Volume constrained minimizers of the fractional perimeter with a potential energy

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Abstract

We consider volume-constrained minimizers of the fractional perimeter with the addition of a potential energy in the form of a volume integral. Such minimizers are solutions of the prescribed fractional curvature problem. We prove existence and regularity of minimizers under suitable assumptions on the potential energy, which cover the periodic case. In the small volume regime we show that minimizers are close to balls, with a quantitative estimate.

1 Introduction

Let $s \in (0,1)$ and let $E \subset \mathbb{R}^N$ be a measurable set, the fractional perimeter $P_s(E)$ of E is defined as the squared $H^{s/2}$-seminorm of the characteristic function of E, i.e.

$$P_s(E) = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\chi_E(x)-\chi_E(y)|^2}{|x-y|^{N+s}}dxdy = \int_E\int_{E^c}\frac{1}{|x-y|^{N+s}}dxdy. \quad (1)$$

This notion has been introduced in [15, 3] and has been widely studied in the last years (see [8, 7] and references therein).

It is well known that balls are the unique minimizers of the fractional perimeter among sets with the same volume. Indeed, the following fractional isoperimetric inequality holds for sets of finite volume (see [3, 8]):

$$P_s(E) \geq \frac{P_s(B)}{|B|^{\frac{N-s}{N}}}|E|^{\frac{N-s}{N}}, \quad (2)$$

where B is the ball of radius 1, and equality holds if and only if E is a ball. The isoperimetric inequality (2) can also be localized in bounded sets with Lipschitz boundary (see [7, Lemma 2.5]).

In this paper we are interested in existence and properties of minimizers of the following isoperimetric problem

$$\min_{|E|=m} \mathcal{F}(E) = \min_{|E|=m} \left(P_s(E) - \int_E g(x)dx \right). \tag{3}$$

In particular we will provide regularity properties of minimizers under the assumption that $g : \mathbb{R}^N \to \mathbb{R}$ is locally Lipschitz continuous and bounded from above, see Corollary 3.5, whereas the existence of a solution of the isoperimetric problem is obtained for g periodic, see Theorem 5.1, or g coercive, that is

$$\lim_{|x|\to+\infty} g(x) = -\infty, \tag{4}$$

see Proposition 5.3.

Our main result is the following.

Theorem 1.1. *Assume that g is locally Lipschitz and either coercive or $\mathbb{Z}^N$-periodic. Then for any $s \in (0,1)$ and $m > 0$ there exists a bounded minimizer E of (3). Moreover, ∂E is of class $C^{2,\alpha}$ for any $\alpha < s$ outside of a closed singular set S of Hausdorff dimension at most $N-3$.*

Existence of such minimizers is related to the problem of finding compact solutions to the geometric equation

$$H_s(x) = g(x), \tag{5}$$

where H_s denotes (up to a multiplicative constant) the s-mean curvature at a point $x \in \partial E$ (see [3, 1]) , that is,

$$H_s(x) = \int_{\mathbb{R}^N} \frac{\chi_{E^c}(y) - \chi_E(y)}{|x-y|^{N+s}} dy.$$

Indeed if E is a critical point of the functional

$$P_s(E) - \int_E g(x)dx, \tag{6}$$

and ∂E is of class $C^{1,\alpha}$ for some $\alpha > s$, then it is easy to prove that E solves the prescribed fractional curvature problem (5). Note that in general there is no existence for minimizers of the problem (6), due to the lack of compactness.

As a corollary of our main result, we get that if E is a minimizer of (3), then there exists a constant μ_m, depending on m, such that

$$H_s(x) = g(x) + \mu_m$$

for $x \in \partial E \setminus S$, where S is the singular set in Theorem 1.1.

We will also show in Proposition 4.1 that, in the small volume regime, the contribution of the volume term $\int_E g(x)dx$ becomes irrelevant, and the minimizers converge, after appropriate rescalings, to a ball. Note that if g is close to a constant, it is known that solutions to (5) are necessarily compact and close to balls in the Hausdorff distance (see [6]).

2 Notation and basic estimates

Given a set $E \subset \mathbb{R}^N$, we denote by E^c its complement, that is, $E^c = \mathbb{R}^N \setminus E$. We denote by B_r the ball of center 0 and radius r, whereas $B(x,r)$ is the ball centered at x and with radius r. We also let $\omega_N = |B_1|$.

Given E, F two sets in $\mathbb{R}^N$, the symmetric difference of E and F is defined as usual as $E\Delta F = (E \setminus F) \cup (F \setminus E)$.

We recall the following computation, that will be useful in the sequel (see [7, Lemma 2.1]). Let $E = E_1 \cup E_2$ be a subset of $\mathbb{R}^N$ with $|E_1 \cap E_2| = 0$, then

$$P_s(E) = P_s(E_1) + P_s(E_2) - 2\int_{E_1}\int_{E_2} \frac{1}{|x-y|^{N+s}}dxdy. \tag{7}$$

It is possible to define the nonlocal perimeter of E in a bounded set Ω as follows:

$$P_s(E,\Omega) = \int_{\mathbb{R}^N\setminus E}\int_{E\cap\Omega} \frac{1}{|x-y|^{N+s}}dxdy + \int_{\Omega\setminus E}\int_{E\setminus\Omega} \frac{1}{|x-y|^{N+s}}dxdy. \tag{8}$$

Finally we recall the following formula (see [7, Lemma 2.4]). Given two disjoint bounded open sets Ω_1, Ω_2, then there holds

$$P_s(E,\Omega_1) + P_s(E,\Omega_2) = P_s(E,\Omega_1\cup\Omega_2) + 2\int_{\Omega_1}\int_{\Omega_2} \frac{1}{|x-y|^{N+s}}dxdy. \tag{9}$$

3 Regularity of minimizers

In this section we shall assume that

$$g \text{ is locally Lipschitz continuous and bounded from above} \tag{10}$$

and we will prove regularity of minimizers.

We start with a nonlocal version of the so-called Almgren's Lemma (see [9, Lemma 2.3]).

Lemma 3.1. *Let $s \in (0,1)$ and let $E \subset \mathbb{R}^N$ be a measurable set with $P_s(E) < +\infty$. Let $x_0 \in \mathbb{R}^n$ and $r > 0$ be such that*

$$|B(x_0,r)\cap E| > 0 \qquad \text{and} \qquad |B(x_0,r)\cap E^c| > 0\,. \tag{11}$$

Then there exist positive constants k_0, C, depending on E, such that for any $k \in (-k_0, k_0)$ there exists a measurable set F with $P_s(F) < +\infty$, satisfying the following properties

1. $E\Delta F \subset\subset B(x_0,r)$,
2. $|F| - |E| = k$,
3. $|P_s(E) - P_s(F)| \le C|k|$.

Proof. Let $T \in C^1_c(B(x_0,r),\mathbb{R}^N)$ be such that

$$M := \int_E \text{div } T(x)\,dx > 0.$$

Notice that such a vector field T necessarily exists since otherwise we would have

$$P(E,B(x_0,r)) = \sup\left\{\int_E \text{div } T(x)\,dx :\ T \in C^1_c(B(x_0,r),\mathbb{R}^N),\ \|T\|_\infty \le 1\right\} = 0\,,$$

which, by the relative isoperimetric inequality, would contradict (11).

For $t \in (-1,1)$, we define the maps $\Phi_t(x) = x + tT(x)$. It is easy to see that Φ_t is a diffeomorphism of $\mathbb{R}^N$ for t sufficiently small, moreover the Jacobian of Φ_t is given by $J\Phi_t(x) = 1 + t \text{ div } T(x) + o(t)$.

By construction $E\Delta\Phi_t(E) \subset\subset B(x_0,r)$, moreover

$$|\Phi_t(E)| = \int_E (1 + t \text{ div } T(x) + o(t))dx = |E| + Mt + o(t).$$

For k sufficiently small, we then let $F := \Phi_{t(k)}(E)$ where $t(k) = k/M + o(k)$ is such that $|F| = |E| + k$, so that Properties 1 and 2 are verified.

We now compute

$$\begin{aligned} P_s(\Phi_t(E)) &= \int_E\int_{E^c} \frac{1 + t\text{div}T(x) + t\text{div}T(y) + o(t)}{|\Phi_t(x) - \Phi_t(y)|^{N+s}}dxdy \\ &= \int_E\int_{E^c} \frac{1 + t\text{div}T(x) + t\text{div}T(y) + o(t)}{|x - y + t(T(x) - T(y))|^{N+s}}dxdy. \end{aligned}$$

Using the regularity of T, we get that there exists a constant C (depending on T) such that

$$(1-C|t|)^{N+s}|x-y|^{N+s} \le |x-y+t(T(x)-T(y))|^{N+s} \le (1+C|t|)^{N+s}|x-y|^{N+s}.$$

Substituting this estimate in the expression for $P_s(\Phi_t(E))$ above, we obtain that

$$P_s(E)(1 - C|t|) \le P_s(\Phi_t(E)) \le P_s(E)(1 + C|t|)$$

where C depends on T, N, s. This shows that the set F also satisfies Property 3, and the proof is concluded.

□

Using this lemma we get boundedness of minimizers (see [7, Lemma 4.1]).

Proposition 3.2. *Let* (10) *hold. Then, every minimizer E of* (3) *is bounded. In particular, there exists $\overline{R}$, depending on E, such that $E \subseteq B_{\overline{R}}$, up to a suitable translation.*

Proof. Let E be a minimizer of (3). For $r \geq 0$ we define

$$f(r) = |E \setminus B_r|.$$

Then $f(r)$ is a nonincreasing function and by the coarea formula, we have

$$f'(r) = -P(E \cap B_r).$$

We claim that there exists $\overline{R}$, such that $f(r) = 0$ for $r \geq \overline{R}$. Let us assume by contradiction that $f(r) > 0$ for any $r > 0$. Without loss of generality we can also assume that $E \cap B_1 \neq \emptyset$ and $E^c \cap B_1 \neq \emptyset$. Moreover, we fix $R_0 \geq 1$ such that $f(r) < k_0$ for any $r \geq R_0$, where k_0 is as in Lemma 3.1. Then by Lemma 3.1 for any $r \geq R_0$ there exists a set F such that:

1. $F \Delta E \subset\subset B_1 \subseteq B_r$,
2. $|F| = |E| + f(r)$,
3. $|P_s(E) - P_s(F)| \leq C f(r)$.

Let $G = F \cap B_r$. By the first two properties in Lemma 3.1, we have that $|G| = |E|$. Therefore, by minimality of E and recalling (7), we get

$$\begin{aligned} &P_s(E) - \int_E g(x)dx < P_s(G) - \int_G g(x)dx \qquad (12)\\ &= P_s(F) - P_s(F \setminus B_r) + 2\int_{F\setminus B_r}\int_{F\cap B_r} \frac{1}{|x-y|^{N+s}}dxdy - \int_{F\cap B_r} g(x)dx. \end{aligned}$$

By Property 3 in Lemma 3.1 we get that

$$P_s(F) \leq P_s(E) + C f(r). \qquad (13)$$

Notice that by the construction in Lemma 3.1, using the locally Lipschitz regularity of g, we have also

$$\begin{aligned} &\left|\int_{F\cap B_r} g(x)dx - \int_{E\cap B_r} g(x)dx\right| = \left|\int_{F\cap B_1} g(x)dx - \int_{E\cap B_1} g(x)dx\right| \\ &= \left|\int_{E\cap B_1} \big(g(x + t(f(r)))T(x))\, J\Phi_{t(f(r))}(x) - g(x)\big)dx\right| \\ &\leq (K_g(1)\|T\|_\infty + \|\mathrm{div} T\|_\infty \|g\|_{L^\infty(B_1)})|E \cap B_1| t(f(r)) + o(f(r)), \end{aligned}$$

where $K_g(1)$ is the Lipschitz constant of g in B_1.

So,

$$-\int_{F\cap B_r} g(x)dx \leq -\int_{E\cap B_r} g(x)dx + Cf(r) \leq -\int_E g(x)dx + (C + \sup g) f(r). \qquad (14)$$

Using the coarea formula and recalling that $E \setminus B_r = F \setminus B_r$, we get

$$\begin{aligned} \int_{F\setminus B_r}\int_{F\cap B_r}\frac{1}{|x-y|^{N+s}}dxdy &\leq \int_{E\setminus B_r}\int_{B_r}\frac{1}{|x-y|^{N+s}}dxdy \\ \leq \int_{E\setminus B_r}\int_{B^c(y,|y|-r)}\frac{1}{|x-y|^{N+s}}dxdy &= \frac{N\omega_N}{s}\int_{E\setminus B_r}\frac{1}{(|y|-r)^s}dy \qquad (15) \\ \leq \frac{N\omega_N}{s}\int_r^{+\infty}\frac{1}{(t-r)^s}P(E\cap B_t)dt &= -\frac{N\omega_N}{s}\int_r^{+\infty}\frac{f'(t)}{(t-r)^s}dt. \end{aligned}$$

Substituting (13), (14), (15) in (12), we eventually obtain

$$P_s(E\setminus B_r) \leq C'f(r) - \frac{N\omega_N}{s}\int_r^{+\infty}\frac{f'(t)}{(t-r)^s}dt,$$

for some $C' > 0$. Hence, by the isoperimetric inequality (2) we get

$$C(N,s)f(r)^{\frac{N-s}{N}} \leq C'f(r) - \frac{N\omega_N}{s}\int_r^{+\infty}\frac{f'(t)}{(t-r)^s}dt.$$

Recalling that $f(r)$ is decreasing to 0 as $r \to +\infty$, we can choose $R_1 > R_0$ such that

$$C'f(r) \leq \frac{C(N,s)}{2}f(r)^{\frac{N-s}{N}}$$

for all $r \geq R_1$. Therefore, for $r \geq R_1$ we obtain that f satisfies the inequality

$$\frac{sC(N,s)}{2N\omega_N}f(r)^{\frac{N-s}{N}} \leq -\int_r^{+\infty}\frac{f'(t)}{(t-r)^s}dt. \qquad (16)$$

We integrate (16) on $(R,+\infty)$, with $R > R_1$, and we exchange the order of integration to get

$$\frac{sC(N,s)}{2N\omega_N}\int_R^{+\infty}f(r)^{\frac{N-s}{N}}dr \leq -\frac{1}{1-s}\int_R^{+\infty}f'(r)(r-R)^{1-s}dr. \qquad (17)$$

We now compute

$$\begin{aligned} & -\frac{1}{1-s}\int_R^{+\infty}f'(r)(r-R)^{1-s}dr \\ = & -\frac{1}{1-s}\int_R^{R+1}f'(r)(r-R)^{1-s}dr - \frac{1}{1-s}\int_{R+1}^{+\infty}f'(r)(r-R)^{1-s}dr \\ \leq & \frac{f(R)}{1-s} - \frac{f(R+1)}{1-s} - \frac{1}{1-s}\int_{R+1}^{+\infty}f'(r)(r-R)^{1-s}dr \\ = & \frac{f(R)}{1-s} + \frac{1}{1-s}\int_{R+1}^{+\infty}f'(r)(1-(r-R)^{1-s})dr \\ \leq & \frac{f(R)}{1-s} + \int_{R+1}^{+\infty}f(r)(r-R)^{-s}dr \leq \frac{f(R)}{1-s} + \int_R^{+\infty}f(r)dr. \end{aligned}$$

Using again the fact that f is decreasing to 0, we can choose R sufficiently large such that

$$\int_R^{+\infty} f(r)dr \leq \frac{sC(N,s)}{4N\omega_N} \int_R^{+\infty} f(r)^{\frac{N-s}{N}} dr.$$

Substituting this inequality in (17), we get that f satisfies the integrodifferential inequality

$$\frac{s(1-s)C(N,s)}{4N\omega_N} \int_R^{+\infty} f(r)^{\frac{N-s}{N}} dr \leq f(R) \tag{18}$$

for all $R \geq R_2$, with R_2 sufficiently large.

Proceeding now exactly as in [7, Lemma 4.1], from (18) we can conclude that there exists $\overline{R}$ such that $f(r) = 0$ for every $r \geq \overline{R}$. □

Once we have boundedness of minimizers, we can obtain regularity.

We will use the following result about regularity of local almost minimizers of the fractional perimeter, proved in a more general setting in [5, Thm 1.1, Thm 1.2]. Moreover, in [5] it is proved that the singular set has Hausdorff dimension at most $N-2$, improved to $N-3$ in [14, Corollary 2].

Theorem 3.3. *Let $s \in (0,1)$, $\delta > 0$, Ω an open set. Let E be a nonlocal almost minimal set. This means that for any $x_0 \in \partial E$, for any $r < \min(\delta, d(x_0, \partial\Omega))$ and for any measurable set F with $E\Delta F \subset B(x_0, r)$, the following holds*

$$P_s(E, \Omega) \leq P_s(F, \Omega) + Kr^N.$$

Then E has boundary of class C^1 outside of a closed singular set S of Hausdorff dimension at most $N-3$.

We start by showing that any solution to the isoperimetric problem (3) is actually also a local minimizer for a suitably defined unconstrained problem.

Lemma 3.4. *Let (10) hold. Let E be a minimizer of (3) with $|E| = m$. Then there exists $R > 0$ and μ_0, depending on E, such that $E \subseteq B_{R/2}$ and E is a solution to*

$$\min_{F \in B_R} \left(P_s(F) - \int_F g(x)dx + \mu \, ||F| - m| \right)$$

for every $\mu \geq \mu_0$.

Proof. First of all, without loss of generality, for simplicity we let $m = 1$. Let E be a minimizer of $\mathcal{F}$ among sets F with $|F| = 1$. Then, by Proposition 3.2 there exist R depending on E, N, s and $\|g\|_\infty$ such that $E \subseteq B_{R/2}$.

We argue by contradiction and we assume there exists a sequence $\mu_n \to +\infty$ and $F_n \subseteq B_R$ such that

$$P_s(F_n) - \int_{F_n} g(x)dx + \mu_n ||F_n| - 1| < P_s(E) - \int_E g(x)dx. \tag{19}$$

We observe that $||F_n| - 1| > 0$, since otherwise we would get a contradiction to the previous inequality by minimality of E among sets of volume 1.

From now on we assume $\mu_n > \|g\|_{L^\infty(B_R)}$ for every n. We observe that

$$\left| \int_{F_n} g(x)dx \right| \leq \|g\|_{L^\infty(B_R)}|F_n| \leq \|g\|_{L^\infty(B_R)}||F_n| - 1| + \|g\|_{L^\infty(B_R)}.$$

Using this computation and minimality of F_n, say (19), we get that there exists C independent of n such that

$$P_s(F_n) \leq P_s(E) - \int_E g(x)dx - (\mu_n - \|g\|_{L^\infty(B_R)})||F_n| - 1| + \|g\|_{L^\infty(B_R)} \leq C,$$

and

$$(\mu_n - \|g\|_{L^\infty(B_R)})||F_n| - 1| \leq P_s(E) - \int_E g(x)dx + \|g\|_{L^\infty(B_R)} \leq C.$$

In particular this implies that $|F_n| \to 1$ as $n \to +\infty$.

Let $\lambda_n = |F_n|^{-1/N}$. Then, by the computation above, $\lambda_n \to 1$ as $n \to +\infty$. We define $\tilde{F}_n = \lambda_n F_n$. So, by definition $|\tilde{F}_n| = 1$ and, by minimality of E, we get

$$\begin{aligned} P_s(E) - \int_E g(x)dx &\leq P_s(\tilde{F}_n) - \int_{\tilde{F}_n} g(x)dx = \lambda_n^{N-s} P_s(F_n) - \lambda_n^N \int_{F_n} g(\lambda_n x)dx \\ &\leq \lambda_n^{N-s} P_s(F_n) - \lambda_n^N \int_{F_n} g(x)dx + \lambda_n^N \int_{F_n} |g(\lambda_n x) - g(x)|dx \\ &\leq \lambda_n^{N-s} P_s(F_n) - \lambda_n^N \int_{F_n} g(x)dx + RK_g(R)|\lambda_n - 1| \qquad (20) \end{aligned}$$

where $K_g(R)$ is the Lipschitz constant of g in B_R. So, using both (19) and (20), we obtain that

$$\mu_n||F_n| - 1| < (\lambda_n^{N-s} - 1)P_s(F_n) - (\lambda_n^N - 1)\int_{F_n} g(x)dx + K_g(R)R|\lambda_n - 1|.$$

So, we divide both sides by $||F_n| - 1| = |\lambda_n^N - 1|\lambda_n^{-N}$ and we obtain, recalling that $P_s(F_n) \leq C$,

$$\mu_n < \frac{|\lambda_n^{N-s} - 1|}{|\lambda_n^N - 1|}\lambda_n^N C + |\lambda_n^N - 1|\|g\|_{L^\infty(B_R)} + \lambda_n^N \|g\|_{L^\infty(B_R)} + RK_g(R)\frac{|\lambda_n - 1|}{|\lambda_n^N - 1|}\lambda_n^N.$$

So, in particular, recalling that $\lambda_n \to 1$ as $n \to +\infty$, we get that

$$\mu_n \leq C$$

for some constant depending on $R, \|g\|_{L^\infty(B_R)}, K_g(R), N, s$, in contradiction with the assumption that $\mu_n \to +\infty$. □

Corollary 3.5. *Assume* (10). *Let E be a minimizer of* (3). *Then ∂E is of class $C^{2,\alpha}$ for every $\alpha < s$, up to a closed singular set S of Hausdorff dimension at most $N-3$.*

Proof. Observe that Lemma 3.4 implies that E is a nonlocal almost minimal set in B_R. Take $\delta < R/2$, $\Omega = B_R$ and $K = (\|g\|_{L^\infty(B_R)} + \mu_0)\omega_N$. Then, for any $x_0 \in \partial E$, any $r < \delta$ and any measurable set F with $E\Delta F \subset B(x_0, r)$, the following inequality holds

$$\begin{aligned} P_s(E) &\leq P_s(F) + \|g\|_{L^\infty(B_R)}|E\Delta F| + \mu_0||E| - |F|| \\ &\leq P_s(F) + (\|g\|_{L^\infty(B_R)} + \mu_0)|E\Delta F| \leq P_s(F) + Kr^N. \end{aligned}$$

Therefore, we can apply Theorem 3.3 and conclude that ∂E is of class C^1, up to a closed singular set S of Hausdorff dimension at most $N-3$. Eventually we use the bootstrap argument in [2, Theorem 1.5] and the Lipschitz regularity of g to improve the regularity of ∂E from C^1 to $C^{2,\alpha}$ for any $\alpha < s$. □

4 Asymptotics of minimizers for small volumes

In this section we discuss the asymptotic behavior of minimizers of (3) in the small volume regime. We will prove in particular that the volume term becomes irrelevant for small volumes.

First of all observe that if E is a minimizer of (3) with mass constraint $|E| = m$, then $E_\lambda = \lambda E$ is a minimizer of

$$\mathcal{F}_\lambda(E) = P_s(E) - \lambda^{-s} \int_E g\left(\frac{x}{\lambda}\right) dx, \tag{21}$$

among all sets of volume $|E| = \lambda^N m$. Indeed $\mathcal{F}_\lambda(E_\lambda) = \lambda^{N-s}\mathcal{F}(E)$.

We show that minimizers of (3), properly rescaled, tend to a ball as the volume goes to zero.

Proposition 4.1. *Assume that $g \in L^\infty$. Then for $\varepsilon \in (0,1)$ let E_ε be a minimizer of* (3) *with volume constraint $|E_\varepsilon| = \varepsilon^N \omega_N$, and let $\tilde{E}_\varepsilon = \varepsilon^{-1} E_\varepsilon$. Then, as $\varepsilon \to 0$, the sets $\tilde{E}_\varepsilon$ converge in the L^1-topology, up to translations, to the unit ball B, and in particular there holds*

$$\min_{x \in \mathbb{R}^N} \left|\tilde{E}_\varepsilon \Delta B(x,1)\right| \leq C\|g\|_\infty \varepsilon^s\,, \tag{22}$$

where the constant C depends only on N, s.

Proof. Note that by the observation above $\tilde{E}_\varepsilon$ is a minimizer of the functional $\mathcal{F}_{\varepsilon^{-1}}$, defined in (21), among sets of volume ω_N. By minimality of $\tilde{E}_\varepsilon$ we then

get, for every $x \in \mathbb{R}^N$,

$$\begin{aligned} P_s(B) \le P_s(\tilde{E}_\varepsilon) &\le P_s(B(x,1)) - \varepsilon^s \int_{B(x,1)} g(\varepsilon y)\, dy + \varepsilon^s \int_{\tilde{E}_\varepsilon} g(\varepsilon y)\, dy \\ &\le P_s(B) - \varepsilon^s \int_{B(x,1)\setminus \tilde{E}_\varepsilon} g(\varepsilon y)\, dy + \varepsilon^s \int_{\tilde{E}_\varepsilon \setminus B(x,1)} g(\varepsilon y)\, dy \\ &\le P_s(B) + \varepsilon^s \|g\|_\infty |\tilde{E}_\varepsilon \Delta B(x,1)|\,, \end{aligned}$$

which gives

$$P_s(\tilde{E}_\varepsilon) - P_s(B) \le \varepsilon^s \|g\|_\infty |\tilde{E}_\varepsilon \Delta B(x,1)|\,.$$

Recalling the quantitative isoperimetric inequality for the fractional perimeter (see [8, Thm 1.1])

$$\min_{x\in\mathbb{R}^N} \left|\tilde{E}_\varepsilon \Delta B(x,1)\right| \le C(N,s) \left(P_s(\tilde{E}_\varepsilon) - P_s(B)\right)^{\frac{1}{2}}\,,$$

where $C(N,s)$ depends only on N, s, we then get

$$\min_{x\in\mathbb{R}^N} \left|\tilde{E}_\varepsilon \Delta B(x,1)\right| \le C(N,s) \|g\|_\infty^{\frac{1}{2}} \varepsilon^{\frac{s}{2}} \min_{x\in\mathbb{R}^N} \left|\tilde{E}_\varepsilon \Delta B(x,1)\right|^{\frac{1}{2}}\,,$$

from which we obtain (22). □

Remark 4.2. The result in Proposition 4.1 also holds if g belongs to L^∞_{loc} and is coercive. Indeed, the proof is the same once we show that the points x in (22) can be chosen in a fixed compact set, independent of ε, and this can be easily proved reasoning as in Proposition 5.3.

5 Existence result

We now prove existence of minimizers of (3) under suitable assumptions on the function g.

5.1 Periodic case

The first case we consider is when g is $\mathbb{Z}^N$ periodic. We follow the same strategy as in the proof of [10, Thm 4.4] (where the analogous result for the classical perimeter is proved) and [7, Thm 7.2], which is based on a concentration compactness type argument.

Theorem 5.1. *Assume* (10) *and that g is a $\mathbb{Z}^N$ periodic function. Then, for every $m > 0$ there exists a bounded minimizer E to* (3).

Proof. Without loss of generality we shall assume that $m = 1/2$, since the argument is the same for all values of $m > 0$.

We recall a technical Lemma proved in [11, Lemma 4.2].

Lemma 5.2. *Let $C > 0$ and let $\{x_i\}_{i\in\mathbb{N}}$ be a non-increasing sequence of positive numbers such that*

$$\sum_{i=1}^{\infty} x_i^{\frac{N-s}{N}} \le C \qquad \text{and} \qquad \sum_{i=1}^{\infty} x_i = \frac{1}{2}\,.$$

Then there exists $k_0 \in \mathbb{N}$ such that, for all $k \ge k_0$ there holds

$$\sum_{i=k+1}^{\infty} x_i \le \frac{1}{(2Ck)^{\frac{s}{N}}}\,.$$

Let now E_n be a minimizing sequence for (3), that is,

$$\lim_{n\to\infty} \mathcal{F}(E_n) = \inf_{|E|=\frac{1}{2}} \mathcal{F}(E).$$

In particular, since the function g is bounded, we have

$$P_s(E_n) \le \mathcal{F}(E_n) + \int_{E_n} g(x)dx \le C\,,$$

where C does not depend on n. For $n \in \mathbb{N}$, we also let $\{Q_{i,n}\}_{i\in\mathbb{N}}$ be a partition of $\mathbb{R}^N$ into disjoint unit cubes such that the quantities $x_{i,n} = |E_n \cap Q_{i,n}|$ are non-increasing in i. In particular, there holds

$$\sum_{i=1}^{\infty} x_{i,n} = m = \frac{1}{2}. \tag{23}$$

Recalling the fractional isoperimetric inequality (2), which can be also localized in Lipschitz domains (see [7, Lemma 2.5]), we also have

$$\sum_{i=1}^{\infty} x_{i,n}^{\frac{N-s}{N}} \le C \sum_{i=1}^{\infty} P_s(E_n, Q_{i,n}) \le 2CP_s(E_n) \le C',$$

for some constants $C, C' > 0$. By Lemma 5.2 we then obtain that

$$\sum_{i=k+1}^{\infty} x_{i,n} \le c\, k^{-\frac{s}{N}}\,, \tag{24}$$

for some $c > 0$ and for all $k \in \mathbb{N}$. By a diagonal argument, up to extracting a subsequence, we can assume that $x_{i,n} \to \alpha_i$ as $n \to \infty$, for some $\alpha_i \in [0, 1/2]$. By (23) and (24) we then get

$$\sum_{i=1}^{\infty} \alpha_i = \frac{1}{2}\,. \tag{25}$$

Fix now $z_{i,n} \in Q_{i,n}$. Up to extracting a further subsequence, we can suppose that $d(z_{i,n}, z_{j,n}) \to c_{ij} \in [0, +\infty]$, and that there exists $G_i \subseteq \mathbb{R}^N$ such that

$$(E_n - z_{i,n}) \to G_i \quad \text{ in the } L^1_{\text{loc}}\text{-convergence} \tag{26}$$

for every $i \in \mathbb{N}$. We say that $i \sim j$ if $c_{ij} < +\infty$ and we denote by $[i]$ the equivalence class of i. Notice that G_i equals G_j up to a translation, if $i \sim j$. Let $\mathcal{A} := \{[i] : i \in \mathbb{N}\}$. We claim that

$$\sum_{[i]\in\mathcal{A}} P_s(G_i) \le \liminf_{n\to+\infty} P_s(E_n)\,. \tag{27}$$

To prove it, we fix $M \in \mathbb{N}$ and $R > 0$. Let $Q_R = [-R, R]^N$. We take different equivalence classes $i_1, \dots, i_M$ and we notice that if $i_k \neq i_j$ then the set $z_{i_k,n}+Q_R$ is moving far apart from the set $z_{i_j,n} + Q_R$, and so we have

$$\lim_{n\to+\infty} \int_{z_{i_k,n}+Q_R} \int_{z_{i_j,n}+Q_R} \frac{dx\,dy}{|x-y|^{N+s}} = 0.$$

By (26), the lower semicontinuity of the perimeter and (9), we obtain

$$\begin{aligned}
&\sum_{k=1}^{M} P_s(G_{i_k}, Q_R) \le \liminf_{n\to+\infty} \sum_{k=1}^{M} P_s(E_n, (z_{i_k,n} + Q_R))\\
&\le \liminf_{n\to+\infty} P_s\left(E_n, \bigcup_{k=1}^{M}(z_{i_k,n} + Q_R)\right) + 2 \sum_{\substack{1\le k,j\le M\\ i_k\neq i_j}} \int_{z_{i_k,n}+Q_R} \int_{z_{i_j,n}+Q_R} \frac{dx\,dy}{|x-y|^{N+s}}\\
&\le \liminf_{n\to+\infty}\ P_s(E_n).
\end{aligned}$$

By sending first $R \to +\infty$ and then $M \to +\infty$, this yields (27).

Now we claim that

$$\sum_{[i]\in\mathcal{A}} |G_i| = \frac{1}{2}. \tag{28}$$

Indeed, for every $i \in \mathbb{N}$ and $R > 0$ we have

$$|G_i| \ge |G_i \cap Q_R| = \lim_{n\to+\infty} |(E_n - z_{i,n}) \cap Q_R|.$$

If j is such that $j \sim i$ and $c_{ij} \le \frac{R}{2}$, possibly increasing R we have $Q_{j,n} - z_{i,n} \subset Q_R$ for all $n \in \mathbb{N}$, so that

$$\begin{aligned}
|(E_n - z_{i,n}) \cap Q_R| &= \sum_{j=1}^{I_n} |(E_n - z_{i,n}) \cap Q_R \cap (Q_{j,n} - z_{i,n})|\\
&\ge \sum_{j:\, c_{ij}\le\frac{R}{2}} |(E_n - z_{i,n}) \cap Q_R \cap (Q_{j,n} - z_{i,n})| = \sum_{j:\, c_{ij}\le\frac{R}{2}} |(E_n - z_{i,n}) \cap (Q_{j,n} - z_{i,n})|\\
&= \sum_{j:\, c_{ij}\le\frac{R}{2}} |E_n \cap Q_{j,n}|,
\end{aligned}$$

and so

$$|G_i| \ge \lim_{n\to+\infty} |(E_n - z_{i,n}) \cap Q_R| \ge \lim_{n\to+\infty} \sum_{j:\, c_{ij}\le\frac{R}{2}} |E_n \cap Q_{j,n}| = \sum_{j:\, c_{ij}\le\frac{R}{2}} \alpha_j.$$

Letting $R \to +\infty$ we then have

$$|G_i| \geq \sum_{j:\, i\sim j} \alpha_j = \sum_{j\in[i]} \alpha_j \,,$$

hence, recalling (25),

$$\sum_{[i]\in\mathcal{A}} |G_i| \geq \frac{1}{2},$$

thus proving (28) (since the other inequality is trivial).

Let now

$$E_n^{[i]} := E_n \cap \bigcup_{j\sim i} Q_{j,n},$$

and observe that we still have that the sets $(E_n^{[i]} - z_{i,n})$ converge to G_i as $n \to +\infty$, in the $L^1_{\rm loc}$-convergence. As a consequence, we obtain

$$\sum_{[i]\in\mathcal{A}} \int_{G_i} g(x)dx = \lim_{n\to+\infty} \sum_{[i]\in\mathcal{A}} \int_{E_n^{[i]}-z_{i,n}} g(x)dx = \lim_{n\to+\infty} \int_{E_n} g(x)dx\,. \tag{29}$$

Putting together (27) and (29) we then get

$$\sum_{[i]\in\mathcal{A}} \mathcal{F}(G_i) \leq \liminf_{n\to+\infty} \mathcal{F}(E_n) = \inf_{|E|=\frac{1}{2}} \mathcal{F}(E)\,. \tag{30}$$

This means in particular that each set G_i is a minimizer of $\mathcal{F}$ among sets of volume equal to $|G_i|$, hence it is bounded thanks to Proposition 3.2.

Assume now that at least two of the sets G_i's have positive volume, and let $F := \cup_{[i]}(G_i + w_i)$, where the vectors $w_i \in \mathbb{Z}^N$ are chosen in such a way that the sets $(G_i + w_i)$ are pairwise disjoint. Then, by $\mathbb{Z}^N$ periodicity of g, and by (7) we get

$$\mathcal{F}(F) < \sum_{[i]\in\mathcal{A}} (P_s(G_i + w_i) - \int_{G_i+w_i} g(x)dx) = \sum_{[i]\in\mathcal{A}} \mathcal{F}(G_i) \leq \inf_{|E|=\frac{1}{2}} \mathcal{F}(E)\,,$$

thus leading to a contradiction. It follows that there exists $\bar{i}$ such that $|G_{\bar{i}}| = 1/2$, so that $G_{\bar{i}}$ is a (bounded) minimizer of the functional $\mathcal{F}$. □

5.2 Coercive case

We now assume that g is coercive.

Proposition 5.3. *Assume that g is a measurable function, bounded from above, and coercive. Then for every $m > 0$ there exists a minimizer to* (3).

Proof. The argument is the same as for local perimeter functionals (see [9, Lemma 6]). First of all observe that, up to adding a constant, we can assume that $g \leq 0$.

Let E_n be a minimizing sequence, then

$$P_s(E_n) \leq P_s(E_n) - \int_{E_n} g(x)dx \leq C. \tag{31}$$

For $R > 0$, we compute

$$-(\sup_{\mathbb{R}^N \setminus B_R} g)|E_n \setminus B_R| \leq -\int_{E_n \setminus B_R} g(x)dx \leq P_s(E_n) - \int_{E_n} g(x)dx \leq C.$$

Since by assumption $\sup_{\mathbb{R}^N \setminus B_R} g \to -\infty$ as $R \to +\infty$, this implies that

$$\sup_n |E_n \setminus B_R| \to 0 \qquad \text{as } R \to +\infty. \tag{32}$$

By (31), (32) and the compact embedding of $H^{s/2}$ into L^1, there exists a set E with $|E| = m$ such that, up to a subsequence, $E_n \to E$ in L^1. By the lower semicontinuity of P_s wth respect to the L^1 convergence, it follows that E is a minimizer of (3). □

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