Time-like minimal submanifolds as singular limits of nonlinear wave equations

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Abstract

We consider the sharp interface limit $\epsilon \to 0^+$ of the semilinear wave equation $\Box \mathbf{u} + \nabla W(\mathbf{u})/\epsilon^2 = 0$ in $\mathbb{R}^{1+n}$, where $\mathbf{u}$ takes values in $\mathbb{R}^k$, $k = 1, 2$, and W is a double-well potential if $k = 1$ and vanishes on the unit circle and is positive elsewhere if $k = 2$. For fixed $\epsilon > 0$ we find some special solutions, constructed around minimal surfaces in $\mathbb{R}^n$. In the general case, under some additional assumptions, we show that the solutions converge to a Radon measure supported on a time-like k-codimensional minimal submanifold of the Minkowski space-time. This result holds also after the appearence of singularities, and enforces the observation made by J. Neu that this semilinear equation can be regarded as an approximation of the Born-Infeld equation.

1 Introduction

In this paper we consider the following system of semilinear hyperbolic equations

$$\Box \mathbf{u} + \frac{1}{\epsilon^2} \nabla W(\mathbf{u}) = 0\,, \tag{1}$$

for

$$\mathbf{u} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^k, \qquad n \geq 1,\ k = 1, 2,$$

where $\Box \mathbf{u} = \mathbf{u}_{tt} - \Delta \mathbf{u} = \partial_{x^0 x^0} \mathbf{u} - \partial_{x^i x^i} \mathbf{u}$ is the wave operator in $\mathbb{R}^{1+n}$ with coordinates $x^0 = t, x^1, \ldots, x^n$, $\epsilon > 0$ is a small parameter, and $W(\mathbf{u}) = \widetilde{W}(|\mathbf{u}|)$, where $\widetilde{W} : \mathbb{R} \to \mathbb{R}^+$ is a double-well potential. Equation (1) is a Lorentz invariant field equation, governing the dynamics of topological defects such as vortices [9]; it is also strictly related to time-like lorentzian minimal submanifolds of codimension k in Minkowski $(1 + n)$-dimensional space-time [10]. We refer to [12] for a discussion on the existence of local and global solutions to (1). The elliptic/parabolic analog of (1) is called the Ginzburg-Landau equation, and has been recently investigated by many authors in connection with euclidean minimal surfaces and mean curvature flow in codimension k (see for instance [2] and references therein). Here we are interested in the asymptotic limit as $\epsilon \to 0^+$ of solutions $\mathbf{u}_\epsilon$ to (1). The case $k = 1$ will be referred to as the scalar case, since (1) reduces to a single equation, and solutions will be denoted by u_ϵ; note that in this case, the vacuum states ± 1 are stable solutions.

For $n = 3$ and $k = 1$, the asymptotic limit as $\epsilon \to 0^+$ of u_ϵ has been formally computed by Neu in [10], using suitable asymptotic expansions. The author shows that there are solutions which take the constant values ± 1 out of a transition layer of thickness ϵ, provided such a layer is suitably close to a one-codimensional time-like lorentzian minimal surface Σ. The one-codimensional time-like minimal surface equation can be described as follows: the points $(x^0, x^1, \cdots, x^n)$ on each time-slice $\Sigma(t) := \Sigma \cap \{x^0 = t\}$ of Σ must satisfy the equation

$$A = (1 - V^2)\kappa \tag{2}$$

in normal direction, where A, V and κ are

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respectively the acceleration, the velocity and the euclidean mean curvature of $\Sigma(t)$ at $(x^0, x^1, \cdots, x^n)$. We point out that Eq. (2) is the Euler-Lagrange equation of the n-dimensional area induced by the Minkowski metric, given by

$$\mathcal{A}(\Sigma) = \int_\Sigma \sqrt{-\nu_0^2 + |\hat{\nu}|^2}\, d\mathcal{H}^n,$$

where $\nu = (\nu_0, \hat{\nu})$ is a unit euclidean normal to Σ, and $\mathcal{H}^n$ is the n-dimensional euclidean Hausdorff measure. We refer to [3], [8], [5] for the analysis of various aspects of Eq. (2). Interestingly, Neu [10] showed also that, due to possible oscillations on a small scale on the initial interface, which are not dissipated in time, solutions to (1) may not converge to a solution of (2), as the oscillation scale tends to zero.

In the first part of the present paper we compute some explicit selfsimilar solutions of (1). In particular, we show that, given any euclidean nondegenerate minimal hypersurface M in $\mathbb{R}^n$, there exists a solution to (1) traveling around M (see Propositions 2.2 and 2.4). In the second part of the paper we adapt to the hyperbolic setting the parabolic strategy followed in [1]. Given a solution $\mathbf{u}_\epsilon$ to (1) let

$$\ell_\epsilon(\mathbf{u}_\epsilon) := c_k(\epsilon)\left(\frac{-|\mathbf{u}_{\epsilon t}|^2 + |\nabla \mathbf{u}_\epsilon|^2}{2} + \frac{W(\mathbf{u}_\epsilon)}{\epsilon^2}\right)$$

be the rescaled lagrangian integrand, where

$$c_k(\epsilon) := \begin{cases} \epsilon & \text{if } k = 1, \\ \frac{1}{|\log \epsilon|} & \text{if } k = 2. \end{cases}$$

In our main result (Theorem 3.3) we show that, under some technical assumptions, $\ell_\epsilon(\mathbf{u}_\epsilon)$ concentrates on a k-codimensional set Γ, as $\epsilon \to 0^+$. Moreover, Γ is a time-like lorentzian minimal submanifold whenever it is smooth. In order to prove this result we suitably extend the notion of rectifiable varifold to the lorentzian setting, and prove that the stress-energy tensor of the solutions of (1) converges to a stationary lorentzian varifold, as $\epsilon \to 0^+$. The proof of Theorem 3.3 naturally leads to Definition 3.1, which generalizes the concept of minimality, with respect to the Minkowski area, to nonsmooth k-codimensional submanifolds. A weak notion of lorentzian minimal submanifold (a lorentzian stationary varifold in our case) seems here to be unavoidable, in view of the presence of singularities.

Finally, we conclude the paper by discussing the validity of our assumptions in relation to the example of Neu [10].

1.1 Notation

Throughout the paper bold letters will refer to the case $k = 2$. The greek indices $\alpha, \beta, \gamma, \delta$ run from 0 to n, while the roman indices i, j run from 1 to n; we adopt the Einstein summation convention over repeated indices.

We let $\eta^{-1} = \mathrm{diag}(-1, 1, \ldots, 1)$ be the inverse Minkowski metric tensor with contravariant components $\eta^{\alpha\beta}$; $\eta_{\alpha\beta}$ are the covariant components of the Minkowski metric tensor η.

Given $\xi = (\xi_0, \hat{\xi}) \in \mathbb{R} \times \mathbb{R}^n$ we set $|\hat{\xi}|^2 := \eta^{ij}\hat{\xi}_i\hat{\xi}_j$,

$$\langle \xi, \xi\rangle_m := -(\xi_0)^2 + |\hat{\xi}|^2 = \eta^{\alpha\beta}\xi_\alpha\xi_\beta,$$

and if $\langle \xi, \xi\rangle_m \neq 0$ we set $|\xi|_m := \frac{\langle \xi,\xi\rangle_m}{|\langle \xi,\xi\rangle_m|^{1/2}}$.

We say that ξ is space-like (resp. time-like) if $\langle \xi, \xi\rangle_m > 0$ (resp. $\langle \xi, \xi\rangle_m < 0$). Given a $(1,1)$-tensor A, we say that A is space-like (resp. time-like) if $A\xi$ is space-like (resp. time-like) for all $\xi \in \mathbb{R} \times \mathbb{R}^n \setminus \{(0,0)\}$.

∇ (resp. $\overline{\nabla}$) indicates the euclidean gradient in $\mathbb{R}^n$ (resp. in $\mathbb{R}^{1+n}$); for a smooth function $g : \mathbb{R}^{1+n} \to \mathbb{R}$ we set $\nabla_m g := (-g_t, \nabla g) = \eta^{\alpha\beta}\frac{\partial g}{\partial x^\beta} = \eta^{-1}\overline{\nabla} g$.

$\mathcal{H}^h$ denotes the h-dimensional euclidean area (i.e. the Hausdorff measure) either in $\mathbb{R}^n$ or in $\mathbb{R}^{1+n}$ for $h \in \{0, \ldots, n\}$; $\llcorner$ is the symbol of restriction of measures and $\rightharpoonup$ denotes the weak convergence of Radon measures. If μ is a measure absolutely continuous with respect to λ, we write $\mu << \lambda$, and we denote by $d\mu/d\lambda$ the Radon-Nikodym derivative of μ with respect to λ.

We recall that a smooth k-codimensional submanifold M of $\mathbb{R}^n$ without boundary is said minimal if M has vanishing mean curvature. A minimal submanifold $M \subset \mathbb{R}^n$ is said nondegenerate if the second variation of its $(n-k)$-dimensional area, represented by the associated Jacobi operator, is injective.

2 Selfsimilar solutions

Unless otherwise specified, in what follows we take $W(\mathbf{u}) = \frac{1}{4}(1-|\mathbf{u}|^2)^2$ if $n \le 4$, and if $n > 4$ we suppose W to be a function of $|\mathbf{u}|$ with the proper growth at infinity in order problem (1) to be well-posed [12].
We let

$$e_\epsilon(\mathbf{u}_\epsilon) := c_k(\epsilon)\left(\frac{|\mathbf{u}_{\epsilon t}|^2 + |\nabla \mathbf{u}_\epsilon|^2}{2} + \frac{W(\mathbf{u}_\epsilon)}{\epsilon^2}\right)$$

be the rescaled energy integrand of a solution $\mathbf{u}_\epsilon$ of (1). By $|\mathbf{u}_{\epsilon t}|^2$ (resp. $|\nabla \mathbf{u}_\epsilon|^2$) we mean the square euclidean norm of $\mathbf{u}_{\epsilon t} \in \mathbb{R}^k$ (resp. of $\nabla \mathbf{u}_\epsilon$, i.e., the sum of the squares of the elements of the matrix $\nabla \mathbf{u}_\epsilon$).
We notice that the following quantity is conserved for any $t \ge 0$:

$$\int_{\mathbb{R}^n} e_\epsilon(\mathbf{u}_\epsilon(t,x))\ dx = \int_{\mathbb{R}^n} e_\epsilon(\mathbf{u}_\epsilon(0,x))\ dx, \tag{3}$$

assuming the proper growth conditions on the right hand side.

2.1 Traveling waves

Let $k = 1, 2$. We construct solutions of (1), which are traveling waves along a prescribed direction $\nu \in \mathbb{R}^n$, $|\nu| = 1$. Up to a rotation of $\mathbb{R}^n$, we can assume $\nu = (0, \ldots, 0, 1)$. Letting $x = (y, z) \in \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$, we look for traveling wave solutions of (1) of the form

$$\mathbf{u}_\epsilon(t,x) = \mathbf{v}(y, z - vt), \tag{4}$$

for some $v \in (-1,1)$ and a suitable map $\mathbf{v} : \mathbb{R}^n \to \mathbb{R}^k$. Then (1) becomes

$$-\Delta_y \mathbf{v} - (1-v^2)\mathbf{v}_{zz} + \frac{1}{\epsilon^2}\nabla W(\mathbf{v}) = 0, \tag{5}$$

where Δ_y is the Laplacian in $\mathbb{R}^{n-1}$ with respect to the $y = (y^1, \ldots, y^{n-1})$-coordinates.
Let

$$\mathbf{f}(y,z) := \mathbf{v}(y, \sqrt{1-v^2}z). \tag{6}$$

Then $\mathbf{f}$ satisfies the elliptic Ginzburg-Landau system

$$-\Delta \mathbf{f} + \frac{1}{\epsilon^2}\nabla W(\mathbf{f}) = 0. \tag{7}$$

Hence traveling wave solutions of (1), with $v \in (-1,1)$, correspond to solutions of the elliptic system (7).
We recall the following result [11].

Theorem 2.1. *For any smooth nondegenerate embedded minimal submanifold $M \subset \mathbb{R}^n$ of codimension 1 without boundary, there exist solutions f_ϵ of (7) such that*

$$\epsilon\left(\frac{|\nabla f_\epsilon|^2}{2} + \frac{W(f_\epsilon)}{\epsilon^2}\right) \rightharpoonup \sigma\, \mathcal{H}^{n-1} \llcorner M$$

as $\epsilon \to 0^+$, where $\sigma = \sigma(W,n)$ is a positive constant independent of M.

As a consequence our first result is the existence of traveling waves close to any nondegenerate minimal hypersurface of $\mathbb{R}^n$.

Proposition 2.2. *Let $k = 1$. Let $M \subset \mathbb{R}^n$ be a smooth nondegenerate embedded minimal submanifold of codimension 1 without boundary, and let $v \in (-1,1)$. Define*

$$\Sigma := \Big\{ \left(t, y, \sqrt{1-v^2}z + vt\right) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} : (y,z) \in M \Big\}.$$

Then there exist traveling wave solutions $u_\epsilon : \mathbb{R}^{1+n} \to \mathbb{R}$ of (1) of the form (4), such that

$$\ell_\epsilon(u_\epsilon) \rightharpoonup \sigma\, \mu \llcorner \Sigma \qquad \epsilon \to 0^+, \tag{8}$$

where the measure μ is the n-dimensional area induced by the Minkowski metric.

Proof. Set $\gamma := (1-v^2)^{-1/2}$. If f_ϵ are as in Theorem 2.1, we define $u_\epsilon(t,x) := f_\epsilon(y, \gamma(z - vt))$. Then $\ell_\epsilon(u_\epsilon) = \epsilon\left(\frac{|\nabla f_\epsilon|^2}{2} + \frac{W(f_\epsilon)}{\epsilon^2}\right)$, hence if $\varphi \in C_c(\mathbb{R}^{1+n})$,

$$\int_{\mathbb{R}}\int_{\mathbb{R}^n} \ell_\epsilon(u_\epsilon)\varphi\ dxdt = \int_{\mathbb{R}}\int_{\mathbb{R}^n} \epsilon\left(\frac{|\nabla f_\epsilon|^2}{2} + \frac{W(f_\epsilon)}{\epsilon^2}\right)\varphi\ dxdt \tag{9}$$

where the integrand in parentheses is evaluated at $(y, \gamma(z - vt))$. Therefore, making the change of variables $(t', y', z') = L(t,y,z)$, where L is the Lorentz transformation given by

$$L(t,y,z) := (\gamma(t - vz), y, \gamma(z - vt)),$$

we have that (9) equals

$$\int_{\mathbb{R}}\int_{\mathbb{R}^n} \epsilon\left(\frac{|\nabla f_\epsilon|^2}{2} + \frac{W(f_\epsilon)}{\epsilon^2}\right)\varphi\ dx'dt' \to \sigma \int_{\mathbb{R}}\int_M \varphi\ d\mathcal{H}^{n-1}(x')\,dt' = \sigma\int_\Sigma \varphi\ d\mu,$$

where μ is the image of $\mathcal{H}^n \llcorner \mathbb{R} \times M$, through the map L^{-1}. $\square$

Remark 2.3. The hypersurface Σ in Proposition 2.2 is a time-like lorentzian minimal hypersurface. Indeed, let $d : \mathbb{R}^n \to \mathbb{R}$ be the signed (euclidean) distance function from M, negative in the interior of M (note that M is the boundary of a smooth open subset of $\mathbb{R}^n$), so that $M = \{(y,z) \in \mathbb{R}^n : d(y,z) = 0\}$, $|\nabla d|^2 = 1$ in a neighbourhood of M, and $\Delta d = 0$ on M. Define $g : \mathbb{R}^{1+n} \to \mathbb{R}$ as $g(t,x) := d(y, \gamma(z - vt))$, $x = (y,z)$. Observe that $\Sigma = \{g = 0\}$, so that the minkowskian mean curvature of Σ is given by the euclidean divergence in $\mathbb{R}^{1+n}$ of $\nabla_m g / |\nabla_m g|_m$, namely by

$$\left(\frac{-g_t}{\sqrt{-(g_t)^2 + |\nabla g|^2}}\right)_t + \left(\frac{g_{x^i}}{\sqrt{-(g_t)^2 + |\nabla g|^2}}\right)_{x^i}$$

evaluated on Σ. The equality $|\nabla d|^2 = 1$ implies $\sqrt{-(g_t)^2 + |\nabla g|^2} = 1$ in a neighbourhood of Σ. Therefore we only have to check that

$$-g_{tt} + g_{x^i x^i} = 0 \qquad \text{on } \Sigma, \tag{10}$$

which is verified because $-g_{tt} + g_{x^i x^i}$ on Σ coincides with Δd on M.

Note that $\ell_\epsilon(u_\epsilon)$ concentrates on Σ in the limit $\epsilon \to 0^+$; the same happens for $e_\epsilon(u_\epsilon)$, since $e_\epsilon(u_\epsilon)$, and $\ell_\epsilon(u_\epsilon)$ in Proposition 2.2 are mutually absolutely continuous.

2.2 Rotating waves

In this section we let $W(\mathbf{u}) = (1 - |\mathbf{u}|^2)^2/4$, $\widetilde{W} : \mathbb{R} \to \mathbb{R}$ be defined as $\widetilde{W}(s) := (1 - s^2)^2/4$, and let $k = 2$; we identify the target space $\mathbb{R}^2$ with the complex plane. We look for solutions of (1) of the form

$$\mathbf{u}_\epsilon(t,x) = \rho(x) e^{i\omega t}, \qquad \rho : \mathbb{R}^n \to \mathbb{R}, \tag{11}$$

for some $\omega \in \mathbb{R}$. Substituting (11) into (1), we get that ρ must satisfy

$$-\Delta\rho - \omega^2 \rho + \frac{1}{\epsilon^2} \widetilde{W}'(\rho) = 0. \tag{12}$$

This scalar equation can be rewritten as

$$-\Delta\rho + \frac{1}{\epsilon^2} \widetilde{W}_\epsilon'(\rho) = 0, \tag{13}$$

where

$$\begin{aligned} \widetilde{W}_\epsilon(\rho) &:= \frac{(1 + \epsilon^2\omega^2 - \rho^2)^2}{4} \\ &= (1 + \epsilon^2\omega^2)^2\, \widetilde{W}\left(\frac{\rho}{\sqrt{1 + \epsilon^2\omega^2}}\right). \end{aligned}$$

Therefore (13) reduces to (7) with $k = 1$ and W replaced by $\widetilde{W}$, after the change of variables

$$f(x) = \frac{\rho\left(\frac{x}{\sqrt{1+\epsilon^2\omega^2}}\right)}{\sqrt{1 + \epsilon^2\omega^2}},$$

and we can still apply Theorem 2.1. In particular, we get the following

Proposition 2.4. *Let $M \subset \mathbb{R}^n$ be a smooth nondegenerate embedded minimal submanifold of codimension 1 without boundary, and let $\omega \in \mathbb{R}$. Define*

$$\Sigma := \mathbb{R} \times M.$$

Then there exist solutions $\mathbf{u}_\epsilon : \mathbb{R}^{1+n} \to \mathbb{R}^2$ of (1) of the form (11), such that

$$\epsilon \left(\frac{-|\mathbf{u}_{\epsilon t}|^2 + |\nabla \mathbf{u}_\epsilon|^2}{2}\right) + \frac{W(\mathbf{u}_\epsilon)}{\epsilon} \rightharpoonup \sigma\, \mu \llcorner \Sigma$$

as $\epsilon \to 0^+$, where the measure μ is the n-dimensional area induced by the Minkowski metric.

Proof. If $\varphi \in C_c^\infty(\mathbb{R}^{1+n})$ we have

$$\begin{aligned} &\int_{\mathbb{R}} \int_{\mathbb{R}^n} \frac{\epsilon}{c_2(\epsilon)} \ell_\epsilon(\mathbf{u}_\epsilon) \varphi \, dx dt \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^n} \Big[\epsilon \frac{|\nabla\rho|^2}{2} \\ &\quad + \frac{1}{\epsilon}\left(\widetilde{W}(\rho) + \epsilon^2 \frac{\rho^2\omega^2}{2}\right)\Big] \varphi \, dx dt \\ &\to \sigma \int_{\mathbb{R}} \int_M \varphi \, d\mathcal{H}^{n-1} dt = \sigma \int_\Sigma \varphi \, d\mu. \end{aligned}$$

$\square$

Note that in Proposition 2.4 $\frac{\epsilon}{c_2(\epsilon)} \ell_\epsilon(\mathbf{u}_\epsilon)$ concentrates on the lorentzian minimal submanifold Σ of codimension 1, even if $k = 2$.

3 Convergence as $\epsilon \to 0^+$

We are interested in passing to the limit in (1), as $\epsilon \to 0^+$. As already mentioned in the introduction, a formal limit has been performed in [10] when $k = 1$. Rigorous asymptotic results for well prepared initial data have been recently announced in [7].
From now on we shall assume that there exists a constant $C > 0$ such that

$$\sup_{\epsilon \in (0,1)} \int_{\mathbb{R}^n} e_\epsilon(\mathbf{u}_\epsilon(0,x))\ dx \le C. \tag{14}$$

3.1 Assumptions on ℓ and e

Under assumption (14), from (3) it follows that the measures $e_\epsilon(\mathbf{u}_\epsilon)\ dtdx$ converge, up to a (not relabelled) subsequence as $\epsilon \to 0^+$, to a Radon measure e in $\mathbb{R}^{1+n}$. Since $|\ell_\epsilon(\mathbf{u}_\epsilon)|$ and $c_k(\epsilon)W(\mathbf{u}_\epsilon)/\epsilon^2$ are both bounded by $e_\epsilon(\mathbf{u}_\epsilon)$, they also weakly converge, up to a subsequence, to two measures ℓ and w respectively,

$$\ell_\epsilon(\mathbf{u}_\epsilon)\ dtdx \rightharpoonup \ell, \tag{15}$$

$$c_k(\epsilon)W(\mathbf{u}_\epsilon)/\epsilon^2\ dtdx \rightharpoonup w, \tag{16}$$

and ℓ and w are absolutely continuous with respect to e, with density less than or equal to 1. In the following, we shall assume that

(A1) e is absolutely continuous with respect to ℓ.

Assumption (A1) is a weak way to say that $|\mathbf{u}_{\epsilon t}|^2$ is controlled by $|\nabla \mathbf{u}_\epsilon|^2$, or equivalently that the tensor $\overline{\nabla}\mathbf{u}_\epsilon$, suitably normalized, becomes space-like in the limit $\epsilon \to 0^+$. Such an assumption essentially implies that the set Γ defined in (18) below is time-like.
Following [1], we shall assume also that

(A2) for ℓ-almost every (t,x) it holds

$$0 < \lim_{\rho \to 0^+} \frac{\ell(B_\rho(t,x))}{\rho^{n+1-k}} < +\infty, \tag{17}$$

where $B_\rho(t,x)$ denotes the euclidean ball of radius ρ centered at (t,x). Recalling Preiss' Theorem [4], from (A2) it follows that the support of the measures e and ℓ

$$\Gamma := \mathrm{spt}(\ell) = \mathrm{spt}(e) \tag{18}$$

is a rectifiable set of dimension $n+1-k$, and

$$\ell = f\, \mathcal{H}^{n+1-k} \llcorner \Gamma$$

in the sense of measures, for some Borel measurable function $f > 0$.
Assumption (A2) also ensures that the lagrangian integrands $\ell_\epsilon(\mathbf{u}_\epsilon)$ concentrate on a time-like rectifiable set Γ of codimension k in the limit $\epsilon \to 0^+$. Hence Γ has the correct codimension, but is not necessarily smooth everywhere. We observe at this point that, in general, time-like lorentzian minimal submanifolds are singular, and that the density f defined above is typically 0 at the singular points of Γ; for instance the limit in (17) vanishes if (t,x) is the vertex of half a light-cone. Note carefully that we are not excluding the presence of zero density points on Γ, since (A2) is required to be valid only for ℓ-almost all points, and not for all points of Γ. Differently with respect to the parabolic case considered in [1], we cannot expect here a uniform lower density bound (where the zero on the left hand side of (17) would be substituted by an absolute positive constant).

3.2 Rectifiable lorentzian varifolds

A matrix P represents a lorentzian orthogonal projection on a time-like subspace of codimension k of $\mathbb{R}^{1+n}$ if there exists a Lorentz transformation L such that

$$L^{-1}\, P\, L = \begin{cases} \mathrm{diag}(1,0,1,\ldots,1) & \text{if } k=1, \\ \mathrm{diag}(1,0,0,1,\ldots,1) & \text{if } k=2. \end{cases}$$

The pair of Radon measures $V = (\mu_V, \delta_P)$ is called *rectifiable lorentzian varifold* (without boundary) of codimension k if $\mathrm{spt}(\mu_V) \subset \mathbb{R}^{1+n}$ is an $(n+1-k)$-rectifiable set whose tangent space is time-like $\mathcal{H}^{n+1-k}$-almost everywhere, and δ_P is the Dirac delta concentrated in P, where P is the lorentzian orthogonal projection onto the tangent space to $\mathrm{spt}(\mu_V)$. Notice that, when $k = 1$ and $\mathrm{spt}(\mu_V)$ is smooth and time-like, the orthogonal lorentzian projection P can be written as

$$P = \mathrm{Id} - \eta^{-1}\nu_m \otimes \nu_m,$$

where ν is a normal (co)vector to $\mathrm{spt}(\mu_V)$, and $\nu_m := \nu/|\nu|_m$.

Definition 3.1. *We say that the rectifiable lorentzian varifold $V = (\mu_V, \delta_P)$ is stationary if*

$$\int_{\mathbb{R}^{1+n}} \mathrm{tr}\left(P\ \overline{\nabla}\mathbf{X}\right)\, d\mu_V = 0 \qquad (19)$$

for all vector fields $\mathbf{X} \in (C_c^1(\mathbb{R}^{1+n}))^{n+1}$.

Notice that (19) is equivalent to require that the generalized varifold (μ_V, δ_P) (in the sense of [1, Def. 3.4]) is stationary.

Remark 3.2. When $\mathrm{spt}(\mu_V)$ is smooth, a direct computation [13] shows that condition (19) implies that $\mathrm{spt}(\mu_V)$ is a time-like minimal submanifold of codimension k, and μ_V coincides, up to a positive constant, with the $(n+1-k)$-dimensional Minkowski area restricted to $\mathrm{spt}(\mu_V)$.

3.3 The stress-energy tensor

We let

$$\begin{aligned} T_\epsilon^{\alpha\beta}(\mathbf{u}) \quad := \quad & -c_k(\epsilon)\eta^{\alpha\gamma}\partial_{x^\gamma}\mathbf{u} \cdot \eta^{\beta\delta}\partial_{x^\delta}\mathbf{u} \\ & +\ell_\epsilon(\mathbf{u})\,\eta^{\alpha\beta} \end{aligned}$$

be the contravariant components of the symmetric stress-energy tensor, where $\cdot$ is the euclidean scalar product in $\mathbb{R}^k$. Notice that

$$|T_\epsilon^{\alpha\beta}(\mathbf{u})| \le e_\epsilon(\mathbf{u}), \qquad (20)$$

for any $\alpha, \beta \in \{0, \dots, n\}$. A direct computation shows that a solution $\mathbf{u}_\epsilon$ of (1) satisfies

$$\partial_{x^\beta} T_\epsilon^{\alpha\beta}(\mathbf{u}_\epsilon) = 0. \qquad (21)$$

As a consequence, for every vector field $X \in C_c^1(\mathbb{R}^{1+n})$ we have

$$\int_{\mathbb{R}^{1+n}} \eta\, T_\epsilon^{\alpha\beta}(\mathbf{u}_\epsilon)\, \partial_{x^\beta} X\ dtdx = 0. \qquad (22)$$

Since $|T_\epsilon^{\alpha\beta}(\mathbf{u}_\epsilon)|$ is bounded by $e_\epsilon(\mathbf{u}_\epsilon)$, for any $\alpha, \beta \in \{0, \dots, n\}$ there exists a measure $T^{\alpha\beta}$ such that

$$T_\epsilon^{\alpha\beta}(\mathbf{u}_\epsilon) dtdx \rightharpoonup T^{\alpha\beta} \qquad (23)$$

as $\epsilon \to 0^+$. We denote by T the measure-valued tensor with components $T^{\alpha\beta}$. Note that $T^{\alpha\beta} = T^{\beta\alpha}$ and $\mathrm{spt}(T) = \Gamma$.
From (20) it follows that $T^{\alpha\beta}$ on the right hand side of (23) is absolutely continuous with respect to e, and therefore is also absolutely continuous with respect to ℓ, thanks to (A1). We denote by $\widetilde{T}^{\alpha\beta}$ the density of the measure $T^{\alpha\beta}$ with respect to the measure ℓ, i.e.,

$$\widetilde{T}^{\alpha\beta} := \frac{dT^{\alpha\beta}}{d\ell}, \qquad (24)$$

and by $\widetilde{T}$ the tensor with components $\widetilde{T}^{\alpha\beta}$.
In addition to (A2), we shall also assume that

(A3) for $\mathcal{H}^{n+1-k}$-almost every $x \in \Gamma$, the tensor $\mathrm{Id} - \eta\widetilde{T}(x)$ is space-like.

Recalling the expression of $T_\epsilon(\mathbf{u}_\epsilon) - \ell_\epsilon(\mathbf{u}_\epsilon)\eta^{-1}$, we point out that (A3) is reminiscent to require that the tensor $\overline{\nabla}\mathbf{u}_\epsilon$, suitably normalized, becomes space-like near Γ as $\epsilon \to 0^+$, and that Γ is time-like $\mathcal{H}^{n+1-k}$-almost everywhere. In particular, if $k = 1$ and Γ is smooth, the tensor $\mathrm{Id} - \eta\widetilde{T}$ turns out to be equal to $\eta^{-1}\nu_m \otimes \nu_m$, so that $\eta\widetilde{T}$ is a lorentzian orthogonal projection, and (A3) is equivalent to require that the normal vector ν_m to Γ is space-like $\mathcal{H}^n$-almost everywhere. This is for instance consistent with the explicit solution corresponding to a singular pulsating sphere considered in [14].

3.4 Main result

We are now in a position to prove the main result of the paper.

Theorem 3.3. *Assume that the initial data $\mathbf{u}_\epsilon(0, x)$ satisfy (14). Let ℓ, w and $\widetilde{T}$ be defined as in (15), (16) and (24) respectively. Under the assumptions (A1)-(A3), the following two statements hold:*

(i) Let $k = 1$, and assume further

$$\text{(A4)} \qquad \frac{dw}{dl} = \frac{1}{2}.$$

Then $(\ell, \delta_{\eta\widetilde{T}})$ is a stationary lorentzian rectifiable varifold of codimension one.

(ii) Let $k = 2$. Then $(\ell, \delta_{\eta\widetilde{T}})$ is a stationary lorentzian rectifiable varifold of codimension two.

As a consequence of Theorem 3.3, the set Γ defined in (18) is a rectifiable set of dimension $n+1-k$, and the tangent space to Γ is time-like for $\mathcal{H}^n$-almost everywhere, by assumption (A3). Moreover, in the regions where it is smooth, Γ is a time-like minimal submanifold of codimension k, and ℓ coincides, up to a constant, with the $(n+1-k)$-Minkowski area restricted to Γ.
Assumption (A4) corresponds to the so-called *equipartition.* In the parabolic case and when $k = 1$, the analog of (A4) turns out to be automatically satisfied [6], and in that framework this property shows that $\int_{\mathbb{R}^n} \epsilon|\nabla u_\epsilon(t,\cdot)|^2 \ dx$ and $\int_{\mathbb{R}^n} \frac{1}{\epsilon} W(u_\epsilon(t,\cdot))\ dx$ equally contribute in the limit $\epsilon \to 0^+$. Still in the parabolic case and when $k = 2$, there is no equipartition, and the contribution of $c_2(\epsilon) \int_{\mathbb{R}^n} \frac{1}{\epsilon^2} W(\mathbf{u}_\epsilon(t,\cdot))\ dx$ turns out to be negligible with respect to $c_2(\epsilon) \int_{\mathbb{R}^n} |\nabla \mathbf{u}_\epsilon(t,\cdot)|^2 \ dx$. This has an analog in our hyperbolic case (see formula (33) below).

Proof. Passing to the limit in the linear condition (22) we have

$$\int_{\mathbb{R}^{1+n}} \partial_{x^\beta} X \, d\,\eta T^{\alpha\beta} = 0, \tag{25}$$

for any $\alpha \in \{0,\dots,n\}$. Note that (25) is (the generic component of) the stationarity condition for the lorentzian varifold $(\ell, \delta_{\eta\widetilde{T}})$. Therefore, it is enough to prove that $(\ell, \delta_{\eta\widetilde{T}})$ is a rectifiable lorentzian varifold, i.e. for $\mathcal{H}^{n+1-k}$-almost every $x \in \Gamma$ the matrix $\eta\widetilde{T}(x)$ is the lorentzian orthogonal projection onto the tangent space to Γ at x.
By a rescaling argument around $\mathcal{H}^{n+1-k}$-almost every point $x \in \Gamma$ as in [1, Eq. (3.6)], from (25) we obtain

$$\eta\widetilde{T}(x) \int_{\mathbb{R}^{1+n}} \overline{\nabla}\phi \, d\nu = 0, \tag{26}$$

for all test functions ϕ supported in the euclidean unit ball of $\mathbb{R}^{1+n}$, and for all ν in the tangent space to ℓ at x. As in [1, Lemma 3.9], from (26) it follows that for $\mathcal{H}^{n+1-k}$-almost every $x \in \Gamma$

$$\text{at least } k \text{ eigenvalues of } \eta\widetilde{T}(x) \text{ are zero.} \tag{27}$$

These eigenvalues correspond to the directions in the normal space to Γ at x.
From the equalities

$$\mathrm{tr}(\partial_{x^\alpha}\mathbf{u}_\epsilon \cdot \eta^{\beta\delta}\partial_{x^\delta}\mathbf{u}_\epsilon) = -|\mathbf{u}_{\epsilon t}|^2 + |\nabla\mathbf{u}_\epsilon|^2$$

and

$$c_k(\epsilon)(|\mathbf{u}_{\epsilon t}|^2 - |\nabla\mathbf{u}_\epsilon|^2) = 2\frac{c_k(\epsilon)W(\mathbf{u}_\epsilon)}{\epsilon^2} - 2\ell_\epsilon(\mathbf{u}_\epsilon),$$

we obtain

$$\mathrm{tr}(\eta T_\epsilon(\mathbf{u}_\epsilon)) = 2\frac{c_k(\epsilon)W(\mathbf{u}_\epsilon)}{\epsilon^2} + (n-1)\ell_\epsilon(\mathbf{u}_\epsilon). \tag{28}$$

Passing to the limit as $\epsilon \to 0^+$ we get

$$\mathrm{tr}(\eta T) = 2w + (n-1)\ell \tag{29}$$

in the sense of measures. Considering the density with respect to ℓ we get

$$\mathrm{tr}(\eta\widetilde{T}) = 2\frac{dw}{d\ell} + (n-1). \tag{30}$$

Thanks to assumption (A3), for $\mathcal{H}^{n+1-k}$-almost every $x \in \Gamma$ the tensor $\mathrm{Id} - \eta\widetilde{T}(x)$, is space-like. Therefore, letting $\lambda_0, \lambda_1, \dots, \lambda_n$ be the eigenvalues of $\eta\widetilde{T}(x)$, there exists a Lorentz transformation $L(x)$ such that

$$\begin{aligned} & L^{-1}(x)(\eta\widetilde{T}(x) - \mathrm{Id})L(x) \\ = \ & L^{-1}(x)\,\eta\widetilde{T}(x)\,L(x) - \mathrm{Id} \\ = \ & \mathrm{diag}(0, \lambda_1 - 1, \dots, \lambda_n - 1). \end{aligned}$$

In particular

$$\lambda_0 = 1.$$

Passing to the limit in the expression of $T_\epsilon(\mathbf{u}_\epsilon) - \ell_\epsilon(\mathbf{u}_\epsilon)\eta^{-1}$ as $\epsilon \to 0^+$, we get that $\widetilde{T} - \eta^{-1} = \eta^{-1}(\eta\widetilde{T} - \mathrm{Id})$ is negative semidefinite (in the euclidean sense), which implies

$$\lambda_i \le 1 \qquad \forall i \in \{1,\dots,n\}. \tag{31}$$

From (31) and (27) we then obtain

$$\mathrm{tr}(\eta\widetilde{T}(x)) = \sum_{i=0}^n \lambda_i \le n+1-k\,. \tag{32}$$

Note that equality in (31) holds if and only if $\eta\widetilde{T}(x)$ is a lorentzian orthogonal projection on a time-like subspace of codimension

k. Consequently, our aim is now to prove equality in (32).

Case (i): $k = 1$. Using (A4), (30) becomes $\mathrm{tr}(\eta\widetilde{T}(x)) = n$.

Case (ii): $k = 2$. From (30) and (32) it follows

$$\frac{dw}{d\ell} \le 0.$$

Since w is a positive measure, we deduce

$$\frac{\mathrm{d}w}{\mathrm{d}\ell} = 0. \tag{33}$$

Therefore, (30) becomes $\mathrm{tr}(\eta\widetilde{T}(x)) = n - 1$. $\square$

In [10] it is shown by a formal asymptotic argument (made rigorous in [7]) that the thesis of Theorem 3.3 holds true when $k = 1$, for well-prepared initial data and before the appearence of singularities. However, the "small ripples" example in [10] suggests that Theorem 3.3 (i) may not be true in general, without assuming (A4). In particular, differently from the parabolic case [6], we expect that (A4) is not always satisfied for not well-prepared initial data.

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