Per}_r(F,\Omega)$$

for any $F \subseteq \mathbb{R}^n$ for which $F \setminus \Omega' = E_o \setminus \Omega'$.

The same holds for the functional $\mathcal{F}_{r,g}$.

1.4. **Class A minimizers.** In this subsection, we present some rigidity results for the nonlocal functionals introduced in (1.2).

Next result shows that half-spaces are always Class A minimizers for Per_r.

Proposition 1.10. *Let $\omega \in \mathbb{R}^n$ and $E = \{x \,|x \cdot \omega < 0\}$. Then E is a Class A minimizer for* Per_r.

In addition, we give the complete characterization of Class A minimizers in dimension 1, according to the following result:

Theorem 1.11. *If E is a Class A minimizer for* Per_r *and $n = 1$, then E is either $\varnothing$ or $\mathbb{R}$ or a halfline of the type either $(a,+\infty)$ or $(-\infty,a)$, for some $a \in \mathbb{R}$.*

It would be interesting to study the Bernstein problem for Per_r. In particular, in analogy with the classical perimeter, one could expect that the the Class A minimizers are only $\varnothing$ or $\mathbb{R}^n$ or half-spaces, at least in small dimension.

1.5. **Isoperimetric inequalities and density estimates.** We now discuss the isoperimetric properties of the functional Per_r. To this end, we first point out that balls are isoperimetric for the functional in (1.2), as a consequence of the Brunn-Minkowski Inequality. Namely, we have that:

Lemma 1.12. (i) *For any $R > 0$ and any measurable set $E \subseteq \mathbb{R}^n$ such that $\mathcal{L}^n(E) = \mathcal{L}^n(B_R)$ it holds that*

$$\mathrm{Per}_r(E) \geqslant \mathrm{Per}_r(B_R). \tag{1.8}$$

(ii) *Viceversa, if $\mathcal{L}^n(E) = \mathcal{L}^n(B_R)$ and*

$$\mathrm{Per}_r(E) = \mathrm{Per}_r(B_R),$$

then $E = B_R(p) \setminus \mathcal{N}$, for some set $\mathcal{N}$ of null measure and some $p \in \mathbb{R}^n$.

We present now a version of the relative isoperimetric inequality for Per_r in an appropriate scale:

Theorem 1.13. *Let assume there exists $\lambda \geqslant 1$ such that*

$$\lambda R \geqslant r > 0. \tag{1.9}$$

There exists $C > 0$, possibly depending on n, such that for all $E \subseteq \mathbb{R}^n$ with

$$\frac{\mathcal{L}^n(E \cap B_R)}{\mathcal{L}^n(B_R)} \leqslant \frac{1}{2}, \tag{1.10}$$

there holds

$$\Big(\mathcal{L}^n(E \cap B_R)\Big)^{\frac{n-1}{n}} \leqslant C\lambda \operatorname{Per}_r(E, B_R). \tag{1.11}$$

For the proof of this result we will need the following technical lemma (which can be seen as a working version of the compactness result in Theorem 1.7).

Lemma 1.14. *Let $\Omega \subseteq \mathbb{R}^n$ be open and bounded. Consider a sequence of sets $\Omega_k \supseteq \Omega$, such that*

$$\partial\Omega_k \text{ is uniformly locally Lipschitz.} \tag{1.12}$$

Let

$$r_k \to 0 \quad \text{as} \quad k \to +\infty \tag{1.13}$$

and $E_k \subseteq \mathbb{R}^n$ such that

$$\sup_{k\in\mathbb{N}} \operatorname{Per}_{r_k}(E_k, \Omega_k) < +\infty. \tag{1.14}$$

Then, there exist $\widehat{E}_k \subseteq \mathbb{R}^n$, $E \subseteq \mathbb{R}^N$ and a constant $C > 0$ only depending on n such that

$$\widehat{E}_k \supseteq E_k, \tag{1.15}$$

$$\operatorname{Per}(\widehat{E}_k,\, \Omega_k) \leqslant C \operatorname{Per}_{r_k}(E_k, \Omega_k) \tag{1.16}$$

$$\int_{\Omega_k} |\chi_{E_k} - \chi_{\widehat{E}_k}|\, dx \leqslant C\, r_k \operatorname{Per}_{r_k}(E_k, \Omega_k) \tag{1.17}$$

$$\text{and} \qquad \chi_{E_k} \to \chi_E \qquad \text{in } L^1(\Omega) \text{ up to a subsequence.} \tag{1.18}$$

Remark 1.15. We stress that (1.11) holds with a constant which depends on λ, that is, on the ratio $\frac{r}{R}$, if $r > R$. Namely, if $r > R$, (1.11) may fail to be true for C just depending on n.

As a simple consequence of Theorem 1.13, we also provide the following nonlocal Poincaré-Wirtinger inequality:

Theorem 1.16. *There exists $C > 0$, only depending on n, such that the following statement holds. Let $\lambda \geqslant 1$, with $\lambda R \geqslant r > 0$, and $u \in L^\infty(B_r) \cap L^1(B_R)$. Let*

$$\langle u \rangle_R := \frac{1}{\mathcal{L}^n(B_R)} \int_{B_R} u.$$

Then,

$$\int_{B_R} \big|u - \langle u \rangle_R\big| \leqslant \frac{CR\lambda}{r} \int_{B_R} \operatorname*{osc}_{B_r(x)} u\, dx. \tag{1.19}$$

Remark 1.17. When $r > R$, the estimate (1.19) does not necessarily hold true with a constant independent of λ.

We address now the density properties of the minimizers of Per_r. Differently than the classical cases, the density properties of the minimizers may depend on the initial density for small scales: nevertheless, we can obtain a density growth in larger balls, and the constants become uniform once a suitable density threshold is reached. More precisely, our result is the following:

Theorem 1.18. *Let $\Omega \subseteq \mathbb{R}^n$, $r > 0$ and E be a minimizer for* Per_r. *Let $R_o > 0$. Suppose that $B_{R_o} \subseteq \Omega$ and*

$$\omega_o := \mathcal{L}^n(E \cap B_{R_o}) > 0. \tag{1.20}$$

Let also $k \in \mathbb{N}$ be such that $B_{R_o+2kr} \subseteq \Omega$. Then,

$$\mathcal{L}^n(E \cap B_{R_o+2kr}) \geqslant (\omega_o^{\frac{1}{n}} + 2c_\star kr)^n, \tag{1.21}$$

for a suitable $c_\star > 0$, possibly depending on n, r and ω_o.

Moreover,

$$\text{if } n = 1,\ c_\star \text{ is a pure number, independent of } r \text{ and } \omega_o. \tag{1.22}$$

Also,

$$\text{if } \omega_o \geqslant \underline{c}\, r^n \text{ for some } \underline{c} > 0, \text{ then } c_\star \text{ only depends on } n \text{ and } \underline{c}, \text{ and it is independent of } \Omega \text{ and } \omega_o. \tag{1.23}$$

In addition, if

$$\mathcal{L}^n(E \cap B_{R_o+2(k-1)r}) \leqslant \overline{C} r^n, \tag{1.24}$$

for some $\overline{C} > 0$, then

$$\mathcal{L}^n(E \cap B_{R_o+2(k-1)r}) \geqslant \omega_o\, (1+\widetilde{c})^k\,, \tag{1.25}$$

for some $\widetilde{c} > 0$, depending on n and $\overline{C}$.

It is interesting to point out that Theorem 1.18 detects two scales of growth (and this fact is different from the case of the classical minimal surfaces, as well as of the nonlocal minimal surfaces in [6], where there is only one type of growth, given by the dimension of the space). Indeed, in our framework, if the initial density is below the threshold prescribed by r^n (as stated in (1.24)), then there is an exponential density growth (as stated in (1.25)), till the density reaches the quantity r^n. Then, once a density of order r^n is reached, the growth reduces to the usual one, that is the radius to the power n (as stated in (1.21)). In such case of polynomial growth away from an initial r^n, the constant become uniform (as stated in (1.23), being the onedimensional case special, in view of (1.22)).

We think that it would be interesting to establish whether or not the growth in (1.25) is optimal or if sharper estimates may be obtained independently on the initial density.

Finally it is interesting to remark that compactness and regularity properties related to Per_r can be problematic, or even fail, at a small scale, also for minimizers. To make a concrete example, we consider $K > 0$ and the function $g(x) := -K\chi_{B_r \setminus B_{r/2}}$. We let

$$\mathcal{F}_K(E) := \mathcal{F}_{r,g}(E) = \mathrm{Per}_r(E) - K\,\mathcal{L}^n\big(E \cap (B_r \setminus B_{r/2})\big).$$

Then, minimizers are not necessarily smooth and sequences of minimizers are not necessarily compact. Indeed, we have:

Theorem 1.19. *There exists $C > 0$, only depending on n, for which the following statement holds true.*

Suppose that

$$K \geqslant \frac{C}{r}. \tag{1.26}$$

Then, there exists $E_* \subseteq \mathbb{R}^n$ satisfying

$$\mathcal{F}_K(E_*) \leqslant \mathcal{F}_K(E)$$

for any bounded set $E \subseteq \mathbb{R}^n$, and such that ∂E_* is not locally a continuous graph (and, in fact, can be "arbitrarily bad" inside $B_{r/2}$).

Moreover, there exists a sequence $E_k \subseteq \mathbb{R}^n$ satisfying

$$\mathcal{F}_K(E_k) \leqslant \mathcal{F}_K(E)$$

for any bounded set $E \subseteq \mathbb{R}^n$, and such that χ_{E_k} is not precompact in $L^1(B_r)$.

Given the negative result in Theorem 1.19, we think that it is an interesting problem to develop a regularity theory for minimizers of Per_r and of functionals such as $\mathcal{F}_{r,g}$.

1.6. **Planelike minimizers in periodic media.** In the spirit of [4], we recall the following definition:

Definition 1.20. We say that a set $E \subseteq \mathbb{R}^n$ is planelike if, up to an appropriate change of coordinates, there exists $K > 0$ such that

$$E \supseteq \{(x_1, \dots, x_n) \text{ s.t. } x_n \leqslant 0\} \quad \text{and} \quad \mathbb{R}^n \setminus E \supseteq \{(x_1, \dots, x_n) \text{ s.t. } x_n \geqslant K\}.$$

To state our result, we recall some notation. We say that a direction $\omega \in S^{n-1}$ is rational if the orthogonal space has maximal dimension over the integers, i.e.

$$\begin{aligned} &\text{there exist } K_1, \dots, K_{n-1} \in \mathbb{Z}^n \text{ which are linearly independent} \\ &\text{and such that } \omega \cdot K_j = 0 \text{ for any } j \in \{1, \dots, n-1\}. \end{aligned} \tag{1.27}$$

Given a rational direction $\omega \in S^{n-1}$, we say that a set E is ω-periodic if, for any $k \in \mathbb{Z}^n$ with $\omega \cdot k = 0$, we have that $E + k = E$. Similarly, a function $u : \mathbb{R}^n \to \mathbb{R}$ is said to be ω-periodic if, for any $k \in \mathbb{Z}^n$ with $\omega \cdot k = 0$, it holds that $u(x + k) = u(x)$ for any $x \in \mathbb{R}^n$.

Then, we state the following:

Theorem 1.21. *There exist $\eta \in (0, 1)$ and $M > 1$, only depending on n, such that the following result holds true. Let $r \in (0, 1)$, $g : \mathbb{R}^n \to \mathbb{R}$ be $\mathbb{Z}^n$-periodic, with zero average in $[0, 1]^n$ and such that $\|g\|_{L^\infty(\mathbb{R}^n)} \leqslant \eta$.*

Let $\omega \in S^{n-1}$. Then, there exists E^*_ω which is a Class A minimizer for $\mathcal{F}_{r,g}$, such that

$$\{\omega \cdot x \leqslant -M\} \subseteq E^*_\omega \subseteq \{\omega \cdot x \leqslant M\}. \tag{1.28}$$

Moreover, if ω is rational, then E^*_ω is ω-periodic.

1.7. **Organization of the paper.** The rest of the paper is devoted to the proofs of our main results. Section 2 contains the proofs of Proposition 1.3.

The Γ-convergence results and the compactness properties for the functional $\mathcal{F}_{r,g}$, together with the proofs of Theorem 1.4, Remarks 1.5 and 1.6, and Theorem 1.8, are presented in Section 3.

The proof of Theorems 1.9 is contained in Section 4.

The characterizations of Class A minimizers in Proposition 1.10 and in Theorem 1.11 are dealt with in Section 5.

We address the isoperimetric inequalities in Section 6, which contains the proofs of Lemma 1.12, Lemma 1.14, Theorem 1.13, Remark 1.15, Theorem 1.16 and Remark 1.17.

The regularity and density estimates, with the proofs of Theorems 1.18 and 1.19, are discussed in Section 7.

Finally, in Section 8, we deal with the construction of the planelike minimizers in periodic media and we prove Theorem 1.21.

2. Basic properties of minimizers of Per_r – Proof of Proposition 1.3

We provide the proof of Proposition 1.3, which justifies our definitions of local and Class A minimizers, given in Definition 1.2.

Proof of Proposition 1.3. First of all, we claim that there exists a universal $\varepsilon > 0$ such that

$$\left\{ \left[\left(\mathbb{R}^n \setminus B_{1/\varepsilon}\left(\frac{e_n}{\varepsilon}\right)\right) \cap (\mathbb{R}^n \setminus B_{1-\varepsilon}) \right] \oplus B_1 \right\} \cap B_\varepsilon(e_n) = \varnothing. \tag{2.1}$$

A picture can easily convince the reader about this simple geometric fact.

From now, we fix ε as in (2.1) and, without loss of generality, we take $\varepsilon \in \left(0, \frac{1}{2}\right]$. As a matter of fact, scaling (2.1), we see that

$$\left\{ \left[\left(\mathbb{R}^n \setminus B_{r/\varepsilon}\left(\frac{re_n}{\varepsilon}\right)\right) \cap \left(\mathbb{R}^n \setminus B_{(1-\varepsilon)r}\right) \right] \oplus B_r \right\} \cap B_{\varepsilon r}(re_n) = \varnothing$$

and therefore

$$\begin{aligned} &\mathcal{L}^n \left(\left\{ \left[\left(\mathbb{R}^n \setminus B_{r/\varepsilon}\left(\frac{re_n}{\varepsilon}\right)\right) \cap \left(\mathbb{R}^n \setminus B_{(1-\varepsilon)r}\right) \right] \oplus B_r \right\} \cap B_r \right) \\ &\qquad \leqslant \mathcal{L}^n\big(B_r \setminus B_{\varepsilon r}(re_n)\big) < \mathcal{L}^n(B_r). \end{aligned} \tag{2.2}$$

Now we take E to be a Class A minimizer for Per_r and we claim that

$$\begin{aligned} &\text{either there exists } p \in \mathbb{R}^n \text{ such that } B_{r/\varepsilon}(p) \subseteq E, \\ &\text{or there exists } p \in \mathbb{R}^n \text{ such that } B_{r/\varepsilon}(p) \subseteq \mathbb{R}^n \setminus E. \end{aligned} \tag{2.3}$$

The proof of (2.3) is by contradiction: if not, any ball of radius r/ε contains both points of E and of its complement, and so it contains at least one point of ∂E.

Let now $M \geqslant 10$ to be taken suitably large in the sequel and $R := Mr/\varepsilon$. We consider N disjoint balls of radius $2r/\varepsilon$ contained in the ball B_R, and we observe that we can take $N \geqslant \frac{c R^n}{(r/\varepsilon)^n} = cM^n$, for some universal $c > 0$. Let us call $B_{2r/\varepsilon}(p_1), \dots, B_{2r/\varepsilon}(p_N)$ such balls. We know that each ball $B_{r/\varepsilon}(p_j)$ contains a point $q_j \in \partial E$ and so $(\partial E) \oplus B_r$ contains at least the balls $B_r(q_j)$ which are disjoint and contained in B_R.

Consequently,

$$2r \operatorname{Per}_r(E, B_R) \geqslant \sum_{j=1}^{N} \mathcal{L}^n(B_r(q_j)) = N\mathcal{L}^n(B_r) \geqslant \bar{c} M^n r^n, \tag{2.4}$$

for some $\bar{c} > 0$.

Now we consider $F := E \cup B_{R-r}$. Notice that $\partial F \subseteq \mathbb{R}^n \setminus B_{R-r}$ and thus $(\partial F) \oplus B_r \subseteq \mathbb{R}^n \setminus B_{R-2r}$. This and the minimality of E give that

$$2r \operatorname{Per}_r(E, B_R) \leqslant 2r \operatorname{Per}_r(F, B_R) \leqslant \mathcal{L}^n(B_R \setminus B_{R-2r}) \leqslant C R^{n-1} r = \frac{C M^{n-1} r^n}{\varepsilon^{n-1}}.$$

From this and (2.4) a contradiction easily follows by taking M appropriately large (possibly also in dependence of the fixed ε). This completes the proof of (2.3).

Now, from (2.3), we can suppose that E contains a ball of radius r/ε and we prove that $E = \mathbb{R}^n$ (if instead $\mathbb{R}^n \setminus E$ contains a ball of radius $B_{r/\varepsilon}$, a similar argument would prove that E is void).

Sliding the ball till it touches the boundary of E, we find a ball of radius r/ε which lies in E and whose boundary contains a point of ∂E. Therefore, up to a rigid motion, we can suppose that $0 \in \partial E$ and $B_{r/\varepsilon}(re_n/\varepsilon) \subseteq E$. We define

$$G := E \cup B_{(1-\varepsilon)r} \supseteq B_{r/\varepsilon}(re_n/\varepsilon) \cup B_{(1-\varepsilon)r}.$$

Notice that

$$\partial G \subseteq \mathbb{R}^n \setminus \big(B_{r/\varepsilon}(re_n/\varepsilon) \cup B_{(1-\varepsilon)r}\big) = \big(\mathbb{R}^n \setminus B_{r/\varepsilon}(re_n/\varepsilon)\big) \cap \big(\mathbb{R}^n \setminus B_{(1-\varepsilon)r}\big)$$

and so

$$(\partial G) \oplus B_r \subseteq \Big[\big(\mathbb{R}^n \setminus B_{r/\varepsilon}(re_n/\varepsilon)\big) \cap \big(\mathbb{R}^n \setminus B_{(1-\varepsilon)r}\big)\Big] \oplus B_r.$$

This, (2.2) and the minimality of E give that

$$2r \operatorname{Per}_r(E, B_r) \leqslant 2r \operatorname{Per}_r(G, B_r) < \mathcal{L}^n(B_r). \tag{2.5}$$

On the other hand, since $0 \in \partial E$, we have that $(\partial E) \oplus B_r \supseteq B_r$ and therefore

$$2r \operatorname{Per}_r(E, B_r) \geqslant \mathcal{L}^n(B_r).$$

This is in contradiction with (2.5). The proof of Proposition 1.3 is thus complete. $\square$

3. Γ-convergence results and compactness properties for the functional $\mathcal{F}_{r,g}$ – Proofs of Theorem 1.4, Remarks 1.5 and 1.6, and Theorem 1.8

We start with a preliminary result on the convergence of characteristic functions.

Lemma 3.1. *Let Ω be an open subset of $\mathbb{R}^n$ and let E_k be a sequence of sets such that χ_{E_k} converges to u weakly in $L^1_{\mathrm{loc}}(\Omega)$. Then, letting $\Sigma := \{x \in \Omega : u(x) \in (0,1)\}$, there holds*

$$\chi_{E_k} \to u \quad \text{in } L^1(\Omega \setminus \Sigma). \tag{3.1}$$

In particular, if u is a characteristic function, then $\chi_{E_k} \to u$ in $L^1_{\mathrm{loc}}(\Omega)$.

Proof. Without loss of generality we can assume that Ω is bounded.

Let $u_k := \chi_{E_k}$. Since $0 \leqslant u_k \leqslant 1$, we have that

$$\begin{aligned}
\lim_{k\to+\infty} \int_{\Omega\setminus\Sigma} |u_k - u| &= \lim_{k\to+\infty} \left(\int_{\Omega\cap\{u=1\}} (1-u_k) + \int_{\Omega\cap\{u=0\}} u_k \right) \\
&= \lim_{k\to+\infty} \left(\mathcal{L}^n\big(\Omega\cap\{u=1\}\big) - \int_\Omega u_k \chi_{\{u=1\}} + \int_\Omega u_k \chi_{\{u=0\}} \right) \\
&= \mathcal{L}^n\big(\Omega\cap\{u=1\}\big) - \int_\Omega u \chi_{\{u=1\}} + \int_\Omega u \chi_{\{u=0\}} = 0,
\end{aligned}$$

which proves (3.1). $\square$

Remark 3.2. Note that a consequence of the previous lemma is the following fact: if F_k is a sequence of sets such that $\chi_{F_k} \to \chi_F$ in $L^1_{\rm loc}(\Omega)$, for some open set Ω and for some $F \subset \mathbb{R}^n$ then $\chi_{\lambda_k F_k} \to \chi_F$ in $L^1_{\rm loc}(\Omega)$ for all $\lambda_k \to 1$. Indeed it is sufficient to prove that $\chi_{\lambda_k F_k}$ converges to χ_F weakly in $L^1_{\rm loc}(\Omega)$ and then apply the previous lemma.

We now provide the proof of the convergence result for Per_r.

Proof of Theorem 1.4. First of all, we prove the claim in (1). For this, we observe that, for every $r > 0$, it holds that

$$\mathcal{L}^n\Big(\partial\big((\partial E) \oplus B_r\big)\Big) = 0. \tag{3.2}$$

This can be obtained e.g. as a consequence of the estimate proved in [23, Theorem 2]: for all closed sets A and all $r > 0$, it holds that $\mathcal{H}^{n-1}(\partial(A \oplus B_r)) \leqslant \frac{C}{r}\mathcal{L}^n\big((A \oplus B_r) \setminus A\big)$, where $C > 0$ is a dimensional constant. Using this with $A := A_m = (\partial E) \cap B_m$, for any fixed $m \in \mathbb{N}$, we find that $\mathcal{H}^{n-1}(\partial(A_m \oplus B_r)) < +\infty$, and therefore

$$\mathcal{L}^n\left(\bigcup_{m\in\mathbb{N}} \partial(A_m \oplus B_r)\right) = 0. \tag{3.3}$$

We conclude observing that

$$\partial\big((\partial E) \oplus B_r\big) \subseteq \bigcup_{m\in\mathbb{N}} \partial(A_m \oplus B_r). \tag{3.4}$$

Using (3.2), we see that $\chi_{(\partial E)\oplus B_{r_k}} \to \chi_{(\partial E)\oplus B_r}$ a.e. in Ω.

Hence, if Ω is bounded, or if $\Omega = \mathbb{R}^n$ and ∂E is bounded, the assertion in (1) follows from the Dominated Convergence Theorem.

To complete the proof of (1), we have only to consider the case in which $\Omega = \mathbb{R}^n$ and ∂E is unbounded. In this case, we can take a sequence $p_j \in \partial E$, with $|p_j| > 2r + 2 + |p_{j-1}|$. In this way $B_\rho(p_j) \cap B_\rho(p_i)$ is void when $j \neq i$ and $\rho \in (0, r+1)$, which gives that $\mathcal{L}^n\big((\partial E) \oplus B_\rho\big) = +\infty$. This says that, in this case,

$$\mathrm{Per}_{r_k}(E, \Omega) = +\infty = \mathrm{Per}_r(E, \Omega),$$

and so (1) holds true.

Now, we prove the claim in (2). By (1), we immediately deduce that

$$\Gamma - \limsup_{r_k\to r} \mathrm{Per}_{r_k}(\cdot, \Omega) \leqslant \mathrm{Per}_r(\cdot, \Omega).$$

We are left to prove that, if $E_k \to E$ in $L^1_{\rm loc}(\Omega)$, then

$$\liminf_{r_k\to r} \mathrm{Per}_{r_k}(E_k, \Omega) \geqslant \mathrm{Per}_r(E, \Omega).$$

To see this, we observe that, if we set

$$\widetilde{E}_k := \frac{r}{r_k} E_k, \qquad \widetilde{\Omega}_k := \frac{r}{r_k}\Omega, \tag{3.5}$$

then

$$\mathrm{Per}_{r_k}(E_k, \Omega) = \left(\frac{r_k}{r}\right)^{n-1} \mathrm{Per}_r(\widetilde{E}_k, \widetilde{\Omega}_k), \tag{3.6}$$

and, recalling Remark 3.2,

$$\chi_{\widetilde{E}_k} \to \chi_E \text{ in } L^1_{\rm loc}(\Omega). \tag{3.7}$$

Notice that, again by Remark 3.2 applied with $F_k := \widetilde{\Omega}_k$ and $F := \Omega$, we have that

$$|\mathrm{Per}_r(\widetilde{E}_k, \widetilde{\Omega}_k) - \mathrm{Per}_r(\widetilde{E}_k, \Omega)| \leqslant \frac{1}{2r}\, |\widetilde{\Omega}_k \Delta \Omega| \to 0 \qquad \text{as } k \to \infty. \tag{3.8}$$

Therefore, by the lower semicontinuity of the functional Per_r, from (3.6), (3.7) and (3.8) we conclude that

$$\liminf_{r_k\to r} \mathrm{Per}_{r_k}(E_{r_k}, \Omega) = \liminf_{r_k\to r} \left(\frac{r_k}{r}\right)^{n-1} \mathrm{Per}_r(\widetilde{E}_{r_k}, \Omega) \geqslant \mathrm{Per}_r(E, \Omega).$$

This completes the proof of (2).

To prove (3), we define $\widetilde{E}_{r_k}$ as in (3.5). Then

$$\ell = \liminf_{r_k \to r} \mathrm{Per}_r(\widetilde{E}_{r_k}, \Omega).$$

Let $u_k := \chi_{\widetilde{E}_{r_k}}$. Then, up to subsequences, u_k converges to u weakly-$\star$ in $L^\infty_{\mathrm{loc}}(\Omega)$ and then also u_k converges to u weakly in $L^1_{\mathrm{loc}}(\Omega)$, with $u : \mathbb{R}^n \to [0,1]$. So, by the lower semicontinuity of the functional $\mathcal{E}_{1,\Omega}$, we have that

$$\int_\Omega \operatorname*{osc}_{B_r(x)} u\,dx \leqslant \liminf_k \int_\Omega \operatorname*{osc}_{B_r(x)} u_k\,dx = 2r \liminf_k \mathrm{Per}_r(\widetilde{E}_{r_k}, \Omega) = 2r\ell,$$

which proves the third inequality in (1.6).

So, by Lemma 3.1, we have

$$u_k \to u \text{ in } L^1(\Omega \setminus \Sigma). \tag{3.9}$$

Also, (3.9) gives that, for all $x \in \Sigma \oplus B_r$, it holds that

$$(\partial E_k) \cap B_r(x) \neq \varnothing \text{ for } k \text{ large enough.} \tag{3.10}$$

Indeed, if this is not true, then either $B_r(x) \subseteq E_k$ or $B_r(x) \subseteq \mathbb{R}^n \setminus E_k$ for infinitely many k's, and so either $u_k = 1$ or $u_k = 0$ a.e. in $B_r(x)$ for infinitely many k. This would imply that either $u = 1$ or $u = 0$ a.e. in $B_r(x)$, in contradiction with the fact that $\Sigma \cap B_r(x) \neq \varnothing$, and so (3.10) is proved.

Using (3.10), we get that $\operatorname{osc}_{B_r(x)} u_k \to 1$ for $x \in \Sigma \oplus B_r$, therefore

$$\mathcal{L}^n((\Sigma \oplus B_r) \cap \Omega) \leqslant \liminf_k \int_{(\Sigma \oplus B_r)\cap\Omega} \operatorname*{osc}_{B_r(x)} u_k\,dx \leqslant 2r \liminf_k \mathrm{Per}_r(\widetilde{E}_{r_k}, \Omega) = 2r\ell,$$

which completes the proof of (1.6).

Then, the proof of Theorem 1.4 is complete. □

We now exhibit the lack of compactness that was claimed in Remark 1.5.

Proof of Remark 1.5. Let us take, for example, $r := 1$ and $\Omega := (-3,3) \times (0,1)^{n-1} \subseteq \mathbb{R}^n$. Let also, for any $k \geqslant 1$,

$$E_k := \bigcup_{j=-2^{k-1}}^{2^{k-1}} \left(\frac{2j}{2^k}, \frac{2j+1}{2^k} \right) \times (0,1)^{n-1}.$$

If χ_{E_k} converged pointwise, it would also converge in $L^1(\Omega)$, due to the Dominated Convergence Theorem. But this is not the case, since the norm in $L^1(\Omega)$ of $\chi_{E_k} - \chi_{E_{k+m}}$ is always bounded from below independently on k and m. □

Now we present the pathological counterexample to Theorem 1.4, as stated in Remark 1.6.

Proof of Remark 1.6. We take $n \geqslant 2$, $r > 0$, $r_k := r + \frac{1}{k}$, $\Omega := \{x_n > 0\}$ and $E := \{x_n < -r\}$. In this way, for any $\rho \geqslant r$,

$$(\partial E) \oplus B_\rho = \{x_n = -r\} \oplus B_\rho = \big\{x_n \in (-r-\rho,\, -r+\rho)\big\}.$$

Therefore

$$\big((\partial E) \oplus B_r\big) \cap \Omega = \{x_n \in (-2r, 0)\} \cap \{x_n > 0\} = \varnothing,$$

$$\text{and} \qquad \big((\partial E) \oplus B_{r_k}\big) \cap \Omega = \left\{ x_n \in \left(-2r - \frac{1}{k}, \frac{1}{k} \right) \right\} \cap \{x_n > 0\} = \left\{ x_n \in \left(0, \frac{1}{k} \right) \right\}.$$

These considerations prove (1.7). □

Now we show that sequences of minimizers for the functional $\mathcal{F}_{r,g}$ produce a limit minimizer.

Proof of Theorem 1.8. First of all we consider the case in which $E = \{u = 1\}$, and we show that it is a local minimizer in Ω and that $g = 0$ a.e. on Σ.

Observe that for all $x \in A := (\Sigma \cup \partial E) \oplus B_r$ it holds that

$$(\partial E_k) \cap B_r(x) \neq \varnothing \text{ for } k \text{ large enough.} \tag{3.11}$$

The proof of this fact is the same as the proof of (3.10) above (in the proof of Theorem 1.4).

Fix $\Omega' \subseteq \Omega \ominus B_r$ and let

$$E_k^\star := (E \cup (\Sigma \cap \{g < 0\}) \cap \Omega') \cup (E_k \setminus \Omega').$$

Observe that

$$\int_{E_k} g\,dx = \int_{E_k^\star} g\,dx + \int_{(E_k\cap(\Sigma\cap\{g\geqslant 0\}))\cap\Omega'} g\,dx - \int_{((\Sigma\cap\{g<0\})\setminus E_k)\cap\Omega'} g\,dx + \omega'_k \tag{3.12}$$

with

$$\omega'_k := \int_{E_k\cap\{u=1\}\cap\Omega'} g\,dx - \int_{\{u=1\}\cap\Omega'} g\,dx + \int_{E_k\cap\{u=0\}\cap\Omega'} g\,dx.$$

Note that

$$\lim_{k\to+\infty}\omega'_k = \lim_{k\to+\infty}\int_{\{u=1\}\cap\Omega'} g\chi_{E_k}\,dx - \int_{\{u=1\}\cap\Omega'} g\,dx + \lim_{k\to+\infty}\int_{\{u=0\}\cap\Omega'} g\chi_{E_k}\,dx = 0.$$

We define $\Sigma_k := (E_k\Delta\{g<0\})\cap\Sigma\cap\Omega'$. So (3.12) reads

$$\int_{E_k\cap\Omega'} g\,dx = \int_{E_k^\star\cap\Omega'} g\,dx + \int_{\Sigma_k}|g|\,dx + \omega'_k. \tag{3.13}$$

We also let

$$C := \big((E_k\setminus\overline{E_k^\star})\cup(E_k^\star\setminus\overline{E_k})\big)\cap\partial\Omega'.$$

Notice that for all $x\in D := (C\oplus B_r)\cap\Omega'$ there holds

$$(\partial E_k)\cap B_r(x)\neq\varnothing \text{ for } k \text{ large enough.} \tag{3.14}$$

To check this, we argue by contradiction and we suppose that, for instance, $B_r(x)\subseteq E_k$ for k large enough. Then, $u_k=1$, and so $u=1$ a.e. in $B_r(x)$, i.e. $B_r(x)\cap\Omega'\subseteq E\cap\Omega'$. Recalling that $B_r(x)\setminus\Omega'\subseteq E_k\setminus\Omega'$, this implies that $B_r(x)\subseteq E_k^\star$. Accordingly, we have that $B_r(x)\cap C=\varnothing$, and so $x\notin D$, against our assumption. This proves (3.14).

Properties (3.11) and (3.14) imply that

$$(A\cup D)\cap\Omega'\subseteq\big(((\partial E_k)\oplus B_r)\cap\Omega'\big)\cup O_k, \tag{3.15}$$

for some $O_k\subseteq\mathbb{R}^n$ with

$$\omega_k := \frac{1}{2r}\mathcal{L}^n\big(O_k\big)\to 0 \qquad \text{as } k\to+\infty. \tag{3.16}$$

By definition of $E_k^\star$, there holds

$$\big((\partial E_k^\star)\oplus B_r\big)\cap\Omega'\subseteq(A\cup D\cup((\partial E_k)\oplus B_r))\cap\Omega', \tag{3.17}$$

Therefore, from (3.15) and (3.17) it follows that

$$\mathcal{L}^n\big(((\partial E_k^\star)\oplus B_r)\cap\Omega'\big)\leqslant\mathcal{L}^n\big(((\partial E_k)\oplus B_r)\cap\Omega'\big)+2r\omega_k$$

and then

$$\mathrm{Per}_r(E_k^\star,\Omega')\leqslant\mathrm{Per}_r(E_k,\Omega')+\omega_k. \tag{3.18}$$

From (3.18) and (3.13) we get

$$\mathrm{Per}_r(E_k^\star,\Omega')+\int_{E_k^\star\cap\Omega'} g\,dx\leqslant\mathrm{Per}_r(E_k,\Omega')+\int_{E_k\cap\Omega'} g\,dx+\omega_k-\int_{\Sigma_k}|g|\,dx-\omega'_k.$$

Therefore, by minimality of E_k we deduce that

$$0=\lim_{k\to+\infty}\int_{\Sigma_k}|g|\,dx=\int_{\Sigma\cap\{g>0\}} gu\,dx-\int_{\Sigma\cap\{g<0\}} g(1-u)\,dx.$$

This implies that $u=0$ on $\Sigma\cap\{g>0\}$ and $u=1$ on $\Sigma\cap\{g<0\}$, which, recalling the definition of Σ, implies that $\mathcal{L}^n(\Sigma\cap\{g>0\})=0=\mathcal{L}^n(\Sigma\cap\{g<0\})$, so $g=0$ almost everywhere on Σ. If $\mathcal{L}^n(\{g=0\})=0$, we deduce that $\mathcal{L}^n(\Sigma)=0$, and then we conclude the strong convergence of χ_{E_k} to χ_E.

Moreover, since $\Sigma_k\subseteq\Sigma$, we conclude that

$$\mathrm{Per}_r(E_k^\star,\Omega')+\int_{E_k^\star\cap\Omega'} g\,dx\leqslant\mathrm{Per}_r(E_k,\Omega')+\int_{E_k\cap\Omega'} g\,dx+\omega_k-\omega'_k. \tag{3.19}$$

Let now F be such that $F\Delta E\subseteq\Omega'\ominus B_r$. We define

$$F_k := (F\cap\Omega')\cup(E_k\setminus\Omega').$$

By construction

$$\mathrm{Per}_r(F_k,\Omega')-\mathrm{Per}_r(E_k^\star,\Omega')=\mathrm{Per}_r(F,\Omega')-\mathrm{Per}_r(E,\Omega').$$

Recalling (7.5) we then get

$$\begin{aligned}\mathrm{Per}_r(F,\Omega')+\int_{F\cap\Omega'}g\,dx-\mathrm{Per}_r(E,\Omega')-\int_{E\cap\Omega'}g\,dx&=\mathrm{Per}_r(F_k,\Omega')+\int_{F_k\cap\Omega'}g\,dx-\mathrm{Per}_r(E_k^\star,\Omega')-\int_{E_k^\star\cap\Omega'}g\,dx\\&\geqslant \mathrm{Per}_r(F_k,\Omega')+\int_{F_k\cap\Omega'}g\,dx-\mathrm{Per}_r(E_k,\Omega')-\omega_k-\int_{E_k\cap\Omega'}g\,dx+\omega_k'\geqslant-\omega_k+\omega_k',\end{aligned}$$

where the last inequality follows by the minimality of E_k. Now we send $k\to+\infty$ and we obtain that

$$\mathrm{Per}_r(F,\Omega')+\int_{F\cap\Omega'}g\,dx-\mathrm{Per}_r(E,\Omega')-\int_{E\cap\Omega'}g\,dx\geqslant 0,$$

thanks to (3.16). This concludes the proof of the local minimality of $E=\{u=1\}$.

Now, let E be any set such that $\{u=1\}\subseteq E\subseteq\Omega\setminus\{u=0\}$. Then we can define $E_k^\star=((E\cup(\Sigma\cap\{g<0\})\cap\Omega')\cup(E_k\setminus\Omega')$ and repeat the same argument as above (recalling that $g=0$ almost everywhere on Σ) to get that (7.5) holds. The proof of Theorem 1.8 is thus complete. ☐

4. The Dirichlet problem – Proof of Theorem 1.9

Proof of Theorem 1.9. Let E_k be a minimizing sequence. Then, up to subsequences, $\chi_{E_k}\rightharpoonup u$ in $L^1_{\mathrm{loc}}(\Omega)$, with $u:\mathbb{R}^n\to[0,1]$. By the lower semicontinuity in L^1 of the functional $v\to\int_\Omega \mathrm{osc}_{B_r(x)}\,v\,dx$ proved in [9] and the coarea formula, we get

$$\liminf_{k\to+\infty}\mathrm{Per}_r(E_k,\Omega)=\liminf_{k\to+\infty}\frac{1}{2r}\int_\Omega\operatorname*{osc}_{B_r(x)}\chi_{E_k}\,dx\geqslant\frac{1}{2r}\int_\Omega\operatorname*{osc}_{B_r(x)}u\,dx=\int_0^1\mathrm{Per}_r(\{u>s\},\Omega)\,ds. \tag{4.1}$$

Notice that

$$\int_0^1\mathrm{Per}_r(\{u>s\},\Omega)\,ds\geqslant\mathrm{Per}_r(\{u>s_\Omega\},\Omega), \tag{4.2}$$

for a suitable $s_\Omega\in(0,1)$. So, we define $E:=\{u>s_\Omega\}$. Since χ_{E_k} does not depend on k outside Ω', we have that $E=E_k$ outside Ω' and thus it is an admissible competitor. Then, (4.1) says that E is a minimizer for Per_r, and this proves Theorem 1.9. ☐

5. Class A minimizers – Proofs of Proposition 1.10 and of Theorem 1.11

Now we prove the results about Class A minimizers. We start showing that half-spaces are Class A minimizers for Per_r in every dimension.

Proof of Proposition 1.10. In this proof, we write $x=(x',x_n)\in\mathbb{R}^n$. Up to translations and rotations, we can assume that $E=\{x\in\mathbb{R}^n \text{ s.t. } x_n<0\}$. We fix B_R with $R>r$, and we consider $F\subseteq\mathbb{R}^N$ such that $F\Delta E\Subset B_R$. Let C_R be the cylinder $\{x'\in\mathbb{R}^{n-1}\text{ s.t. } |x'|\leqslant R\}\times[-R,R]$, and observe that $\mathrm{Per}_r(E,C_R)=n\omega_nR^{n-1}$.

For any fixed $x'\in\mathbb{R}^{n-1}$, let also $\ell_{x'}=\{(x',x_n)\in\mathbb{R}^n\text{ s.t. } x_n\in\mathbb{R}\}$. We compute

$$2r\mathrm{Per}_r(F,C_R)=\int_{|x'|\leqslant R}\mathcal{H}^1((\partial F\oplus B_r)\cap\ell_{x'})\,dx'\geqslant 2r\int_{|x'|\leqslant R}dx'=2r\mathrm{Per}_r(E,C_R),$$

where we used the observation that $\mathcal{H}^1((\partial F\oplus B_r)\cap\ell_{x'})\geqslant 2r$, for every x'. This proves Proposition 1.10. ☐

Now we characterize the Class A minimizers of the nonlocal perimeter functional in dimension 1

Proof of Theorem 1.11. Suppose that $E\subseteq\mathbb{R}$ is a Class A minimizer for Per_r. Assume also that $E\neq\varnothing$ and $E\neq\mathbb{R}$. Observe that this implies that $E\not\subseteq(a,b)$ and $\mathbb{R}^n\setminus E\not\subseteq(a,b)$ for every $-\infty<a<b<+\infty$. Indeed, if $E\subseteq(a,b)$ with $-\infty<a<b<+\infty$, then the empty set would be an admissible competitor for E in $(a-r,b+r)$ and this would contradict the minimality of E. Similarly for $\mathbb{R}^n\setminus E$.

To conclude, it is sufficient to show that E is connected:

$$\text{if } p,\, q\in E \text{ with } p<q, \text{ then } (p,q)\subseteq E. \tag{5.1}$$

We prove (5.1) by contradiction.

Assume it is not true, then there exists a point $\beta \in (\partial E) \cap (p,q)$. We define $F := E \cup (p,q)$ and we observe that F and E coincide outside (p,q). Also,

$$(\partial F) \cap (p,q) = \varnothing \text{ while } (\partial E) \cap (p,q) \ni \beta. \tag{5.2}$$

We also observe that

$$(\partial F) \setminus [p,q] = (\partial E) \setminus [p,q]. \tag{5.3}$$

We claim that

$$(\partial F) \setminus (p,q) \subseteq (\partial E) \setminus (p,q). \tag{5.4}$$

Indeed, if $\zeta \in (\partial F) \setminus (p,q)$ then either $\zeta \in (\partial F) \setminus [p,q]$, or $\zeta \in \{p,q\}$. If $\zeta \in (\partial F) \setminus [p,q]$, then, by (5.3), we have that $\zeta \in (\partial E) \setminus [p,q] \subseteq (\partial E) \setminus (p,q)$, and we are done.

Hence, we can focus on the case in which, for instance, $\zeta = p$. Since F contains (p,q), the fact that $\zeta \in \partial F$ implies that there exists $\zeta_k \in \mathbb{R}^n \setminus F$ with $\zeta_k \leqslant \zeta = p$. Then, by the definition of F, we see that $\xi_k \in \mathbb{R}^n \setminus E$. On the other hand, we know that $\xi = p \in E$ (recall (5.1)). These observations imply that $\zeta = p \in \partial E$. This proves (5.4) also in this case.

From (5.2) and (5.4) we get that

$$\begin{aligned}
& \mathcal{L}^n\Big(\big((\partial E) \oplus (-r,r)\big) \cap (p-r,q+r)\Big) - \mathcal{L}^n\Big(\big((\partial F) \oplus (-r,r)\big) \cap (p-r,q+r)\Big) \\
= \;& \mathcal{L}^n\Big(\big((\partial E) \oplus (-r,r)\big) \cap (p,q)\Big) - \mathcal{L}^n\Big(\big((\partial F) \oplus (-r,r)\big) \cap (p,q)\Big) \\
& + \mathcal{L}^n\Big(\big((\partial E) \oplus (-r,r)\big) \cap \big((p-r,q+r) \setminus (p,q)\big)\Big) \\
& - \mathcal{L}^n\Big(\big((\partial F) \oplus (-r,r)\big) \cap \big((p-r,q+r) \setminus (p,q)\big)\Big) \\
\geqslant \;& \mathcal{L}^n\big((\beta - r, \beta + r)\big) - \mathcal{L}^n\big((0,r)\big) \\
> \;& 0.
\end{aligned}$$

This implies that $\mathrm{Per}_r(E,(p-r,q+r)) > \mathrm{Per}_r(F,(p-r,q+r))$, which is against minimality, and so the proof of (5.1) is completed. $\square$

6. Isoperimetric inequalities – Proofs of Lemma 1.12, Lemma 1.14, Theorem 1.13, Remark 1.15, Theorem 1.16 and Remark 1.17

Now, we deal with the isoperimetric problems.

Proof of Lemma 1.12. First of all, we prove (i). To this end, we remark that, without loss of generality, we can suppose that ∂E is bounded (if not, there would exist a sequence $x_j \in \partial E$ such that $|x_j| \geqslant j$ and $|x_{j+1} - x_j| \geqslant 2r+1$, and thus $\partial E \oplus B_r$ would contain the disjoint balls $B_r(x_j)$, thus yielding that $\mathrm{Per}_r(E) = +\infty$).

In addition, we notice that $(\partial B_R) \oplus B_r = B_{R+r} \setminus B_{(R-r)^+}$ and therefore

$$2r\,\mathrm{Per}_r(B_R) = \mathcal{L}^n\Big((\partial B_R) \oplus B_r\Big) = \mathcal{L}^n(B_{R+r} \setminus B_{(R-r)^+}).$$

By the Brunn-Minkowski Inequality (see e.g. [31] or Theorem 4.1 in [20]) we have that

$$\begin{aligned}
\Big(\mathcal{L}^n\big(E \oplus B_r\big)\Big)^{1/n} &\geqslant \Big(\mathcal{L}^n(E)\Big)^{1/n} + \Big(\mathcal{L}^n(B_r)\Big)^{1/n} \\
&= \Big(\mathcal{L}^n(B_R)\Big)^{1/n} + \Big(\mathcal{L}^n(B_r)\Big)^{1/n} = \Big(\mathcal{L}^n(B_{R+r})\Big)^{1/n}.
\end{aligned} \tag{6.1}$$

As a consequence, we get

$$\mathcal{L}^n\big(E \oplus B_r\big) - \mathcal{L}^n(E) \geqslant \mathcal{L}^n\big(B_{R+r}\big) - \mathcal{L}^n(B_R). \tag{6.2}$$

We observe that if $R < r$, then $B_{R-r} = \emptyset$ and $E \ominus B_r = \emptyset$. Therefore (6.2) implies (i).

On the other hand, if $R \geqslant r$, let us take $\widetilde{R} \in [0,R]$ such that

$$\mathcal{L}^n\big(E \ominus B_r\big) = \mathcal{L}^n\big(B_{\widetilde{R}}\big).$$

Also, recalling that $\big(E \ominus B_r\big) \oplus B_r \subseteq E$, we have that

$$\mathcal{L}^n\big((E \ominus B_r) \oplus B_r\big) \leqslant \mathcal{L}^n(E) = \mathcal{L}^n(B_R).$$

Accordingly, applying again the Brunn-Minkowski Inequality we get that

$$\begin{aligned}\mathcal{L}^n(B_R)^{1/n} &\geqslant \mathcal{L}^n\big((E\ominus B_r)\oplus B_r\big)^{1/n}\\ &\geqslant \Big(\mathcal{L}^n(E\ominus B_r)\Big)^{1/n}+\Big(\mathcal{L}^n(B_r)\Big)^{1/n}=\Big(\mathcal{L}^n(B_{\widetilde{R}+r})\Big)^{1/n},\end{aligned}$$

which implies that $\widetilde{R}\leqslant R-r$.

From this, we obtain that if $R\geqslant r$

$$\begin{aligned}\mathcal{L}^n(E)-\mathcal{L}^n\big(E\ominus B_r\big) &=\mathcal{L}^n(B_R)-\mathcal{L}^n(B_{\widetilde{R}})\\ &\geqslant \mathcal{L}^n(B_R)-\mathcal{L}^n\big(B_{R-r}\big).\end{aligned}\tag{6.3}$$

Putting together (6.2) and (6.3) if $R\geqslant r$ we obtain

$$\begin{aligned}2r\,\mathrm{Per}_r(E)&=\mathcal{L}^n\big(E\oplus B_r\big)-\mathcal{L}^n(E\ominus B_r)\\ &\geqslant \mathcal{L}^n\big(B_{R+r}\big)-\mathcal{L}^n\big(B_{R-r}\big)=2r\,\mathrm{Per}_r(B_R),\end{aligned}$$

thus proving (i).

Now, we prove (ii). For this, we observe that if equality holds, then all the previous equalities hold true with equal sign. In particular, formula (6.1) would give that

$$\Big(\mathcal{L}^n\big(E\oplus B_r\big)\Big)^{1/n}=\Big(\mathcal{L}^n(E)\Big)^{1/n}+\Big(\mathcal{L}^n(B_r)\Big)^{1/n}.$$

Hence (see e.g. page 363 in [20]), since equality holds in the Brunn-Minkowski inequality if and only if the two sets are homothetic convex bodies (up to removing sets of measure zero), we have that $E=B_{\lambda R}(p)\setminus\mathcal{N}$, for some set $\mathcal{N}$ of null measure, some $p\in\mathbb{R}^n$ and some $\lambda>0$. Since

$$\mathcal{L}^n(B_R)=\mathcal{L}^n(E)=\mathcal{L}^n\big(B_{\lambda R}(p)\setminus\mathcal{N}\big)=\lambda^n\mathcal{L}^n(B_R),$$

we obtain that $\lambda=1$, which establishes (ii). □

Having settled the global isoperimetric problem, we now deal with the proof of the relative isoperimetric inequality. First of all we give the proof of the technical lemma.

Proof of Lemma 1.14. We consider a partition of $\mathbb{R}^n$ into adjacent cubes of side $\frac{r_k}{4\sqrt{n}}$ (hence, the diameter of each cube is $\frac{r_k}{4}$). These cubes will be denoted by $\{Q_j\}_{j\in\mathbb{N}}$. For any $k\in\mathbb{N}$, we set

$$I_k:=\{j\in\mathbb{N}\text{ s.t. }Q_j\cap E_k\neq\varnothing\}.\tag{6.4}$$

Let also

$$\widehat{E}_k:=\bigcup_{j\in I_k}Q_j.$$

Notice that (1.15) is obvious in this setting. We now prove (1.16). For this, we say that Q_j is a k-boundary cube if $j\in I_k$ and there exists a cube Q_i that is adjacent to Q_j with $i\notin I_k$. We let β_k be the number of k-boundary cubes which intersect Ω_k.

We remark that

$$\mathrm{Per}(\widehat{E}_k,\Omega_k)\leqslant C\beta_k r_k^{n-1},\tag{6.5}$$

for some $C>0$. We also claim that

$$\beta_k\leqslant\frac{C\,\mathrm{Per}_{r_k}(E_k,\Omega_k)}{r_k^{n-1}}.\tag{6.6}$$

up to renaming $C>0$. To this end, let Q_j be a k-boundary cube and Q_i be its adjacent cube with $j\in I_k$ and $i\notin I_k$. Thus, by (6.4), there exists $p_{j,k}\in Q_j\cap E_k$ and $p_{i,k}\in Q_i\setminus E_k$. Consequently, we find a point $p_k^\star\in\partial E_k$ which lies at distance at most $r_k/4$ from Q_j. Therefore

$$(\partial E_k)\oplus B_{r_k}\supseteq B_{r_k}(p_k^\star)\supseteq Q_j\oplus B_{\frac{r_k}{100}}.\tag{6.7}$$

In addition, if Q_j intersects Ω_k, it follows from (1.12) that (for large k)

$$\mathcal{L}^n\big((Q_j\oplus B_{\frac{r_k}{100}})\cap\Omega_k\big)\geqslant\frac{r_k^n}{C},$$

for some $C > 0$. Hence, if $Q_j^\star$ denotes the dilation of Q_j by a factor 2 with respect to its center, we have that $Q_j^\star \supseteq Q_j \oplus B_{\frac{r_k}{100}}$ and

$$\mathcal{L}^n\big((Q_j \oplus B_{\frac{r_k}{100}}) \cap \Omega_k \cap Q_j^\star\big) = \mathcal{L}^n\big((Q_j \oplus B_{\frac{r_k}{100}}) \cap \Omega_k\big) \geqslant \frac{r_k^n}{C}.$$

This and (6.7) give that

$$\mathcal{L}^n\Big(\big((\partial E_k) \oplus B_{r_k}\big) \cap \Omega_k \cap Q_j^\star\Big) \geqslant \frac{r_k^n}{C}. \tag{6.8}$$

Our goal is now to sum up (6.8) for all the indices j for which Q_j is a boundary cube that intersects Ω_k. Notice that the family $\{Q_j^\star\}_{j\in\mathbb{N}}$ is overlapping (differently from the original nonoverlapping family $\{Q_j\}_{j\in\mathbb{N}}$), but the number of overlappings is finite, say bounded by some $C^\star > 0$. Hence, since (6.8) is valid for any k-boundary cube Q_j which intersect Ω_k, summing up (6.8) over the indices j gives that

$$\begin{aligned} C^\star \mathcal{L}^n\Big(\big((\partial E_k) \oplus B_{r_k}\big) \cap \Omega_k\Big) &\geqslant \sum_{j\in\mathbb{N}} \mathcal{L}^n\Big(\big((\partial E_k) \oplus B_{r_k}\big) \cap \Omega_k \cap Q_j^\star\Big) \\ &\geqslant \sum_{\substack{k\text{-boundary cube } Q_j \\ \text{which intersect } \Omega_k}} \mathcal{L}^n\Big(\big((\partial E_k) \oplus B_{r_k}\big) \cap \Omega_k \cap Q_j^\star\Big) \geqslant \frac{\beta_k r_k^n}{C} \end{aligned}$$

and thus

$$C^\star \operatorname{Per}_{r_k}(E_k, \Omega_k) \geqslant \frac{\beta_k r_k^{n-1}}{C},$$

that establishes (6.6), up to renaming constants.

From (6.5) and (6.6) it follows that (1.16) holds true, as desired.

In addition, from (1.14) and (1.16), we obtain a uniform bound for $\operatorname{Per}(\widehat{E}_k, \Omega_k)$ and thus on $\operatorname{Per}(\widehat{E}_k, \Omega)$, so by compactness, up to a subsequence we have that there exists $E \subseteq \mathbb{R}^n$ for which

$$\chi_{\widehat{E}_k} \to \chi_E \quad \text{in } L^1(\Omega). \tag{6.9}$$

Now we prove (1.17). For this, let

$$\begin{aligned} & J_k := \big\{ j \in I_k \text{ s.t. } Q_j \cap \Omega_k \neq \varnothing \text{ and } Q_j \setminus E_k \neq \varnothing \big\} \\ \text{and} \quad & H_k := \bigcup_{j \in J_k} Q_j. \end{aligned}$$

Notice that

$$(\widehat{E}_k \setminus E_k) \cap \Omega_k \subseteq H_k. \tag{6.10}$$

To check this, let $x \in (\widehat{E}_k \setminus E_k) \cap \Omega_k$. Then, there exists $j \in I_k$ such that $x \in Q_j$. Notice that $x \in Q_j \setminus E_k$ and $x \in Q_j \cap \Omega_k$, which means that $j \in J_k$, and so $x \in H_k$, thus proving (6.10).

Now we prove that

$$\text{the cardinality of } J_k \text{ is bounded by } \frac{C \operatorname{Per}_{r_k}(E_k, \Omega_k)}{r_k^{n-1}}, \tag{6.11}$$

Indeed, if $j \in J_k$, then also $j \in I_k$, therefore $Q_j \cap E_k \neq \varnothing$ and also $Q_j \setminus E_k \neq \varnothing$. Hence there exists $x_{j,k} \in Q_j \cap (\partial E_k)$. Notice that

$$B_{r_k}(x_{j,k}) \supseteq Q_j \oplus B_{\frac{r_k}{100}} \tag{6.12}$$

Also, $Q_j \cap \Omega_k \neq \varnothing$. Consequently, making use of (1.12) and (6.12), we see that

$$\begin{aligned} \mathcal{L}^n\Big(\big((\partial E_k) \oplus B_{r_k}\big) \cap \Omega_k \cap (Q_j \oplus B_{\frac{r_k}{100}})\Big) &\geqslant \mathcal{L}^n\big(B_{r_k}(x_{j,k}) \cap \Omega_k \cap (Q_j \oplus B_{\frac{r_k}{100}})\big) \\ &\geqslant \mathcal{L}^n\big((Q_j \oplus B_{\frac{r_k}{100}}) \cap \Omega_k\big) \geqslant \frac{r_k^n}{C}, \end{aligned}$$

up to renaming $C > 0$. Since this is valid for any $j \in J_k$ and there is a finite number of overlaps between different $Q_j \oplus B_{\frac{r_k}{100}}$, we conclude that

$$\mathcal{L}^n\Big(\big((\partial E_k) \oplus B_{r_k}\big) \cap \Omega_k\Big) \geqslant \frac{r_k^n \,\#J_k}{C},$$

up to renaming $C > 0$ that implies (6.11).

Now, in view of (6.10) and (6.11), we find that

$$\begin{aligned}\mathcal{L}^n\big((\widehat{E}_k \setminus E_k) \cap \Omega_k\big) &\leqslant \mathcal{L}^n(H_k) \leqslant \sum_{j\in J_k} \mathcal{L}^n(Q_j)\\ &\leqslant C\, r_k^n \,\#J_k \leqslant C\, r_k \operatorname{Per}_{r_k}(E_k, \Omega_k).\end{aligned}$$

This implies (1.17).

Finally, (1.13), (1.14) and (1.17) give that

$$\chi_{\widehat{E}_k} - \chi_{E_k} \to 0 \quad \text{in } L^1(\Omega),$$

and this, combined with (6.9), implies (1.18), as desired. $\square$

With this, we can now complete the proof of Theorem 1.13.

Proof of Theorem 1.13. First of all we consider the case in which $R < r \leqslant \lambda R$ for $\lambda > 1$. By assumption we know that $\mathcal{L}^n(E \cap B_R) \leqslant \frac{1}{2}\mathcal{L}^n(B_R) = \frac{1}{2}\omega_n R^n$. So either $\mathcal{L}^n(E \cap B_R) = 0$ and there is nothing to prove, or $\mathcal{L}^n(E \cap B_R) \neq 0$. In this case $\partial E \cap B_R \neq \emptyset$, and so $\operatorname{Per}_r(E, B_R) \geqslant \frac{1}{2r}\frac{\mathcal{L}^n(B_R)}{3} = \frac{\omega_n R^n}{6r}$. Summarizing we get

$$\operatorname{Per}_r(E, B_R) \geqslant \frac{\omega_n R^n}{6r} \geqslant \frac{\omega_n R^{n-1}}{6\lambda} \geqslant \frac{\omega_n}{6\lambda}\left(\frac{2\mathcal{L}^n(E \cap B_R)}{\omega_n}\right)^{\frac{n-1}{n}} = \frac{C}{\lambda}\left(\mathcal{L}^n(E \cap B_R)\right)^{\frac{n-1}{n}}.$$

We consider now the case $\lambda = 1$, so $r \leqslant R$, and we argue by contradiction. If (1.11) were not true, recalling also (1.9) and (1.10), we would infer that there exist sequences

$$R_k \geqslant r_k > 0 \tag{6.13}$$

and $E_k \subseteq \mathbb{R}^n$ such that

$$\begin{gathered}\frac{\mathcal{L}^n(E_k \cap B_{R_k})}{\mathcal{L}^n(B_{R_k})} \leqslant \frac{1}{2}\\ \text{and } \Big(\mathcal{L}^n(E_k \cap B_{R_k})\Big)^{\frac{n-1}{n}} > k \operatorname{Per}_{r_k}(E_k, B_{R_k}).\end{gathered} \tag{6.14}$$

We define $\lambda_k := \big(\mathcal{L}^n(E_k \cap B_{R_k})\big)^{-\frac{1}{n}}$, $\widetilde{E}_k := \lambda_k E_k$, $\widetilde{r}_k := \lambda_k r_k$ and $\widetilde{R}_k = \lambda_k R_k$. With this scaling, we have that

$$\mathcal{L}^n(\widetilde{E}_k \cap B_{\widetilde{R}_k}) = \mathcal{L}^n\big(\lambda_k(E_k \cap B_{R_k})\big) = \lambda_k^n \mathcal{L}^n(E_k \cap B_{R_k}) = 1. \tag{6.15}$$

Moreover,

$$\operatorname{Per}_{\widetilde{r}_k}(\widetilde{E}_k, B_{\widetilde{R}_k}) = \operatorname{Per}_{\lambda r_k}(\lambda_k E_k, \lambda_k B_{R_k}) = \lambda_k^{n-1}\operatorname{Per}_{r_k}(E_k, B_{R_k}).$$

Therefore (6.14) becomes

$$\mathcal{L}^n(B_{\widetilde{R}_k}) \geqslant 2 \qquad \text{and} \qquad \operatorname{Per}_{\widetilde{r}_k}(\widetilde{E}_k, B_{\widetilde{R}_k}) = \lambda_k^{n-1}\operatorname{Per}_{r_k}(E_k, B_{R_k}) < \frac{1}{k}\lambda_k^{n-1}\Big(\mathcal{L}^n(E_k \cap B_{R_k})\Big)^{\frac{n-1}{n}} = \frac{1}{k}. \tag{6.16}$$

Thanks to the first inequality in (6.16), setting

$$\widetilde{R}_o := \liminf_{k\to+\infty} \widetilde{R}_k,$$

we have that $R_o \in (0, +\infty]$ and

$$\mathcal{L}^n(B_{\widetilde{R}_o}) \geqslant 2. \tag{6.17}$$

Here, the obvious notation $B_{\widetilde{R}_o} = \mathbb{R}^n$ if $R_o = +\infty$ has been used.

Now we claim that

$$\widetilde{r}_k \to 0. \tag{6.18}$$

For this, we observe that $\widetilde{R}_k \geqslant \widetilde{r}_k$, thanks to (6.13).

In addition,

$$\mathcal{L}^n(\widetilde{E}_k \cap B_{\widetilde{R}_k}) = 1 < 2 \leqslant \mathcal{L}^n(B_{\widetilde{R}_k}),$$

thanks to (6.15) and (6.16). Therefore both $\widetilde{E}_k \cap B_{\widetilde{R}_k}$ and $B_{\widetilde{R}_k} \setminus \widetilde{E}_k$ are nonvoid, and so there exists $p_k \in (\partial \widetilde{E}_k) \cap B_{\widetilde{R}_k}$. Accordingly,

$$\mathrm{Per}_{\widetilde{r}_k}(\widetilde{E}_k, B_{\widetilde{R}_k}) \geqslant \frac{1}{2\widetilde{r}_k} \mathcal{L}^n(B_{\widetilde{r}_k}(p_k) \cap B_{\widetilde{R}_k}) \geqslant \frac{c \, \min\{\widetilde{r}_k^n, \ \widetilde{R}_k^n\}}{\widetilde{r}_k} = c \, \widetilde{r}_k^{n-1},$$

for some $c > 0$. From this and (6.16) we deduce that

$$c \, \widetilde{r}_k^{n-1} \leqslant \mathrm{Per}_{\widetilde{r}_k}(\widetilde{E}_k, B_{\widetilde{R}_k}) < \frac{1}{k},$$

which proves (6.18), as desired.

In light of (6.18), we can now exploit Lemma 1.14 (with $\Omega_k := B_{\widetilde{R}_k}$ and $\Omega := \cap_k B_{\widetilde{R}_k}$, which is nontrivial thanks to (6.17)). In particular, from (1.15) and (1.16), we know that there exists $\widehat{E}_k \subseteq \mathbb{R}^n$ such that

$$\widehat{E}_k \supseteq \widetilde{E}_k \tag{6.19}$$

and

$$\mathrm{Per}(\widehat{E}_k, \, B_{\widetilde{R}_k}) \leqslant C \, \mathrm{Per}_{\widetilde{r}_k}(\widetilde{E}_k, B_{\widetilde{R}_k}).$$

Therefore, recalling (6.16),

$$\mathrm{Per}(\widehat{E}_k, \, B_{\widetilde{R}_k}) \leqslant \frac{C}{k}. \tag{6.20}$$

Moreover, using (1.17),

$$\int_{B_{\widetilde{R}_k}} |\chi_{\widetilde{E}_k} - \chi_{\widehat{E}_k}| \, dx \leqslant C \, \widetilde{r}_k \, \mathrm{Per}_{\widetilde{r}_k}(\widetilde{E}_k, B_{\widetilde{R}_k}) \leqslant \frac{C \widetilde{r}_k}{k}. \tag{6.21}$$

Using (6.15) and (6.21), we see that

$$\begin{aligned} \mathcal{L}^n(\widehat{E}_k \cap B_{\widetilde{R}_k}) &\leqslant \mathcal{L}^n(\widetilde{E}_k \cap B_{\widetilde{R}_k}) + \mathcal{L}^n\big((\widehat{E}_k \setminus \widetilde{E}_k) \cap B_{\widetilde{R}_k}\big) \\ &\leqslant 1 + \frac{C \widetilde{r}_k}{k}. \end{aligned}$$

This and (6.17) imply that

$$\lim_{k \to +\infty} \frac{\mathcal{L}^n(\widehat{E}_k \cap B_{\widetilde{R}_k})}{\mathcal{L}^n(B_{\widetilde{R}_k})} \leqslant \frac{1}{\mathcal{L}^n(B_{\widetilde{R}_o})} \leqslant \frac{1}{2}. \tag{6.22}$$

So, we can assume that, for large k,

$$\frac{\mathcal{L}^n(\widehat{E}_k \cap B_{\widetilde{R}_k})}{\mathcal{L}^n(B_{\widetilde{R}_k})} \leqslant \frac{3}{4},$$

hence we can apply the classical relative isoperimetric inequality and find that

$$\Big(\mathcal{L}^n(\widehat{E}_k \cap B_{\widetilde{R}_k})\Big)^{\frac{n-1}{n}} \leqslant C \, \mathrm{Per}(\widehat{E}_k, B_{\widetilde{R}_k}).$$

Consequently, recalling (6.19) and (6.20),

$$\Big(\mathcal{L}^n(\widetilde{E}_k \cap B_{\widetilde{R}_k})\Big)^{\frac{n-1}{n}} \leqslant \frac{C}{k}.$$

From this, sending $k \to +\infty$ and recalling (6.15), we obtain a contradiction that proves Theorem 1.13. $\square$

Now we check that (1.11) cannot hold with a constant independent of λ.

Proof of Remark 1.15. As an example, let $n = 2$, $R = 100$ and $E := B_1$. Notice that (1.10) is satisfied, but (1.11) cannot be true for arbitrarily large r for some constant C independent of the rate $\frac{r}{R}$. Indeed, we have that $\partial E \subseteq B_{100}$, hence

$$\big((\partial E) \oplus B_r\big) \cap B_R \subseteq B_{100+r} \cap B_R = B_{100}.$$

As a consequence, if r is sufficiently large,

$$\mathrm{Per}_r(E, B_R) \leqslant \frac{1}{2r}\mathcal{L}^n(B_{100}) < \frac{1}{C}\Big(\mathcal{L}^n(B_1)\Big)^{\frac{n-1}{n}} = \frac{1}{C}\Big(\mathcal{L}^n(E \cap B_R)\Big)^{\frac{n-1}{n}},$$

thus violating (1.11). □

Now, we can provide the easy proof of the Poincaré-Wirtinger inequality in Theorem 1.16:

Proof of Theorem 1.16. Up to a vertical translation, we may and do suppose that

$$u \text{ has zero average in } B_R. \tag{6.23}$$

Moreover,

$$\mathcal{L}^n(\{u > 0\} \cap B_R) + \mathcal{L}^n(\{u < 0\} \cap B_R) \leqslant \mathcal{L}^n(B_R).$$

Thus, possibly exchanging u with $-u$, we may and do suppose that

$$\mathcal{L}^n(\{u > 0\} \cap B_R) \leqslant \frac{\mathcal{L}^n(B_R)}{2}. \tag{6.24}$$

Let also $u^+ := \max\{u, 0\}$. Then, using (6.23), we see that

$$\begin{aligned}\int_{B_R} |u| &= \int_{B_R\cap\{u>0\}} u - \int_{B_R\cap\{u<0\}} u\\ &= 2\int_{B_R\cap\{u>0\}} u - \int_{B_R} u = 2\int_{B_R\cap\{u>0\}} u^+ - 0.\end{aligned}$$

Hence, integrating with respect to the distribution function (see e.g. Theorem 5.51 in [34]), we have that

$$\int_{B_R} |u| = 2\int_{B_R\cap\{u>0\}} u^+ = 2\int_0^{+\infty} \mathcal{L}^n(\{u^+ > s\} \cap B_R)\, ds. \tag{6.25}$$

In addition, from (6.24), for any $s \geqslant 0$ we have that

$$\mathcal{L}^n(\{u^+ > s\} \cap B_R) = \mathcal{L}^n(\{u > s\} \cap B_R) \leqslant \mathcal{L}^n(\{u > 0\} \cap B_R) \leqslant \frac{\mathcal{L}^n(B_R)}{2}.$$

Consequently, we can exploit our relative isoperimetric inequality in Theorem 1.13 with $E := \{u^+ > s\}$ and conclude that, for any $s \geqslant 0$,

$$\Big(\mathcal{L}^n(\{u^+ > s\} \cap B_R)\Big)^{\frac{n-1}{n}} \leqslant C\lambda\,\mathrm{Per}_r(\{u^+ > s\}, B_R),$$

for some $C > 0$. Multiplying this estimate by $\Big(\mathcal{L}^n(\{u^+ > s\} \cap B_R)\Big)^{\frac{1}{n}}$, we obtain that, for any $s \geqslant 0$,

$$\begin{aligned}\mathcal{L}^n(\{u^+ > s\} \cap B_R) &\leqslant C\lambda\,\mathrm{Per}_r(\{u^+ > s\}, B_R)\Big(\mathcal{L}^n(\{u^+ > s\} \cap B_R)\Big)^{\frac{1}{n}}\\ &\leqslant C\lambda\, R\,\mathrm{Per}_r(\{u^+ > s\}, B_R),\end{aligned}$$

up to renaming $C > 0$. Accordingly,

$$\begin{aligned}\int_0^{+\infty} \mathcal{L}^n(\{u^+ > s\} \cap B_R)\, ds &\leqslant C\lambda R\int_0^{+\infty} \mathrm{Per}_r(\{u^+ > s\}, B_R)\, ds\\ &\leqslant C\lambda R\int_{\mathbb{R}} \mathrm{Per}_r(\{u^+ > s\}, B_R)\, ds = \frac{C\lambda R}{r}\int_{B_R} \operatorname*{osc}_{B_r(x)} u,\end{aligned}$$

thanks to the coarea formula in (1.4). Hence, recalling (6.25), we conclude that

$$\int_{B_R} |u| \leqslant \frac{2C\lambda R}{r}\int_{B_R} \operatorname*{osc}_{B_r(x)} u,$$

which is the desired result, up to renaming constants. □

Now we check that Theorem 1.16 does not hold in general when $r > R$ with a constant independent of the rate $\frac{r}{R}$:

Proof of Remark 1.17. Let $R = 1$ and

$$u(x) := \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

Notice that u has zero average and its oscillation is always bounded by 2. Therefore, if r is large enough,

$$\frac{CR}{r}\int_{B_R} \operatorname*{osc}_{B_r(x)} u\,dx \leqslant \frac{C}{r} 2\mathcal{L}^n(B_1) < \mathcal{L}^n(B_1) = \int_{B_R} \big|u - \langle u\rangle_R\big|,$$

which violates (1.19). □

7. Regularity issues and density estimates – Proofs of Theorems 1.18 and 1.19

In this section we prove the nonlocal density estimates in Theorem 1.18:

Proof of Theorem 1.18. We set $f(R) := \mathcal{L}^n(E \cap B_R)$. We notice that if $R - r \geqslant r$ and $f(R-r) \leqslant \frac{\mathcal{L}^n(B_R)}{2}$, then we can apply the relative isoperimetric inequality in Theorem 1.13 and obtain that

$$\Big(f(R-r)\Big)^{\frac{n-1}{n}} \leqslant C\,\mathrm{Per}_r(E, B_{R-r}). \tag{7.1}$$

Furthermore,

$$\partial(E \setminus B_R) \subseteq \big((\partial E) \setminus B_R\big) \cup \big((\partial B_R) \cap E\big).$$

Observe that

$$(E \oplus B_r) \cap (B_{R+r} \setminus B_{R-r}) = \big(E \cap (B_{R+r} \setminus B_{R-r})\big) \cup \big((\partial E \oplus B_r) \cap (B_{R+r} \setminus B_{R-r})\big).$$

Consequently

$$\begin{aligned} \big(\partial(E \setminus B_R)\big) \oplus B_r \ &\subseteq\ \Big(((\partial E) \oplus B_r) \setminus B_{R-r}\Big) \cup \big((E \oplus B_r) \cap (B_{R+r} \setminus B_{R-r})\big) \\ &\subseteq\ \Big(((\partial E) \oplus B_r) \cap (B_{R+r} \setminus B_{R-r})\Big) \cup \big(E \cap (B_{R+r} \setminus B_{R-r})\big) \end{aligned}$$

and therefore

$$\begin{aligned} &\mathcal{L}^n\Big(\big((\partial(E \setminus B_R)) \oplus B_r\big) \cap B_{R+r}\Big) \\ &\leqslant \mathcal{L}^n\Big(\big((\partial E) \oplus B_r\big) \cap (B_{R+r} \setminus B_{R-r})\Big) + \mathcal{L}^n\big(E \cap (B_{R+r} \setminus B_{R-r})\big) \\ &= 2r\,\mathrm{Per}_r(E,\, B_{R+r} \setminus B_{R-r}) + \big(f(R+r) - f(R-r)\big). \end{aligned} \tag{7.2}$$

Assume also that $B_{R+r} \subseteq \Omega$. Then, the minimality of E in B_{R+r} and (7.2) give that

$$\begin{aligned} 0 \ &\leqslant\ 2r\Big[\mathrm{Per}_r(E \setminus B_R,\, B_{R+r}) - \mathrm{Per}_r(E,\, B_{R+r})\Big] \\ &=\ \mathcal{L}^n\Big(\big((\partial(E \setminus B_R)) \oplus B_r\big) \cap B_{R+r}\Big) - 2r\,\mathrm{Per}_r(E,\, B_{R+r}) \\ &=\ \big(f(R+r) - f(R-r)\big) + 2r\Big[\mathrm{Per}_r(E,\, B_{R+r} \setminus B_{R-r}) - \mathrm{Per}_r(E,\, B_{R+r})\Big] \\ &=\ \big(f(R+r) - f(R-r)\big) - 2r\,\mathrm{Per}_r(E,\, B_{R-r}). \end{aligned}$$

This and (7.1) give that, if $B_{R+2r} \subseteq \Omega$, $R \geqslant 2r$ and $f(R-r) \leqslant \frac{\mathcal{L}^n(B_R)}{2}$, then

$$0 \leqslant \big(f(R+r) - f(R-r)\big) - \frac{2r\Big(f(R-r)\Big)^{\frac{n-1}{n}}}{C}.$$

That is, if $R \geqslant r$ and $f(R) \leqslant \frac{\mathcal{L}^n(B_R)}{2}$,

$$f(R+2r) \geqslant f(R) + \frac{2r}{C}\Big(f(R)\Big)^{\frac{n-1}{n}}. \tag{7.3}$$

Now we define, for $k \in \mathbb{N}$, the sequence $x_k := f(R_o + 2kr)$, and we claim that, if $B_{R_o+2kr} \subseteq \Omega$, then

$$x_k \geqslant (\omega_o^{\frac{1}{n}} + 2c_\star kr)^n \tag{7.4}$$

where

$$c_\star := \frac{1}{C\left(n + \frac{2(n-1)r}{C\omega_o^{\frac{1}{n}}}\right)}, \tag{7.5}$$

being ω_o as in (1.20) and C as in (7.3). The proof of (7.4) is by induction. First of all, from (1.20) we have that

$$x_0 = f(R_o) = \omega_o,$$

and so (7.4) holds true when $k = 0$. Now we suppose that it holds true for $k - 1$, namely

$$x_{k-1} \geqslant (\omega_o^{\frac{1}{n}} + 2c_\star(k-1)r)^n.$$

Thus, from (7.3),

$$\begin{aligned}
x_k &= f(R_o + 2(k-1)r + 2r) \\
&\geqslant f(R_o + 2(k-1)r) + \frac{2r}{C}\Big(f(R_o + 2(k-1)r)\Big)^{\frac{n-1}{n}} \\
&= x_{k-1} + \frac{2r}{C}\Big(x_{k-1}\Big)^{\frac{n-1}{n}} \\
&= \Big(x_{k-1}\Big)^{\frac{n-1}{n}}\left(\Big(x_{k-1}\Big)^{\frac{1}{n}} + \frac{2r}{C}\right) \\
&\geqslant (\omega_o^{\frac{1}{n}} + 2c_\star(k-1)r)^{n-1}\left(\omega_o^{\frac{1}{n}} + 2c_\star(k-1)r + \frac{2r}{C}\right) \\
&= (\omega_o^{\frac{1}{n}} + 2c_\star kr)^n \frac{(\omega_o^{\frac{1}{n}} + 2c_\star(k-1)r)^{n-1}}{(\omega_o^{\frac{1}{n}} + 2c_\star kr)^{n-1}} \, \frac{\omega_o^{\frac{1}{n}} + 2c_\star(k-1)r + \frac{2r}{C}}{\omega_o^{\frac{1}{n}} + 2c_\star kr} \\
&= (\omega_o^{\frac{1}{n}} + 2c_\star kr)^n\left(1 - \frac{2c_\star r}{\omega_o^{\frac{1}{n}} + 2c_\star kr}\right)^{n-1}\left(1 + \frac{\frac{2r}{C} - 2c_\star r}{\omega_o^{\frac{1}{n}} + 2c_\star kr}\right).
\end{aligned} \tag{7.6}$$

Now, by a first order Taylor expansion, we see that, for any $\tau \in [0,1]$,

$$(1-\tau)^{n-1} \geqslant 1 - (n-1)\tau$$

and therefore

$$\left(1 - \frac{2c_\star r}{\omega_o^{\frac{1}{n}} + 2c_\star kr}\right)^{n-1} \geqslant 1 - \frac{2(n-1)\,c_\star r}{\omega_o^{\frac{1}{n}} + 2c_\star kr}.$$

As a consequence,

$$\begin{aligned}
&\left(1 - \frac{2c_\star r}{\omega_o^{\frac{1}{n}} + 2c_\star kr}\right)^{n-1}\left(1 + \frac{\frac{2r}{C} - 2c_\star r}{\omega_o^{\frac{1}{n}} + 2c_\star kr}\right) \\
\geqslant\ &\left(1 - \frac{2(n-1)\,c_\star r}{\omega_o^{\frac{1}{n}} + 2c_\star kr}\right)\left(1 + \frac{\frac{2r}{C} - 2c_\star r}{\omega_o^{\frac{1}{n}} + 2c_\star kr}\right) \\
=\ &1 + \frac{\frac{2r}{C} - 2c_\star r - 2(n-1)\,c_\star r}{\omega_o^{\frac{1}{n}} + 2c_\star kr} - \frac{2(n-1)\,c_\star r}{\omega_o^{\frac{1}{n}} + 2c_\star kr} \cdot \frac{\frac{2r}{C} - 2c_\star r - 2(n-1)\,c_\star r}{\omega_o^{\frac{1}{n}} + 2c_\star kr} \\
\geqslant\ &1 + \frac{\frac{2r}{C} - 2n\,c_\star r}{\omega_o^{\frac{1}{n}} + 2c_\star kr} - \frac{2(n-1)\,c_\star r}{\omega_o^{\frac{1}{n}} + 2c_\star kr} \cdot \frac{\frac{2r}{C}}{\omega_o^{\frac{1}{n}}} \\
=\ &1 + \frac{\frac{2r}{C} - 2n\,c_\star r - \frac{4(n-1)\,c_\star r^2}{C\omega_o^{\frac{1}{n}}}}{\omega_o^{\frac{1}{n}} + 2c_\star kr} = 1,
\end{aligned}$$

thanks to (7.5). This and (7.6) give that $x_k \geqslant (\omega_o^{\frac{1}{n}} + 2c_\star kr)^n$, which completes the inductive proof of (7.4).

From (7.4) and (7.5), we obtain (1.21), (1.22) and (1.23).

Now, we prove (1.25). To this end, we take k as in (1.24) and we observe that

$$x_0 \leqslant \ldots \leqslant x_{k-1} \leqslant \overline{C} r^n.$$

Hence, for any $j \in \{1, \ldots, k\}$,

$$r\,(x_{j-1})^{-\frac{1}{n}} \geqslant \overline{C}^{-\frac{1}{n}},$$

thus we deduce from (7.3) that

$$\begin{aligned} x_j = f(R_o + 2(j-1)r + 2r) &\geqslant f(R_o + 2(j-1)r) + \frac{2r}{C}\Big(f(R_o + 2(j-1)r)\Big)^{\frac{n-1}{n}} \\ &= x_{j-1} + \frac{2r}{C}\,(x_{j-1})^{\frac{n-1}{n}} \geqslant x_{j-1}\left(1 + \frac{1}{2C\,\overline{C}^{\frac{1}{n}}}\right). \end{aligned}$$

Iterating, we thus obtain

$$x_k \geqslant x_0 \left(1 + \frac{1}{2C\,\overline{C}^{\frac{1}{n}}}\right)^k,$$

that establishes (1.25). This completes the proof of Theorem 1.18. □

Now we address the compactness and lack of regularity issues exemplified in Theorem 1.19:

Proof of Theorem 1.19. We start with some preliminary observations. First of all, if we denote by $\{e_1, \ldots, e_n\}$ the Euclidean basis of $\mathbb{R}^n$, it is clear that

$$\mathcal{L}^n\big(B_{1/8}(e_1/2) \cap (B_1 \setminus B_{1/2})\big) > 0 \text{ and } \mathcal{L}^n\big(B_{1/8}(e_1) \cap (B_1 \setminus B_{1/2})\big) > 0. \tag{7.7}$$

Moreover, there exists a constant $c_\star > 0$, only depending on n, such that, for any $x \in \overline{B_{3/2}}$ it holds that

$$\mathcal{L}^n\big(B_1(x) \cap (B_1 \setminus B_{1/2})\big) \geqslant c_\star. \tag{7.8}$$

To prove (7.8), we argue for a contradiction: if not, there exists a sequence of points $x_k \in \overline{B_{3/2}}$ such that

$$\mathcal{L}^n\big(B_1(x_k) \cap (B_1 \setminus B_{1/2})\big) \leqslant \frac{1}{k}. \tag{7.9}$$

Up to a subsequence, we may assume that $x_k \to \bar{x}$ as $k \to +\infty$, for some $\bar{x} \in \overline{B_{3/2}}$, and, passing to the limit (7.9), we obtain that

$$\mathcal{L}^n\big(B_1(\bar{x}) \cap (B_1 \setminus B_{1/2})\big) = 0. \tag{7.10}$$

Up to a rotation, we can assume that $\bar{x}$ is parallel to e_1, namely $\bar{x} = \lambda e_1$, for some $\lambda \in \big[0, \frac{3}{2}\big]$. We define

$$\lambda_\star := \begin{cases} 1/2 & \text{if } \lambda \in \big[0, \frac{3}{4}\big] \\ 1 & \text{if } \lambda \in \big(\frac{3}{4}, \frac{3}{2}\big]. \end{cases}$$

Notice that, by (7.7), we have that

$$\mathcal{L}^n\big(B_{1/8}(\lambda_\star\, e_1) \cap (B_1 \setminus B_{1/2})\big) > 0. \tag{7.11}$$

In addition, we note that $|\bar{x} - \lambda_\star e_1| = |\lambda - \lambda_\star| \leqslant 1/2$. Consequently, if $p \in B_{1/8}(\lambda_\star e_1)$, we have that $|\bar{x} - p| \leqslant |\bar{x} - \lambda_\star e_1| + |\lambda_\star e_1 - p| \leqslant \frac{1}{2} + \frac{1}{8} < 1$, which gives that $B_{1/8}(\lambda_\star e_1) \subseteq B_1(\bar{x})$.

Therefore, we conclude that $B_{1/8}(\lambda_\star\, e_1) \cap (B_1 \setminus B_{1/2}) \subseteq B_1(\bar{x}) \cap (B_1 \setminus B_{1/2})$. From this and (7.11), we obtain that $\mathcal{L}^n\big(B_1(\bar{x}) \cap (B_1 \setminus B_{1/2})\big) > 0$, and this is in contradiction with (7.10). The proof of (7.8) is thus completed.

We also notice that, by scaling (7.8), it holds that, for any $x \in \overline{B_{3r/2}}$,

$$\mathcal{L}^n\big(B_r(x) \cap (B_r \setminus B_{r/2})\big) \geqslant c_\star\, r^n. \tag{7.12}$$

Now we claim that there exists $\delta_\star > 0$, only depending on n, such that

$$\begin{aligned} &\text{if } H \subseteq B_r \text{ and } \mathcal{L}^n\big(H \cap (B_r \setminus B_{r/2})\big) \geqslant (1 - \delta_\star)\,\mathcal{L}^n(B_1)\left(1 - \frac{1}{2^n}\right) r^n, \\ &\text{then } \{0\} \cup \big((\partial H) \oplus B_r\big) \supseteq \big(\partial(H \cup B_{r/2})\big) \oplus B_r. \end{aligned} \tag{7.13}$$

To prove this, let

$$x \in \big(\partial(H \cup B_{r/2})\big) \oplus B_r. \tag{7.14}$$

Our aim is to show that

$$\text{either } x = 0 \text{ or } B_r(x) \cap (\partial H) \neq \varnothing, \tag{7.15}$$

since this would imply that $x \in \{0\} \cup \big((\partial H) \oplus B_r\big)$, thus establishing (7.13).

Also, since (7.15) is obvious when $x = 0$, we can assume that

$$x \neq 0. \tag{7.16}$$

Notice that, from (7.14), we know that there exists $y \in B_r(x) \cap \big(\partial(H \cup B_{r/2})\big)$. Consequently, we can find $\xi_k \in (H \cup B_{r/2})$ and $\eta_k \in (\mathbb{R}^n \setminus H) \cap (\mathbb{R}^n \setminus B_{r/2})$ with the property that $\xi_k \to y$ and $\eta_k \to y$ as $k \to +\infty$.

We observe that $\eta_k \in \mathbb{R}^n \setminus H$: hence, if $\xi_k \in H$, it follows that $y \in \partial H$ and so (7.15) holds true. Therefore, we can restrict ourselves to the case in which $\xi_k \in (B_{r/2} \setminus H)$. In particular

$$\frac{r}{2} \leqslant \lim_{k\to+\infty} |\eta_k| = |y| = \lim_{k\to+\infty} |\xi_k| \leqslant \frac{r}{2},$$

and so $y \in \partial B_{r/2}$.

Consequently, we see that $|x| \leqslant |y| + |x-y| \leqslant \frac{r}{2} + r = \frac{3r}{2}$, and so we are in the position of exploiting (7.12). Accordingly, we have that

$$\mathcal{L}^n\big(B_r(x) \cap (B_r \setminus B_{r/2})\big) \geqslant c_\star\, r^n. \tag{7.17}$$

In addition, from the hypothesis of (7.13), we find that

$$\begin{aligned}
\mathcal{L}^n(B_1)\left(1-\frac{1}{2^n}\right) r^n &= \mathcal{L}^n(B_r \setminus B_{r/2}) \\
&= \mathcal{L}^n\big((B_r \setminus B_{r/2}) \cap H\big) + \mathcal{L}^n\big((B_r \setminus B_{r/2}) \setminus H\big) \\
&\geqslant (1-\delta_\star)\,\mathcal{L}^n(B_1)\left(1-\frac{1}{2^n}\right) r^n + \mathcal{L}^n\big((B_r \setminus B_{r/2}) \setminus H\big).
\end{aligned}$$

This says that

$$\mathcal{L}^n\big((B_r \setminus B_{r/2}) \setminus H\big) \leqslant \delta_\star\,\mathcal{L}^n(B_1)\left(1-\frac{1}{2^n}\right) r^n \leqslant \frac{c_\star}{2}\, r^n,$$

as long as we choose $\delta_\star$ appropriately small. Thus, recalling (7.17), we find that

$$\begin{aligned}
c_\star\, r^n &\leqslant \mathcal{L}^n\big(B_r(x) \cap (B_r \setminus B_{r/2})\big) \\
&\leqslant \mathcal{L}^n\big(B_r(x) \cap (B_r \setminus B_{r/2}) \cap H\big) + \mathcal{L}^n\Big(\big(B_r(x) \cap (B_r \setminus B_{r/2})\big) \setminus H\Big) \\
&\leqslant \mathcal{L}^n\big(B_r(x) \cap (B_r \setminus B_{r/2}) \cap H\big) + \mathcal{L}^n\big((B_r \setminus B_{r/2}) \setminus H\big) \\
&\leqslant \mathcal{L}^n\big(B_r(x) \cap (B_r \setminus B_{r/2}) \cap H\big) + \frac{c_\star}{2}\, r^n,
\end{aligned}$$

which gives that $\mathcal{L}^n\big(B_r(x) \cap (B_r \setminus B_{r/2}) \cap H\big) \geqslant \frac{c_\star}{2}\, r^n$. In particular, we have that

$$B_r(x) \cap H \neq \varnothing. \tag{7.18}$$

So, we claim that

$$B_r(x) \cap (\partial H) \neq \varnothing. \tag{7.19}$$

To prove (7.19), we suppose the contrary, namely that $B_r(x) \cap (\partial H) = \varnothing$. Then, from (7.18) we have that $B_r(x) \subseteq H$. In particular, recalling (7.16), we have that, if $p_j := x + \left(r - \frac{1}{j}\right) \frac{x}{|x|}$, then

$$|p_j - x| = \left| r - \frac{1}{j} \right| = r - \frac{1}{j} < r,$$

for large j. Accordingly, we obtain that $p_j \in B_r(x) \subseteq H \subseteq B_r$, where one assumption in (7.13) has been used for the latter inclusion.

So, we have found that

$$r \geqslant \lim_{j\to+\infty} |p_j| = \lim_{j\to+\infty} \left| x + \left(r - \frac{1}{j}\right) \frac{x}{|x|} \right| = \lim_{j\to+\infty} \left| |x| + \left(r - \frac{1}{j}\right) \right| = |x| + r.$$

This is a contradiction with (7.16), and so we have proved (7.19).

Then, since (7.19) implies (7.15), we have thus completed the proof of (7.13).

Now, we deal with the core of the proof of Theorem 1.19. For this, we observe that

$$\begin{aligned} \mathcal{F}_K(B_r \setminus B_{r/2}) &:= \mathrm{Per}_r(B_r \setminus B_{r/2}) - K\,\mathcal{L}^n(B_r \setminus B_{r/2}) \\ &= \frac{\mathcal{L}^n\big((\partial(B_r \setminus B_{r/2})) \oplus B_r\big)}{2r} - K\,\mathcal{L}^n(B_r \setminus B_{r/2}) \\ &= \frac{\mathcal{L}^n(B_{2r})}{2r} - K\,\mathcal{L}^n(B_r \setminus B_{r/2}) \\ &= 2^{n-1}\,\mathcal{L}^n(B_1)\, r^{n-1} - \mathcal{L}^n(B_1)\left(1 - \frac{1}{2^n}\right) K r^n. \end{aligned} \tag{7.20}$$

Now we claim that

$$\mathcal{F}_K(B_r \setminus B_{r/2}) \leqslant \mathcal{F}_K(E) \tag{7.21}$$

for any bounded set $E \subseteq \mathbb{R}^n$. To this end, we distinguish two cases, namely

$$\text{either} \qquad \mathcal{L}^n\big(E \cap (B_r \setminus B_{r/2})\big) \leqslant (1-\delta_\star)\,\mathcal{L}^n(B_1)\left(1 - \frac{1}{2^n}\right) r^n \tag{7.22}$$

$$\text{or} \qquad \mathcal{L}^n\big(E \cap (B_r \setminus B_{r/2})\big) > (1-\delta_\star)\,\mathcal{L}^n(B_1)\left(1 - \frac{1}{2^n}\right) r^n, \tag{7.23}$$

being $\delta_\star$ the constant in (7.13).

When (7.22) holds true, we have that

$$-\mathcal{F}_K(E) \leqslant K\,\mathcal{L}^n\big(E \cap (B_r \setminus B_{r/2})\big) \leqslant (1-\delta_\star)\,\mathcal{L}^n(B_1)\left(1 - \frac{1}{2^n}\right) K\, r^n.$$

Accordingly, from (7.20), we have that

$$\begin{aligned} & \mathcal{F}_K(B_r \setminus B_{r/2}) - \mathcal{F}_K(E) \\ \leqslant\; & 2^{n-1}\,\mathcal{L}^n(B_1)\, r^{n-1} - \mathcal{L}^n(B_1)\left(1 - \frac{1}{2^n}\right) K r^n + (1-\delta_\star)\,\mathcal{L}^n(B_1)\left(1 - \frac{1}{2^n}\right) K\, r^n \\ =\; & 2^{n-1}\,\mathcal{L}^n(B_1)\, r^{n-1} - \delta_\star\,\mathcal{L}^n(B_1)\left(1 - \frac{1}{2^n}\right) K\, r^n \\ \leqslant\; & 0, \end{aligned}$$

provided that K is large enough, as prescribed by (1.26). This proves (7.21) when (7.22) holds true, hence we can now focus on the case in which (7.23) is satisfied.

Thanks to (7.23), we can exploit (7.13) with

$$H := E \cap B_r. \tag{7.24}$$

In this way, setting

$$G := H \cup B_{r/2}, \tag{7.25}$$

we have that $\{0\} \cup \big((\partial H) \oplus B_r\big) \supseteq (\partial G) \oplus B_r$. In particular, we have that

$$\mathrm{Per}_r(H) \geqslant \mathrm{Per}_r(G). \tag{7.26}$$

We also point out that $\mathcal{L}^n\big(G \cap (B_r \setminus B_{r/2})\big) = \mathcal{L}^n\big(H \cap (B_r \setminus B_{r/2})\big)$, thanks to (7.25). Hence, exploiting (7.26), we find that

$$\mathcal{F}_K(H) \geqslant \mathcal{F}_K(G). \tag{7.27}$$

In addition, we claim that

$$\mathrm{Per}_r(H) \leqslant \mathrm{Per}_r(E). \tag{7.28}$$

To check this, we recall (see formulas (2.4)-(2.5) in [10]) that

$$\mathrm{Per}_r(E\cap B_r)+\mathrm{Per}_r(E\cup B_r)\leqslant \mathrm{Per}_r(E)+\mathrm{Per}_r(B_r). \tag{7.29}$$

Let now $R\geqslant r$ be such that $\mathcal{L}^n(E\cup B_r)=\mathcal{L}^n(B_R)$. Then, from the isoperimetric inequality in (1.8), we see that

$$\mathrm{Per}_r(E\cup B_r)\geqslant \mathrm{Per}_r(B_R)=\frac{\mathcal{L}^n(B_{R+r})}{2r}\geqslant\frac{\mathcal{L}^n(B_{2r})}{2r}=\mathrm{Per}_r(B_r).$$

Hence, we insert this inequality into (7.29) and we obtain (7.28), as desired.

We also notice that, by (7.24), we have that $\mathcal{L}^n\big(H\cap(B_r\setminus B_{r/2})\big)=\mathcal{L}^n\big(E\cap(B_r\setminus B_{r/2})\big)$, and so

$$\mathcal{F}_K(H)\leqslant\mathcal{F}_K(E), \tag{7.30}$$

thanks to (7.28).

Let also $\rho\geqslant 0$ be such that

$$\mathcal{L}^n(G)=\mathcal{L}^n(B_\rho). \tag{7.31}$$

We point out that, by (7.24) and (7.25),

$$B_{r/2}\subseteq G\subseteq B_r, \tag{7.32}$$

and so

$$\rho\in\left[\frac{r}{2},r\right]. \tag{7.33}$$

Also, making use of the isoperimetric inequality in (1.8), we see that

$$\mathrm{Per}_r(G)\geqslant\mathrm{Per}_r(B_\rho)=\frac{\mathcal{L}^n(B_{r+\rho})}{2r}=\frac{\mathcal{L}^n(B_1)\,(r+\rho)^n}{2r}. \tag{7.34}$$

Furthermore,

$$\begin{aligned}\mathcal{L}^n\big(G\cap(B_r\setminus B_{r/2})\big)&=\mathcal{L}^n(G\cap B_r)-\mathcal{L}^n(G\cap B_{r/2})=\mathcal{L}^n(G)-\mathcal{L}^n(B_{r/2})\\&=\mathcal{L}^n(B_\rho)-\mathcal{L}^n(B_{r/2})=\mathcal{L}^n(B_1)\left(\rho^n-\left(\frac{r}{2}\right)^n\right),\end{aligned} \tag{7.35}$$

thanks to (7.31) and (7.32).

Hence, by (7.34) and (7.35), we have that

$$\mathcal{F}_K(G)\geqslant\frac{\mathcal{L}^n(B_1)\,(r+\rho)^n}{2r}-\mathcal{L}^n(B_1)\,K\left(\rho^n-\left(\frac{r}{2}\right)^n\right)=:\Phi(\rho). \tag{7.36}$$

We notice that, for any $t\in\left[\frac{r}{2},r\right]$,

$$\begin{aligned}\Phi'(t)&=n\,\mathcal{L}^n(B_1)\left(\frac{(r+t)^{n-1}}{2r}-K\,t^{n-1}\right)\\&=n\,\mathcal{L}^n(B_1)\,t^{n-1}\left(\frac{1}{2r}\left(\frac{r}{t}+1\right)^{n-1}-K\right)\\&\leqslant n\,\mathcal{L}^n(B_1)\,t^{n-1}\left(\frac{1}{2r}\left(\frac{r}{r/2}+1\right)^{n-1}-K\right)\\&\leqslant 0,\end{aligned}$$

as long as K is large enough, as prescribed in (1.26). Therefore, recalling (7.20) and (7.33), we have that

$$\mathcal{F}_K(B_r\setminus B_{r/2})=\mathcal{L}^n(B_1)\left[2^{n-1}r^{n-1}-K\left(1-\frac{1}{2^n}\right)r^n\right]=\Phi(r)=\inf_{t\in\left[\frac{r}{2},r\right]}\Phi(t)\leqslant\Phi(\rho).$$

Hence, we insert this information into (7.36), and we conclude that $\mathcal{F}_K(G)\geqslant\mathcal{F}_K(B_r\setminus B_{r/2})$. From this, (7.30) and (7.27), we conclude that

$$\mathcal{F}_K(E)\geqslant\mathcal{F}_K(H)\geqslant\mathcal{F}_K(G)\geqslant\mathcal{F}_K(B_r\setminus B_{r/2}),$$

which completes the proof of (7.21).

Now, for any (arbitrarily ugly) set $U\subseteq B_{r/2}$, we set $E_U:=(B_r\setminus B_{r/2})\cup U$. We notice that $(\partial E_U)\oplus B_r=B_{2r}=(B_r\setminus B_{r/2})\oplus B_r$, and also $\mathcal{L}^n\big(E_U\cap(B_r\setminus B_{r/2})\big)=\mathcal{L}^n(B_r\setminus B_{r/2})$, and therefore

$$\mathcal{F}_K(E_U)=\mathcal{F}_K(B_r\setminus B_{r/2}).$$

Hence, from (7.21), we have that E_U is also a minimizer for $\mathcal{F}_K$, from which the claims in Theorem 1.19 plainly follow. □

8. Planelike minimizers in periodic media – Proof of Theorem 1.21

In this section we establish the existence of planelike minimizers for periodic volume perturbations of Per_r.

Proof of Theorem 1.21. The proof is given in two steps: in the first one, we fix a rational slope ω and we provide the construction of a planelike minimizer E^*_ω which is also ω- periodic. Then, in the second step, we consider irrational slopes by means of an approximation procedure.

Step 1: construction of planelike minimizers with rational slope. The idea of the proof is to perform an argument based on a constrained minimal minimizer procedure, as in [4]. A major difference with [4] here is that optimal density estimates at small scales do not hold, hence the width of the strip may depend, in principle, on r. Indeed, roughly speaking, here one needs an initial density to improve the density in the large, and so, to let the density reach a uniform threshold, a large number (in dependence of r) of fundamental cubes may be needed, and this has a rather strong consequence on the energy estimates when r is small.

Hence, the proof of this step will be performed in two parts: first, we obtain an initial bound on the width of the strip that depends on r, and then we improve this bound up to a uniform scale. This method will combine the minimal minimizer argument in [4] with an ad-hoc procedure of finely selecting appropriating cubes and performing a cut at a suitable level. These estimates will be based on a fine analysis of cubes, to detect local densities and energy contributions.

The details of the proof go like this. We consider a "fundamental domain" for the ω-periodicity, i.e. we take $K_1, \dots, K_{n-1} \in \mathbb{Z}^n$ which are linearly independent and such that $\omega \cdot K_j = 0$ for any $j \in \{1, \dots, n-1\}$, and we set

$$F_\omega := \big\{ t_1 K_1 + \dots + t_{n-1} K_{n-1}, \quad t_1, \dots, t_{n-1} \in (0,1) \big\}.$$

Notice that the existence of $K_1, \dots, K_{n-1}$ is a consequence of the rationality of ω in (1.27).

Given $M \geqslant 2$, we also consider the parallelepipedon

$$\begin{aligned} S_{\omega,M} &:= \big\{ t_1 K_1 + \dots + t_{n-1} K_{n-1} + t_n \omega, \quad t_1, \dots, t_{n-1} \in (0,1), \quad t_n \in (-M, M) \big\} \\ &= \big\{ p + t_n \omega, \quad p \in F_\omega, \quad t_n \in (-M, M) \big\}. \end{aligned}$$

We consider the functional

$$\mathcal{F}_{\omega,M}(E) := \mathrm{Per}_r(E, S_{\omega,2M}) + \int_{E \cap S_{\omega,2M}} g(x)\, dx.$$

We now introduce the set of periodic constrained minimizers for this functional. Namely we define $\mathcal{C}_{\omega,M}$ the family of sets $E \subseteq \mathbb{R}^n$ which are ω-periodic and such that

$$\{\omega \cdot x \leqslant -M\} \subseteq E \subseteq \{\omega \cdot x \leqslant M\}.$$

Let also $L_\omega := \{\omega \cdot x \leqslant 0\}$. Then

$$\mathrm{Per}_r(L_\omega, S_{\omega,2M}) \leqslant C \mathcal{H}^{n-1}(F_\omega), \tag{8.1}$$

for some $C > 0$.

We also consider the family of finite overlapping dilated cubes

$$\mathcal{Q} := \{ j + [0,n]^n, \quad j \in \mathbb{Z}^n \}.$$

We define $\mathcal{Q}_M$ the family of cubes $Q \in \mathcal{Q}$ which intersect $\{\omega \cdot x = \pm M\}$. The fact that g has zero average in each $Q \in \mathcal{Q}$ implies that

$$\left| \int_{E \cap S_{\omega,2M}} g(x)\, dx \right| \leqslant \sum_{Q \in \mathcal{Q}_{2M}} \int_Q |g(x)|\, dx \leqslant \|g\|_{L^\infty(\mathbb{R}^n)} \sum_{Q \in \mathcal{Q}_{2M}} \mathcal{L}^n(Q) \leqslant C\eta\, \mathcal{H}^{n-1}(F_\omega),$$

up to renaming $C > 0$, and therefore, in view of (8.1), it holds that

$$\mathcal{F}_{\omega,M}(L_\omega) \leqslant C \mathcal{H}^{n-1}(F_\omega). \tag{8.2}$$

This says that there exists at least one set in $\mathcal{C}_{\omega,M}$ with finite energy, hence we can proceed to the minimization of the functional. The existence of the minimum in this case follows along the lines of Theorem 1.9.

So we define $\mathcal{M}_{\omega,M}$ as the family of sets $E \in \mathcal{C}_{\omega,M}$ such that

$$\mathcal{F}_{\omega,M}(E) = \inf_{F \in \mathcal{C}_{\omega,M}} \mathcal{F}_{\omega,M}(F).$$

Following a classical idea of [4], we now define the minimal minimizer as

$$E^*_{\omega,M} := \bigcap_{E \in \mathcal{M}_{\omega,M}} E.$$

We remark that

$$\mathrm{Per}_r(E \cap F, \Omega) + \mathrm{Per}_r(E \cup F, \Omega) \leqslant \mathrm{Per}_r(E, \Omega) + \mathrm{Per}_r(F, \Omega), \tag{8.3}$$

for any E, $F \subseteq \mathbb{R}^n$ and any domain Ω, and thus

$$\mathcal{F}_{\omega,M}(E \cap F) + \mathcal{F}_{\omega,M}(E \cup F) \leqslant \mathcal{F}_{\omega,M}(E) + \mathcal{F}_{\omega,M}(F). \tag{8.4}$$

By (8.4), we have that $E^*_{\omega,M} \in \mathcal{M}_{\omega,M}$, that is the minimal minimizer is indeed a minimizer. Moreover, $E^*_{\omega,M}$ satisfies the inclusion properties (for a proof of this we refer to [4, Lemma 6.5])

$$\begin{aligned} &\text{if } k \in \mathbb{Z}^n \text{ and } \omega \cdot k \leqslant 0, \text{ then } E^*_{\omega,M} + k \subseteq E^*_{\omega,M};\\ &\text{if } k \in \mathbb{Z}^n \text{ and } \omega \cdot k \geqslant 0, \text{ then } E^*_{\omega,M} + k \supseteq E^*_{\omega,M}. \end{aligned} \tag{8.5}$$

Consequently, since $E^*_{\omega,M}$ is the smallest possible minimizers,

$$\begin{aligned} &\text{if } B_n(p) \cap E^*_{\omega,M} = \varnothing, \text{ then } E^*_{\omega,M} \subseteq \{\omega \cdot (p - x) \leqslant n\}\\ \text{and} \quad &\text{if } B_n(p) \subseteq E^*_{\omega,M}, \text{ then } E^*_{\omega,M} \supseteq \{\omega \cdot (p - x) \geqslant -n\}. \end{aligned}$$

We now divide the cubes in $\mathcal{Q}$ according to their "color", i.e. their density properties with respect to the set $E^*_{\omega,M}$ (pictorially, we think that the set $E^*_{\omega,M}$ is "black" and its complement is "white").

Namely, we consider the "family of black cubes" given by

$$\mathcal{Q}_{\mathrm{Bl}} := \{Q \in \mathcal{Q} \text{ s.t. } Q \subseteq E^*_{\omega,M}\}$$

and the "family of white cubes"

$$\mathcal{Q}_{\mathrm{Wh}} := \{Q \in \mathcal{Q} \text{ s.t. } Q \cap E^*_{\omega,M} = \varnothing\}.$$

We also take into account the "family of grey cubes"

$$\begin{aligned} \mathcal{Q}_{\mathrm{Gr}} &:= \mathcal{Q} \setminus \big(\mathcal{Q}_{\mathrm{Bl}} \cup \mathcal{Q}_{\mathrm{Wh}}\big)\\ &= \{Q \in \mathcal{Q} \text{ s.t. } Q \setminus E^*_{\omega,M} \neq \varnothing \text{ and } Q \cap E^*_{\omega,M} \neq \varnothing\}. \end{aligned}$$

We also subdivide the grey cubes into the ones which are "foggy black" and the ones which are "foggy white": the first family contains cubes with a sufficient density of $E^*_{\omega,M}$, while the second family contains cubes with a sufficient density of the complement of $E^*_{\omega,M}$, being the notion of "sufficient density" the one compatible with uniform scales in the density estimates of Theorem 1.18. That is, we define

$$\begin{aligned} &\mathcal{Q}_{\mathrm{f.Bl}} := \{Q \in \mathcal{Q}_{\mathrm{Gr}} \text{ s.t. } \mathcal{L}^n(Q \cap E^*_{\omega,M}) \geqslant r^n\}\\ \text{and} \qquad &\mathcal{Q}_{\mathrm{f.Wh}} := \{Q \in \mathcal{Q}_{\mathrm{Gr}} \text{ s.t. } \mathcal{L}^n(Q \setminus E^*_{\omega,M}) \geqslant r^n\}. \end{aligned}$$

Notice that, since $r \in (0, 1)$,

$$\mathcal{Q}_{\mathrm{Gr}} = \mathcal{Q}_{\mathrm{f.Bl}} \cup \mathcal{Q}_{\mathrm{f.Wh}}.$$

On the other hand, in general, we have that $\mathcal{Q}_{\mathrm{f.Bl}} \cap \mathcal{Q}_{\mathrm{f.Wh}} \neq \varnothing$, since there might be cubes with sufficiently high density of both $E^*_{\omega,M}$ and its complement (these cubes are, in some sense, "multicolored" inside). So, we define

$$\begin{aligned} \mathcal{Q}_{\mathrm{Mu}} &:= \{Q \in \mathcal{Q}_{\mathrm{f.Bl}} \cap \mathcal{Q}_{\mathrm{f.Wh}}\}\\ &= \Big\{Q \in \mathcal{Q}_{\mathrm{Gr}} \text{ s.t. } \min\big\{\mathcal{L}^n(Q \cap E^*_{\omega,M}),\, \mathcal{L}^n(Q \setminus E^*_{\omega,M})\big\} \geqslant r^n\Big\}. \end{aligned}$$

Notice that the cubes in $\mathcal{Q}_{\mathrm{f.Bl}} \setminus \mathcal{Q}_{\mathrm{Mu}}$ have a sufficiently high density of $E^*_{\omega,M}$ and a rather low density of its complement, so they "look almost black". For this reason, we set

$$\begin{aligned} \mathcal{Q}_{\mathrm{a.Bl}} &:= \{Q \in \mathcal{Q}_{\mathrm{f.Bl}} \setminus \mathcal{Q}_{\mathrm{Mu}}\}\\ &= \big\{Q \in \mathcal{Q}_{\mathrm{Gr}} \text{ s.t. } \mathcal{L}^n(Q \cap E^*_{\omega,M}) \geqslant r^n > \mathcal{L}^n(Q \setminus E^*_{\omega,M})\big\}. \end{aligned}$$

Similarly, we define the family of almost white cubes as

$$\begin{aligned}\mathcal{Q}_{\rm a.Wh} &:= \{Q \in \mathcal{Q}_{\rm f.Wh} \setminus \mathcal{Q}_{\rm Mu}\}\\ &= \big\{Q \in \mathcal{Q}_{\rm Gr} \text{ s.t. } \mathcal{L}^n(Q \setminus E^*_{\omega,M}) \geqslant r^n > \mathcal{L}^n(Q \cap E^*_{\omega,M})\big\}.\end{aligned}$$

We are now going to show that the strip is divided into five ordered "color layers": on the bottom stay all the black cubes, then the almost black ones, then cubes of multicolor type, then almost white cubes and finally white cubes on the top (rigorous statements below). We also estimate carefully the width of these layers.

To this end, we observe that the "color density" of the cubes is monotone with respect to ω, in the sense that the color of an upper translation is more pale than the color of a lower translation. The precise statement goes as follows: we claim that, for any $k \in \mathbb{Z}^n$ with $\omega \cdot k \geqslant 0$, we have that

$$\mathcal{L}^n\big((Q+k) \cap E^*_{\omega,M}\big) \leqslant \mathcal{L}^n\big(Q \cap E^*_{\omega,M}\big) \leqslant \mathcal{L}^n\big((Q-k) \cap E^*_{\omega,M}\big). \tag{8.6}$$

To check this, we exploit (8.5) to see that $E^*_{\omega,M} - k \subseteq E^*_{\omega,M} \subseteq E^*_{\omega,M} + k$ and therefore

$$\begin{aligned}&\big((Q+k) \cap E^*_{\omega,M}\big) - k = Q \cap (E^*_{\omega,M} - k) \subseteq Q \cap E^*_{\omega,M}\\ \text{and}\qquad &\big((Q-k) \cap E^*_{\omega,M}\big) + k = Q \cap (E^*_{\omega,M} + k) \supseteq Q \cap E^*_{\omega,M}.\end{aligned}$$

Accordingly

$$\begin{aligned}&\mathcal{L}^n\big((Q+k) \cap E^*_{\omega,M}\big) = \mathcal{L}^n\Big(\big((Q+k) \cap E^*_{\omega,M}\big) - k\Big) \leqslant \mathcal{L}^n\big(Q \cap E^*_{\omega,M}\big)\\ \text{and}\qquad &\mathcal{L}^n\big((Q-k) \cap E^*_{\omega,M}\big) = \mathcal{L}^n\Big(\big((Q-k) \cap E^*_{\omega,M}\big) + k\Big) \geqslant \mathcal{L}^n\big(Q \cap E^*_{\omega,M}\big),\end{aligned}$$

thus proving (8.6).

As a consequence of (8.6), we have that, for any $k \in \mathbb{Z}^n$ with $\omega \cdot k \geqslant 0$,

$$\begin{aligned}&\text{if } Q \in \mathcal{Q}_{\rm Bl}, \text{ then } Q+k \in \mathcal{Q}_{\rm Bl} \cup \mathcal{Q}_{\rm Gr} \cup \mathcal{Q}_{\rm Wh},\\ &\text{if } Q \in \mathcal{Q}_{\rm Gr}, \text{ then } Q+k \in \mathcal{Q}_{\rm Gr} \cup \mathcal{Q}_{\rm Wh},\\ &\text{if } Q \in \mathcal{Q}_{\rm Wh}, \text{ then } Q+k \in \mathcal{Q}_{\rm Wh},\\ &\text{if } Q \in \mathcal{Q}_{\rm a.Bl}, \text{ then } Q+k \in \mathcal{Q} \setminus \mathcal{Q}_{\rm Bl},\\ &\text{if } Q \in \mathcal{Q}_{\rm a.Wh}, \text{ then } Q+k \in \mathcal{Q}_{\rm a.Wh} \cup \mathcal{Q}_{\rm Wh},\end{aligned} \tag{8.7}$$

and similar statements hold in the case $\omega \cdot k \leqslant 0$.

We now point out that, if $Q \in \mathcal{Q}_{\rm Gr}$, then $Q \cap (\partial E^*_{\omega,M}) \neq \varnothing$ and therefore

$$\mathrm{Per}_r(E^*_{\omega,M} \cap Q, S_{\omega,2M}) \geqslant c r^{n-1}, \tag{8.8}$$

for some $c > 0$. We also observe that, for any $E \subseteq \mathbb{R}^n$,

$$\left|\int_{E\cap Q} g(x)\,dx\right| \leqslant \|g\|_{L^\infty(\mathbb{R}^n)} \min\big\{\mathcal{L}^n(E \cap Q),\, \mathcal{L}^n(Q \setminus E)\big\}. \tag{8.9}$$

To check this, let us first suppose that $\mathcal{L}^n(E \cap Q) \leqslant \mathcal{L}^n(Q \setminus E)$. Then,

$$\left|\int_{E\cap Q} g(x)\,dx\right| \leqslant \int_{E\cap Q} |g(x)|\,dx \leqslant \|g\|_{L^\infty(\mathbb{R}^n)}\,\mathcal{L}^n(E \cap Q),$$

which gives (8.9) in this case. Conversely, if $\mathcal{L}^n(E \cap Q) > \mathcal{L}^n(Q \setminus E)$ we use that g has zero average and we write

$$\begin{aligned}\left|\int_{E\cap Q} g(x)\,dx\right| &= \left|\int_Q g(x)\,dx - \int_{E\cap Q} g(x)\,dx\right|\\ &= \left|\int_{Q\setminus E} g(x)\,dx\right| \leqslant \|g\|_{L^\infty(\mathbb{R}^n)}\,\mathcal{L}^n(Q \setminus E),\end{aligned}$$

thus completing the proof of (8.9).

In view of (8.8) and (8.9), we know that

$$\begin{aligned}&\text{for any } Q \in \mathcal{Q}_{\rm Gr},\\ &\mathcal{F}_{\omega,M}(E^*_{\omega,M} \cap Q, S_{\omega,2M}) \geqslant c r^{n-1} - \|g\|_{L^\infty(\mathbb{R}^n)}\, r^n \geqslant r^{n-1}(c - \eta r) \geqslant \frac{c r^{n-1}}{2},\end{aligned} \tag{8.10}$$

provided that η is small enough.

On the other hand, if $Q \in \mathcal{Q}_{\rm Mu}$, we are in the uniform density setting of (1.23) and, consequently, by (1.21) we can write that

$$\min\left\{\mathcal{L}^n(E^*_{\omega,M}\cap Q'),\mathcal{L}^n(Q'\setminus E^*_{\omega,M}),\right\}\geqslant c, \tag{8.11}$$

up to renaming $c>0$, where Q' is the dilation of Q with respect to its center by a factor 2 (we stress that the condition $Q\in\mathcal{Q}_{\rm Mu}$ has been used here to guarantee an initial estimate on the density, which makes the constants in Theorem 1.18 uniform).

Then, from (8.11) and the relative isoperimetric inequality in Theorem 1.13, up to renaming $c>0$, we have that if $Q\in\mathcal{Q}_{\rm Mu}$, then

$$\mathrm{Per}_r(E^*_{\omega,M}\cap Q',S_{\omega,2M})\geqslant c.$$

Therefore

$$\begin{split}&\text{for any } Q\in\mathcal{Q}_{\rm Mu},\\ &\mathcal{F}_{\omega,M}(E^*_{\omega,M}\cap Q',S_{\omega,2M})\geqslant c-\|g\|_{L^\infty(\mathbb{R}^n)}\,\mathcal{L}^n(Q')\geqslant\frac{c}{2},\end{split} \tag{8.12}$$

provided that η is small enough.

Now we denote by $J_{\rm Mu}$, $J_{\rm a.Bl}$ and $J_{\rm a.Wh}$ the number of cubes in $\mathcal{Q}_{\rm Mu}$, $\mathcal{Q}_{\rm a.Bl}$ and $\mathcal{Q}_{\rm a.Wh}$, respectively. Then, up to renaming constants, we deduce from (8.10) and (8.12) that

$$\mathcal{F}_{\omega,M}(E^*_{\omega,M},S_{\omega,2M})\geqslant cr^{n-1}\,(J_{\rm a.Bl}+J_{\rm a.Wh})+c\,J_{\rm Mu}.$$

Comparing with (8.2) and using minimality, we thus obtain that

$$r^{n-1}\,(J_{\rm a.Bl}+J_{\rm a.Wh})+c\,J_{\rm Mu}\leqslant C\mathcal{H}^{n-1}(F_\omega),$$

up to renaming $C>0$. Hence, in view of the layer structure described in (8.7), we have that

$$\text{the family of cubes in } \mathcal{Q}_{\rm Mu} \text{ lies in a strip of width at most } C, \tag{8.13}$$

while

$$\text{the families of cubes in } \mathcal{Q}_{\rm a.Bl} \text{ and in } \mathcal{Q}_{\rm a.Wh} \text{ lie in strips of width at most } \tfrac{C}{r^{n-1}}. \tag{8.14}$$

We observe that the bound in (8.13) is already satisfactory, but the one in (8.14) needs to be improved if we want to arrive at a strip of uniform width (independent of r). That is, we are now in a situation in which "almost white" or "almost black" cubes may have a long tail in the strip when r is small, and we want to rule out this possibility.

For this, we need a careful procedure of cutting cubes in $\mathcal{Q}_{\rm a.Wh}$. The idea is that once we have a cube which is "almost white" we can color the region above it in full white, gaining energy.

The goal is thus to replace (8.14) with

$$\text{the families of cubes in } \mathcal{Q}_{\rm a.Bl} \text{ and in } \mathcal{Q}_{\rm a.Wh} \text{ lie in strips of width at most } C, \tag{8.15}$$

up to renaming C (the situation of formulas (8.13) and (8.15) is graphically depicted in Figure 1). So, if we can bound the width in (8.14) with a uniform bound, we are done; otherwise suppose that, for instance, $\mathcal{Q}_{\rm a.Wh}$ occupies a strip of width $W_r\geqslant 2n$, possibly depending on r (from (8.14), we only know that $W_r\leqslant C/r^{n-1}$), say $\{C_o\leqslant\omega\cdot x\leqslant C_o+W_r\}$ (notice that the position of the lower boundary of this strip is uniformly bounded, thanks to (8.13), so we denoted it by C_o for the sake of clarity).

The idea is now to replace $E^*_{\omega,M}$ with $E^*_{\omega,M}\cap\{\omega\cdot x\leqslant C_o+\sqrt{n}\}$. To compute the effect of this cut, let us consider that, at levels $\{\omega\cdot x\in[C_o,C_o+n]\}$, we may have created additional r-perimeter adding portions of $\{\omega\cdot x=C_o+\sqrt{n}\}$ to the boundary of the set. Since this portion is flat, the cut procedure has produced an energy increasing for the r-perimeter of size at most $C\,\mathcal{H}^{n-1}(F_\omega)\,r^{n-1}$. As for the bulk energy produced by g, in each cube Q in $\{\omega\cdot x=C_o+\sqrt{n}\}$, we have produced an energy increasing of at most

$$\|g\|_{L^\infty(\mathbb{R}^n)}\,\min\left\{\mathcal{L}^n(E^*_{\omega,M}\cap Q),\,\mathcal{L}^n(Q\setminus E^*_{\omega,M})\right\}\leqslant\eta\,\mathcal{L}^n(E^*_{\omega,M}\cap Q)\leqslant\eta r^n,$$

thanks to (8.9) and to the fact that $Q\in\mathcal{Q}_{\rm a.Wh}$. That is, the total bulk energy increased at levels $\{\omega\cdot x\in[C_o,C_o+n]\}$ is bounded by $C\,\eta\,\mathcal{H}^{n-1}(F_\omega)\,r^n$. Summarizing, the modifications of the cubes in $\mathcal{Q}_{\rm a.Wh}$ at levels $\{\omega\cdot x\in[C_o,C_o+n]\}$ produce an energy increasing bounded by

$$C\,\mathcal{H}^{n-1}(F_\omega)\,r^{n-1}+C\,\eta\,\mathcal{H}^{n-1}(F_\omega)\,r^n\leqslant C\,\mathcal{H}^{n-1}(F_\omega)\,r^{n-1}\big(1+\eta r\big). \tag{8.16}$$

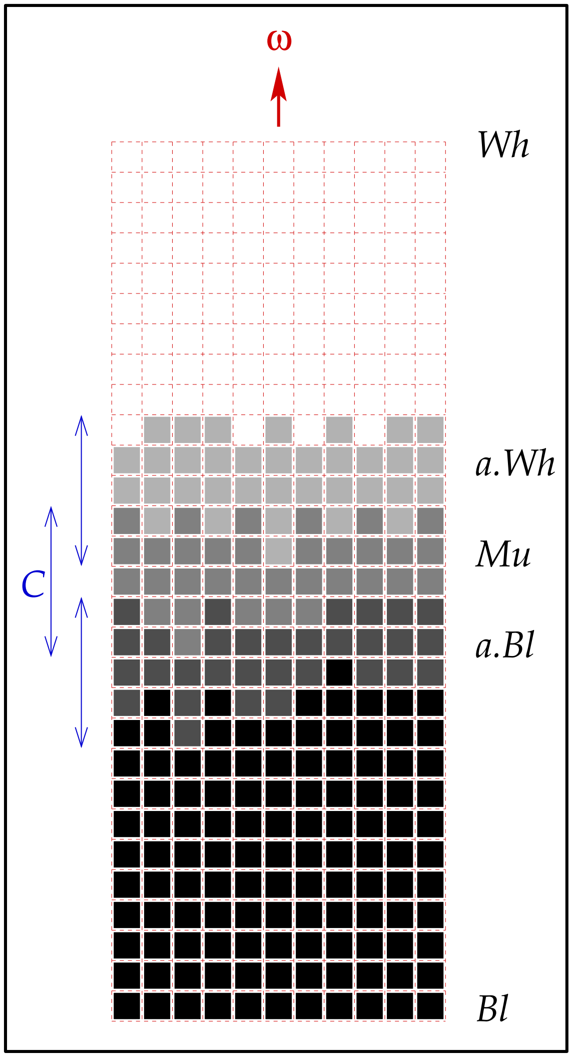

FIGURE 1. *The geometry of the colored cubes in* (8.13) *and* (8.15).

To prove that the total energy has in fact decreased, we now check that the cut procedure produces a considerable gain at the other levels $\{\omega \cdot x \in [C_o + n, C_o + W_r]\}$. For this, notice that

$$\{\omega \cdot x \in [C_o + n, C_o + W_r]\} \text{ contains at least } c\, W_r\, \mathcal{H}^{n-1}(F_\omega) \text{ cubes of } \mathcal{Q}_{\rm a.Wh}. \tag{8.17}$$

In each of these cubes, the cut has produced an energy gain, due to the r-perimeter, and possibly an energy loss due to the bulk energy of g. From (8.8), we know that the energy gain in each of these cubes is at least cr^{n-1}, up to renaming $c > 0$. On the other hand, from (8.9) and the fact that the cube belongs to $\mathcal{Q}_{\rm a.Wh}$, we deduce an upper bound of the bulk energy loss in each cube of the form $C\eta r^n$, for some $C > 0$. Hence, the variation of energy in each of these cubes is of the form $-cr^{n-1} + C\eta r^n$ (which is negative for small η).

Summarizing, and recalling (8.17), we have that the cut procedure has produced in $\{\omega \cdot x \in [C_o + n, C_o + W_r]\}$ an energy variation bounded from above by

$$c\, W_r\, \mathcal{H}^{n-1}(F_\omega)(-cr^{n-1} + C\eta r^n) \leqslant c\, W_r\, \mathcal{H}^{n-1}(F_\omega)\, r^{n-1}(-1 + C\eta r),$$

up to renaming c and C. From this and (8.16), up to renaming constants line after line, we obtain that the variation of the energy produced by the cut is in total bounded from above by

$$\begin{aligned} & C\, \mathcal{H}^{n-1}(F_\omega)\, r^{n-1}\big(1 + \eta r\big) + c\, W_r\, \mathcal{H}^{n-1}(F_\omega)\, r^{n-1}(-1 + C\eta r) \\ \leqslant\ & \mathcal{H}^{n-1}(F_\omega)\, r^{n-1}\big(C + C\eta r - cW_r + CW_r\, \eta r\big) \\ \leqslant\ & \mathcal{H}^{n-1}(F_\omega)\, r^{n-1}\big(C + CW_r\, \eta r - cW_r\big). \end{aligned}$$

Since, by the minimal property of $E^*_{\omega,M}$, this energy variation has to be positive, we conclude that

$$0 \leqslant C + CW_r\,\eta r - cW_r \leqslant C + W_r\,(C\,\eta - c)$$

and thus, for small η, we obtain that W_r is bounded uniformly, by a constant independent of r.

This proves (8.15) for the families of cubes in $\mathcal{Q}_{\rm a.Wh}$ (the cases of the cubes in $\mathcal{Q}_{\rm a.Bl}$ is similar).

From (8.15), one can exploit the methods in [4], namely find that there exists a uniform $M_0 > 0$ such that if $M \geqslant M_0$, then $E^*_{\omega,M} = E^*_{\omega,M_0}$ and then, checking that the minimal minimizer is stable with respect to multiples of the period, establish that it is a Class A minimizer, thus completing the proof of Theorem 1.21 for rational slopes ω.

Step 2: planelike minimizers with irrational slopes. Since the quantity M is a universal constant, independent of n, in order to construct minimizers with an irrational slope $\omega \in S^{n-1}$ we approximate ω with rational frequencies ω_k, which produce planelike minimizers $E^*_{\omega_k}$ and then pass to the limit in k, using Theorem 1.8, which applies in particular to the Class A planelike minimizers. $\square$

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