An example of three dimensional fattening for linked space curves evolving by curvature

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Abstract

We prove that two linked circles in $\mathbb{R}^3$ evolving by curvature develop three dimensional fattening at finite time. The same result holds for two planar convex closed linked curves.

1 Introduction

The so called “fattening phenomenon” is a peculiar type of singularity which appears in motion by mean curvature of boundaries [17], and more generally in geometric evolutions of manifolds. In case of motion by mean curvature of boundaries, fattening can be defined [23], [10] as follows: if we let evolve an initial set E, and we denote by v the unique mean curvature viscosity solution having the signed distance function from ∂E as initial datum, then fattening occurs if $\{x : v(t,x) = 0\}$ has nonempty interior part.
A more intrinsic definition (consistent with the above one) of fattening can be given by using the theory of barriers and minimal barriers of De Giorgi [20], [14]; such a definition is meaningful also in case of nonuniqueness of viscosity solutions, and for evolutions of manifolds of arbitrary codimension.

The complete characterization of those boundaries which fatten up under motion by mean curvature is clearly a difficult open problem; let us briefly recall what is known in this respect. Examples of fattening in two dimensions for curvature flow can be given if the initial set E is not smooth (Evans and Spruck [23] provided an example with the set bounded by the figure eight curve), or if the boundary ∂E is not compact (Ilmanen [28] provided an example of a smooth non compact curve for which the signed distance function restricted to the tubular neighbourhood of width ρ is not smooth for any ρ). On the other hand, if $E \subseteq \mathbb{R}^2$ has compact smooth boundary, fattening does not occur, as a consequence of a theorem of Grayson [26] (see also [10] and [15] for related results). In dimension $n \geq 3$ the situation is much more complicated: first of all, as a consequence of a result of Huisken [27], a smooth bounded strictly convex set $E \subseteq \mathbb{R}^n$ flowing by mean curvature does not develop fattening. A few years ago De Giorgi [17] conjectured that a suitable torus in $\mathbb{R}^n$ flowing by mean curvature should develop fattening, but the conjecture was disproved by Soner-Souganidis in [30] (see also [29], [1]). In three dimensions there are no examples of smooth *compact* sets which develop fattening at finite time under mean curvature flow, while for non compact smooth initial sets, results in this direction can be found in the paper of Angenent-Chopp-Ilmanen [7] (see also [18], [31]). Numerical simulations by Fierro and Paolini in [24] suggest the existence of a torus in $\mathbb{R}^4$ which should develop fattening. Angenent-Ilmanen-Velazquez [8], [9] gave examples of fattening in dimension four through seven of smooth non compact hypersurfaces.

We are concerned with the evolution of curves in $\mathbb{R}^3$ in the generalized sense of De Giorgi [20], developped further by Ambrosio and Soner [4]; see also [13], and [2], [21] concerning singularity formation.

For this evolution law we can speak of k-dimensional fattening, $k > 1$, when the k-dimensional Hausdorff measure of the evolving curve becomes non zero at some positive time. The special case of the figure eight curve in $\mathbb{R}^3$ provides an easy example of 2-dimensional fattening, whereas almost nothing is known concerning 3-dimensional fattening. In this paper, using the theory of minimal barriers, we show that an initial datum E which is the union of two disjoint linked circles in $\mathbb{R}^3$ develops three-dimensional fattening (see Theorem 4.1). The study of this example was suggested by De Giorgi, as a first step in order to understand the relations between the possible singularities of the evolution and the codimension of the flowing manifold.

We prove, more generally, that two convex planar linked space closed curves evolving by curvature develop three-dimensional fattening. We expect that, differently from the codimension one situation, the present example of singularity is stable with respect to small perturbations of the initial datum.

The main tools are the theory of minimal barriers, the comparison principle, and some results on the evolution by curvature of Lipschitz continuous curves.

2 Notation and main definitions

In the following for simplicity we let $I := [0, +\infty[$. We denote by $\mathcal{P}(\mathbb{R}^3)$ the family of all subsets of $\mathbb{R}^3$. Given $\rho > 0$ and $x \in \mathbb{R}^3$ we denote by $B_\rho(x)$ the open ball centered at x with radius ρ; by $\langle \cdot, \cdot \rangle$ we mean the standard scalar product in $\mathbb{R}^3$. If γ is a planar or space curve, we denote by (γ) the trace of γ. We define $\mathrm{dist}(\cdot, \emptyset) := +\infty$ and, given a set $E \subseteq \mathbb{R}^3$, we set,

$$d_E(x) := \mathrm{dist}(x, E) - \mathrm{dist}(x, \mathbb{R}^n \setminus E), \qquad \eta_E(x) := \frac{1}{2}\big(\mathrm{dist}(x, E)\big)^2.$$

It is well known that, if E has smooth compact boundary, then d_E is smooth in a suitable tubular neighbourhood U of ∂E, ∇d_E is, on ∂E, the exterior unit normal to ∂E, and the restriction of $\nabla^2 d_E$ to the tangent space to ∂E coincides with the second fundamental form of ∂E.

The following results on the square distance function have been proved in [3], [4]. Let γ be a smooth closed embedded curve in $\mathbb{R}^3$; then η_γ is smooth in a suitable tubular neighbourhood Ω of γ. On γ the matrix $\nabla^2 \eta_\gamma$ represents the orthogonal projection on the normal space to γ; if $y \in \Omega$, $\nabla^2 \eta_\gamma(y)$ has exactly 2 eigenvalues equal to one, and the remaining eigenvalue is strictly smaller than one. Precisely, if $\pi(y) := y - \nabla \eta_\gamma(y)$ is the (unique) orthogonal projection of y on γ, then

$$\mu_1(y) = \frac{d_\gamma(y)\kappa(y)}{1 + d_\gamma(y)\kappa(y)}, \qquad \mu_2(y) = \mu_3(y) = 1, \tag{2.1}$$

where $\mu_1(y), \mu_2(y), \mu_3(y)$ are the eigenvalues of $\nabla^2 \eta_\gamma(y)$ and $\kappa(y)$ is the curvature of γ at $\pi(y)$ along $\nabla d_\gamma(y)$.

Notice that, if $y \in \Omega \setminus \gamma$, then $\nabla^2 d_\gamma(y) = \dfrac{1}{d_\gamma(y)} \big(\nabla^2 \eta_\gamma(y) - \nabla d_\gamma(y) \otimes \nabla d_\gamma(y)\big)$.

Therefore,

$$\lambda_1(y) = \frac{\kappa(y)}{1 + d_\gamma(y)\kappa(y)}, \qquad \lambda_2(y) = \frac{1}{d_\gamma(y)}, \qquad \lambda_3(y) = 0, \tag{2.2}$$

where $\lambda_1(y), \lambda_2(y)$ are the eigenvalues of $\nabla^2 d_\gamma(y)$ corresponding to eigenvectors orthogonal to $\nabla d_\gamma(y)$, and $\lambda_3(y)$ is the vanishing eigenvalue corresponding to $\nabla d_\gamma(y)$.

Finally, $-\Delta \nabla \eta_\gamma$ coincides, on γ, with the curvature vector of γ.

Given a map $\phi : L \to \mathcal{P}(\mathbb{R}^3)$, where $L \subseteq \mathbb{R}$ is a closed interval, we denote by $d_\phi, \eta_\phi : L \times \mathbb{R}^3 \to \mathbb{R}$ the functions defined as

$$\begin{aligned} d_\phi(t, x) &:= \mathrm{dist}\big(x, \phi(t)\big) - \mathrm{dist}\big(x, \mathbb{R}^n \setminus \phi(t)\big) = d_{\phi(t)}(x), \\ \eta_\phi(t, x) &:= \frac{1}{2}\big(\mathrm{dist}(x, \phi(t)\big)^2 = \eta_{\phi(t)}(x). \end{aligned}$$

We notice that $-\frac{\partial d_\phi}{\partial t}(t,x)$ corresponds to the normal expansion rate of $\phi(t)$ at $x \in \partial\phi(t)$.
If $\phi_1, \phi_2 : L \to \mathcal{P}(\mathbb{R}^3)$, by $\phi_1 \subseteq \phi_2$ (resp. $\phi_1 = \phi_2$, $\phi_1 \cap \phi_2$) we mean $\phi_1(t) \subseteq \phi_2(t)$ (resp. $\phi_1(t) = \phi_2(t)$, $\phi_1(t) \cap \phi_2(t)$) for any $t \in L$.
Let p be a given vector of $\mathbb{R}^3 \setminus \{0\}$ and set $P_p := \mathrm{Id} - p \otimes p/|p|^2$; if Sym(3) stands for the space of all symmetric (3×3)-matrices, we denote by $F : (\mathbb{R}^3 \setminus \{0\}) \times \mathrm{Sym}(3) \to \mathbb{R}$ the function defined as follows:

$$F(p, X) := -\lambda_1(p, X), \tag{2.3}$$

where $\lambda_1(p,X) \leq \lambda_2(p,X)$ are the eigenvalues of the matrix $P_p X P_p$ which correspond to eigenvectors orthogonal to p.

We now define the family $\mathcal{F}$ of all smooth evolutions of compact surfaces without boundary, expanding with normal velocity less than or equal to $F(\cdot,\cdot)$.

Definition 2.1. *Let $a, b \in \mathbb{R}$, $a < b$, $[a,b] \subseteq I$ and let $f : [a,b] \to \mathcal{P}(\mathbb{R}^3)$. We write $f \in \mathcal{F}$ if and only if the following conditions hold:*

(i) $f(t)$ is closed and $\partial f(t)$ is compact for any $t \in [a,b]$;

(ii) there exists an open set $A \subseteq \mathbb{R}^3$ such that $d_f \in \mathbf{C}^\infty([a,b] \times A)$ and $\partial f(t) \subseteq A$ for any $t \in [a,b]$;

(iii) the following inequality holds on $\partial f(t)$:

$$\frac{\partial d_f}{\partial t}(t,x) + F\big(\nabla d_f(t,x), \nabla^2 d_f(t,x)\big) \geq 0, \quad t \in [a,b],\ x \in \partial f(t). \tag{2.4}$$

We write $f \in \mathcal{F}^{\leq}$ (resp. $f \in \mathcal{F}^{=}$) if the inequality $\leq$ (resp. the equality) holds in (2.4).

We also define the family $\mathcal{G}$ of all smooth local curvature evolutions of closed embedded curves.

Definition 2.2. *Let $a, b \in \mathbb{R}$, $a < b$, $[a,b] \subseteq I$, and let $\gamma : [a,b] \to \mathcal{P}(\mathbb{R}^3)$. We write $\gamma \in \mathcal{G}$ if and only if the following conditions hold:*

(i) $\gamma(t)$ is compact for any $t \in [a,b]$;

(ii) there exists an open set $A \subseteq \mathbb{R}^3$ such that $\eta_\gamma \in \mathbf{C}^\infty\big([a,b] \times A\big)$, $\gamma(t) \subseteq A$ for any $t \in [a,b]$, and $\mathrm{rank}(\nabla^2 \eta_\gamma) = 2$ for any $t \in [a,b]$, $x \in \gamma(t)$;

(iii) the following system of equalities holds on $\gamma(t)$:

$$\frac{\partial \nabla \eta_\gamma}{\partial t}(t,x) - \Delta \nabla \eta_\gamma(t,x) = 0, \qquad t \in [a,b],\ x \in \gamma(t). \tag{2.5}$$

The properties of η_γ listed at the beginning of this section, together with the observation that $-\frac{\partial \nabla \eta_\gamma}{\partial t}$ represents the projection of the velocity onto the normal space (see [4]), motivates (iii) of Definition 2.2. Short time existence for curvature flow of smooth initial closed space curves is a consequence of a general theorem proved in [25].

Remark 2.3. *The families $\mathcal{F}$ and $\mathcal{G}$ are translation invariant in space, that is, if $f \in \mathcal{F}$ (resp. $\gamma \in \mathcal{G}$) then $f + y \in \mathcal{F}$ (resp. $\gamma + y \in \mathcal{G}$) for any $y \in \mathbb{R}^3$. Using this fact one can check that, if $A \subseteq \mathbb{R}^3$ is an open set, then $\mathcal{M}(A, \mathcal{F})(t)$ and $\mathcal{M}(A, \mathcal{G})(t)$, which will be defined in (2.6), are open sets for any $t \in I$.*

Let us recall the definitions of geometric barrier and inner barrier in the sense of De Giorgi [19], [20], [12].

Definition 2.4. *A map ϕ is a barrier with respect to $\mathcal{F}$ if and only if ϕ maps a closed interval $L \subseteq I$ into $\mathcal{P}(\mathbb{R}^3)$ and the following property holds: if $f : [a,b] \subseteq I \to \mathcal{P}(\mathbb{R}^3)$ belongs to $\mathcal{F}$ and $f(a) \subseteq \phi(a)$, then $f(b) \subseteq \phi(b)$. Given such a map ϕ, we shall write $\phi \in \mathcal{B}(\mathcal{F}, L)$. When $L = I$, we simply write $\phi \in \mathcal{B}(\mathcal{F})$.*
A function ψ is an inner barrier with respect to $\mathcal{F}^{\leq}$ if and only if ψ maps a closed interval set $L \subseteq I$ into $\mathcal{P}(\mathbb{R}^3)$ and the following property holds: if $f : [a,b] \subseteq L \to \mathcal{P}(\mathbb{R}^3)$ belongs to $\mathcal{F}^{\leq}$ and $\psi(a) \subseteq int\big(f(a)\big)$ then $\psi(b) \subseteq int\big(f(b)\big)$. Given such a map ψ, we shall write $\psi \in \widetilde{\mathcal{B}}(\mathcal{F}^{\leq}, L)$. When $L = I$, we simply write $\psi \in \widetilde{\mathcal{B}}(\mathcal{F}^{\leq})$.

A barrier $\phi \in \mathcal{B}(\mathcal{F})$ corresponds to a weak notion of a set whose boundary $\partial\phi$ expands with normal velocity greater than or equal to $F(\cdot,\cdot)$.

Definition 2.5. *Let $t_0 \in \mathbb{R}$ and let $E \subseteq \mathbb{R}^3$ be a given set. The* minimal barrier *$\mathcal{M}(E, \mathcal{F}, t_0) : [t_0, +\infty[\to \mathcal{P}(\mathbb{R}^3)$, with origin at $E \subseteq \mathbb{R}^n$ (at time t_0), is defined as*

$$\mathcal{M}(E, \mathcal{F}, t_0)(t) := \bigcap \Big\{\phi(t) : \ \phi \in \mathcal{B}(\mathcal{F}, [t_0, +\infty[), \phi(t_0) \supseteq E\Big\}. \tag{2.6}$$

The maximal inner barrier *$\mathcal{N}(E, \mathcal{F}^{\leq}, t_0) : [t_0, +\infty[\to \mathcal{P}(\mathbb{R}^3)$ is defined by*

$$\mathcal{N}(E, \mathcal{F}^{\leq}, t_0)(t) := \bigcup \Big\{\psi(t) : \ \psi \in \widetilde{\mathcal{B}}(\mathcal{F}^{\leq}, [t_0, +\infty[), \ \psi(t_0) \subseteq E\Big\}.$$

If $\rho > 0$, we set $E^+_\rho := \big\{x \in \mathbb{R}^n : \mathrm{dist}(x, E) < \rho\big\}$, and

$$\mathcal{M}^*\big(E, \mathcal{F}, t_0\big) := \bigcap_{\rho>0} \mathcal{M}\big(E^+_\rho, \mathcal{F}, t_0\big), \qquad \mathcal{N}^*(E, \mathcal{F}^{\leq}, t_0) := \bigcap_{\rho>0} \mathcal{N}(E^+_\rho, \mathcal{F}^{\leq}, t_0). \tag{2.7}$$

Notice that $\mathcal{M}^*(E,\mathcal{F},t_0) \in \mathcal{B}(\mathcal{F},[t_0,+\infty[)$; moreover, one can prove that $\mathcal{N}^*(E,\mathcal{F}^{\leq},t_0) \in \widetilde{\mathcal{B}}(\mathcal{F}^{\leq},[t_0,+\infty[)$, see [12].
We define $\mathcal{B}(\mathcal{G},L)$, $\mathcal{M}(E,\mathcal{G},t_0)$, $\mathcal{M}^*(E,\mathcal{G},t_0)$ by replacing $\mathcal{F}$ with $\mathcal{G}$ in Definitions 2.4, 2.5 and in (2.7), respectively.
In the following, to simplify the notation, we often omit the explicit dependence on t_0 of all barriers (and hence on $[t_0,+\infty[$); we always omit this dependence when $t_0 = 0$.

Remark 2.6. *The following observations hold.*

(i) *If E is contained in a affine subspace V, then $\mathcal{M}^*(E,\mathcal{F})$ and $\mathcal{N}^*(E,\mathcal{F}^{\leq})$ are contained in V.*

(ii) *$\mathcal{M}(E,\mathcal{F})$, $\mathcal{N}(E,\mathcal{F}^{\leq})$, $\mathcal{M}^*(E,\mathcal{F})$ and $\mathcal{N}^*(E,\mathcal{F}^{\leq})$ verify the semigroup property in time, i.e.,*

$$\mathcal{M}(E,\mathcal{F},t_1)(t_3) = \mathcal{M}\big(\mathcal{M}(E,\mathcal{F},t_1)(t_2),\mathcal{F},t_2\big)(t_3), \qquad t_0 \leq t_1 \leq t_2 \leq t_3,$$

and similarly for $\mathcal{N}(E,\mathcal{F}^{\leq})$, $\mathcal{M}^*(E,\mathcal{F})$, $\mathcal{N}^*(E,\mathcal{F}^{\leq})$, see [12].

(iii) *$\mathcal{M}^*(\cdot,\mathcal{F})$ extends the family $\mathcal{G}$, that is, if $\gamma : [a,b] \to \mathbb{R}^3$ belongs to $\mathcal{G}$, then $\mathcal{M}^*(\gamma(a),\mathcal{F})(t) = \gamma(t)$ for any $t \in [a,b]$, see [4].*

The next theorem shows (i) the connection between the minimal barrier and the maximal inner barrier (see [12]), and (ii) the connection between the minimal barriers and the viscosity solutions (see [11]).

Theorem 2.7. *For any bounded set $E \subseteq \mathbb{R}^n$ there holds*

$$\mathcal{M}^*\big(E,\mathcal{F}\big) = \mathcal{N}^*\big(E,\mathcal{F}^{\leq}\big) = V(E), \tag{2.8}$$

where $V(E)$ denotes the zero sublevel set of the continuous viscosity solution of $\dfrac{\partial u}{\partial t} + F(\nabla u, \nabla^2 u) = 0$, in the sense of [16], having $\min(d_E(x),1)$ as initial datum.

We now define the fattening phenomenon.

Definition 2.8. *Let $E \subseteq \mathbb{R}^3$ be a set with empty interior. We say that E develops α-dimensional fattening, $\alpha \in]1,3]$, if, for some $t_1 \geq \overline{t}$, there holds*

$$\mathcal{H}^\alpha\Big(\mathcal{M}^*(E,\mathcal{F},\overline{t})(t)\Big) = 0 \qquad \text{for } t \in [\overline{t},t_1], \tag{2.9}$$

$$\mathcal{H}^\alpha\Big(\mathcal{M}^*(E,\mathcal{F},\overline{t})(t)\Big) > 0 \qquad \text{for some } t \in]t_1,+\infty[\ , \tag{2.10}$$

where $\mathcal{H}^\alpha$ denotes the α-dimensional Hausdorff measure in $\mathbb{R}^3$.

References concerning examples of fattening for mean curvature flow in codimension one have been listed in the Introduction; in higher codimension, we recall the following observation, which is proved in [12].

Remark 2.9. *In $\mathbb{R}^3$, the union of the three coordinate axes (istantly) develops three-dimensional fattening. The same argument applies to show that if the set E is the union of n lines in $\mathbb{R}^n$, meeting at a given point, and not contained in an affine hyperplane, then E (istantly) develops n-dimensional fattening.*

3 Preliminary lemmas

In this section we give some preliminary results needed to prove Theorem 4.1. The following lemma shows that, if γ is a closed space curve smoothly evolving by curvature, then the boundaries of small tubular neighbourhoods of $\gamma(t)$ are flowing surfaces verifying (2.4).

Lemma 3.1. *Let $\gamma : [a,b] \subseteq I \to \mathcal{P}(\mathbb{R}^3)$, $\gamma \in \mathcal{G}$ and choose $\sigma > 0$ such that η_γ is smooth on $\Omega := \{(t,x) : t \in [a,b], \eta_\gamma(t,x) < \sigma^2\}$. Then the map $f : [a,b] \to \mathcal{P}(\mathbb{R}^3)$ defined by $f(t) := \{x \in \mathbb{R}^3 : dist(x, \gamma(t)) \leq \sigma\}$ belongs to $\mathcal{F}$. Hence if $\phi : I \to \mathcal{P}(\mathbb{R}^3)$ is a map such that $\phi(t)$ is an open set for any $t \in I$, then*

$$\phi \in \mathcal{B}(\mathcal{F}) \Rightarrow \phi \in \mathcal{B}(\mathcal{G}). \tag{3.1}$$

It follows that, if $E \subseteq \mathbb{R}^3$, then $\mathcal{M}^*(E,\mathcal{F}) \supseteq \mathcal{M}^*(E,\mathcal{G})$.

Proof. See [4, Theorem 3.8 and Remark 6.2]. □

Actually there holds $\mathcal{M}^*(E,\mathcal{F}) = \mathcal{M}^*(E,\mathcal{G})$, see [13]; we will not need this more refined result in the sequel.

The next lemma concerns an immersed closed space curve smoothly evolving by curvature with the exception of a single point which belongs to the curve for all times and remains fixed.

Lemma 3.2. *Let $G : [0,T[\to \mathcal{P}(\mathbb{R}^3)$ be a time dependent family of closed immersed curves verifying the following properties:*

- *(i) there exists $z \in \mathbb{R}^3$ such that $z \in G(t)$ for any $t \in [0,T[$;*
- *(ii) $G(t) \setminus \{z\}$ smoothly evolves by curvature for any $t \in [0,T[$, in the sense that, for any $t \in [0,T[$, each point of $G(t) \setminus \{z\}$ has a neighbourhood where η_G is smooth and the system in (2.5) holds at that point;*
- *(iii) for any open ball B such that $z \in \partial B$, there holds $B \cap G(t) \neq \emptyset$ for any $t \in]0,T[$.*

Then

$$G \in \widetilde{\mathcal{B}}\big(\mathcal{F}^{\leq}, [0,T[\big). \tag{3.2}$$

Roughly speaking, condition (iii) means that, around z, the evolving curves are not contained in a half space for *strictly* positive times. Notice that we do not assign any kinematic condition at the point z (also, at z the curves are not necessarily smooth).

Proof. Let $f:\ [a,b]\subseteq[0,T[\to\mathcal{P}(\mathbb{R}^3)$, $f\in\mathcal{F}^{\leq}$, $\mathrm{int}\big(f(a)\big)\supseteq G(a)$. We have to prove that $\mathrm{int}\big(f(b)\big)\supseteq G(b)$. Let $\delta:\ [a,b]\to[0,+\infty[$ be defined as $\delta(t):=\mathrm{dist}\big(\partial f(t),G(t)\big)$, $t\in[a,b]$. To prove the lemma it is enough to show that δ is nondecreasing on $[a,b]$; as δ is continuous, it is sufficient to prove that δ is nondecreasing on $]a,b[$.

Fix $t\in\,]a,b[$ and choose $x\in G(t)$, $y\in\partial f(t)$ such that $\delta(t)=|x-y|$. Assumption (iii) implies that $x\in G(t)\setminus\{z\}$, since the open ball of radius $|z-y|$ and centered at y must intersect $G(t)$. Hence η_G is smooth in a suitable neighbourhood of (t,x). Let us show that

$$\frac{\partial d_f}{\partial t}(t,y)\leq\Big\langle\frac{\partial\nabla\eta_G}{\partial t}(t,x),\nabla d_f(t,y)\Big\rangle \tag{3.3}$$

(recall that $-\frac{\partial d_f(t,y)}{\partial t}$ is the expanding velocity of $f(t)$ at y, and $-\frac{\partial\nabla\eta_G}{\partial t}(t,x)$ is the projection of the velocity of $G(t)$ at x onto the normal space).

Let $\lambda(t,y)$ be the smallest eigenvalue of $\nabla^2 d_f(t,y)$ corresponding to an eigenvector orthogonal to $\nabla d_f(t,y)$. Since $f\in\mathcal{F}^{\leq}$, we have

$$\frac{\partial d_f}{\partial t}(t,y)\leq\lambda(t,y).$$

We claim now that, by the choice of x and y, there holds

$$\lambda(t,y)\leq\big\langle\nabla\Delta\eta_G(t,x),\nabla d_f(t,y)\big\rangle. \tag{3.4}$$

Indeed, for any $\epsilon\in\,]0,|x-y|[$ we have that, in a suitable local orthonormal basis of $\mathbb{R}^3$, the matrix $\nabla^2 d_G(t,y)$ is diagonal, and precisely by (2.2)

$$\nabla^2 d_{G^+_\epsilon}(t,y)=\nabla^2 d_G(t,y)=\mathrm{diag}\left(\frac{\kappa(y)}{1+d_G(t,y)\kappa(y)},\frac{1}{d_G(t,y)},0\right),$$

where $d_{G^+_\epsilon}(t,z)=\mathrm{dist}(z,G(t)^+_\epsilon)-\mathrm{dist}(z,\mathbb{R}^3\setminus G(t)^+_\epsilon)=d_G(t,z)-\epsilon$, and $\kappa(y)$ is the curvature of $G(t)$ at x along $\nabla d_f(t,y)$. As $G(t)^+_\epsilon\subseteq f(t)$ for any $\epsilon\in\,]0,|x-y|[$ and $\mathrm{dist}(G(t)^+_\epsilon,\mathbb{R}^3\setminus f(t))=|x-y|-\epsilon$, we have

$$\nabla^2 d_f(t,y)\leq\nabla^2 d_{G^+_\epsilon}(t,\pi_\epsilon(x)),$$

where $\pi_\epsilon(x):=x+\epsilon\nabla d_G(t,y)$. Letting $\epsilon\to 0$ we get (3.4).

Since by hypothesis (ii) we have $\nabla\Delta\eta_G(t,\cdot)=\frac{\partial\nabla\eta_G}{\partial t}(t,\cdot)$ on $G(t)\setminus\{z\}$, inequality (3.3) follows from (3.4). Using (3.3) one can check that $\liminf\limits_{\tau\to 0^+}\frac{\delta(t+\tau)-\delta(t)}{\tau}\geq 0$, which in turn implies that δ is nondecreasing on $]a,b[$. □

Lemma 3.3. *Let $\gamma(0) \subseteq \mathbb{R}^2$ be a bounded planar curve with endpoints in p, q, whose trace is the graph of a Lipschitz continuous function. Then $\gamma(0)$ admits a smooth Dirichlet evolution $\gamma(t)$ by curvature, with fixed extremes in p, q, for any $t \in \,]0, +\infty[$.*

Proof. See [22]. □

Lemma 3.4. *Let $P, Q \in \mathbb{R}^3$, and let $\gamma(0)$ be a given Lipschitz continuous embedded curve with endpoints in P, Q. Assume that, for $t \in \,]0, T[$, $\gamma(t)$ is a smooth Dirichlet evolution by curvature, with fixed extremes in P, Q, having $\gamma(0)$ as initial datum. Let E be a subset of $\mathbb{R}^3$ such that*

$$E \supseteq \gamma(0) \qquad \text{and} \qquad P, Q \in \mathcal{M}^*(E, \mathcal{F})(t) \quad \text{for any } t \in [0, T[.$$

Then

$$\mathcal{M}^*(E, \mathcal{F})(t) \supseteq \gamma(t) \qquad \text{for any } t \in [0, T[.$$

Proof. It is enough to show that the map $t \in [0, T[\to \psi(t) := \gamma(t) \cup \mathcal{M}^*(E, \mathcal{F})(t)$ belongs to $\widetilde{\mathcal{B}}\big(\mathcal{F}^{\leq}, [0, T[\big)$. Indeed by Definition 2.5 and the first equality in (2.8) this fact would imply $\psi \subseteq \mathcal{N}\big(E, \mathcal{F}^{\leq}\big) \subseteq \mathcal{M}^*(E, \mathcal{F})$ which entails the result. We now prove the claim $\psi \in \widetilde{\mathcal{B}}\big(\mathcal{F}^{\leq}, [0, T[\big)$.
Let $f : [a, b] \subseteq [0, T[\to \mathcal{P}(\mathbb{R}^3)$, $f \in \mathcal{F}^{\leq}$, $\psi(a) \subseteq \mathrm{int}\big(f(a)\big)$; we have to show that $\psi(b) \subseteq \mathrm{int}\big(f(b)\big)$. Let $\delta : [a, b] \to [0, +\infty]$ be defined as $\delta(t) := \mathrm{dist}\big(\partial f(t), \psi(t)\big)$; it will be enough to show that $\delta(t) \geq \delta(a)$ for any $t \in [a, b]$. As the family $\mathcal{F}$ is translation invariant in space, the distance from f to $\mathcal{M}^*(E, \mathcal{F})$ is nondecreasing in time, so that, for any $\xi \in \mathcal{M}^*(E, \mathcal{F})(t)$ and $t \in [a, b]$, we have

$$\begin{aligned} &\mathrm{dist}\big(\xi, \partial f(t)\big) \geq \mathrm{dist}\big(\mathcal{M}^*(E, \mathcal{F})(t), \partial f(t)\big) \\ &\geq \mathrm{dist}\big(\mathcal{M}^*(E, \mathcal{F})(a), \partial f(a)\big) \geq \delta(a) > 0, \qquad t \in [a, b]. \end{aligned} \tag{3.5}$$

Assume now by contradiction that

$$\delta(t) < \delta(a) \qquad \text{for some } t \in \,]a, b]. \tag{3.6}$$

Choose $x \in \psi(t)$ and $y \in \partial f(t)$ such that $\delta(t) = |x - y|$. From (3.5) it follows that $x \in \gamma(t) \setminus \{P, Q\}$. Then, reasoning as in the proof of Lemma 3.2, we get

$$\liminf_{\tau \to 0^+} \frac{\delta(t + \tau) - \delta(t)}{\tau} \geq 0.$$

Hence, for any $s \in \,]a, b]$ such that $\delta(s) < \delta(a)$ there holds $\liminf_{\tau \to 0^+} \frac{\delta(s+\tau) - \delta(s)}{\tau} \geq 0$, which implies $\delta(s) \geq \delta(a)$ on $]a, b]$, contradicting (3.6). □

4 The example

The next theorem illustrates the example of the two linked circles.

Theorem 4.1. *Let $0 < c < R$ and let $E \subseteq \mathbb{R}^3$ be the union of the two disjoint linked circles E_-, E_+,*

$$E := E_- \cup E_+,$$
$$E_- := \left\{(0, x_2, x_3) : \ (x_2 + c)^2 + x_3^2 = R^2\right\}$$
$$E_+ := \left\{(x_1, x_2, 0) : \ x_1^2 + (x_2 - c)^2 = R^2\right\}.$$

Then E develops 3-dimensional fattening at finite time, i.e.,

$$\mathcal{H}^3\big(\mathcal{M}^*(E, \mathcal{F})(t)\big) > 0,$$

for $t > \bar{t}$ arbitrarily close to a certain $\bar{t} > 0$.

Proof. Denote by $E_-(t)$ (resp. $E_+(t)$) the smooth evolution by curvature of E_- (resp. of E_+), for $t \in [0, t^\dagger[$, where $t^\dagger$ is the extinction time, and let $0 < \bar{t} < t^\dagger$ be the collision time of the two evolving circles. We also set $T := t^\dagger - \bar{t}$.
By (iii) of Remark 2.6 we have

$$\mathcal{M}^*(E, \mathcal{F})(t) = E_-(t) \cup E_+(t), \qquad t \in [0, \bar{t}\,[.$$

It is easy to prove that $\mathcal{M}^*(E, \mathcal{F})(\bar{t}) = E_-(\bar{t}) \cup E_+(\bar{t})$. Clearly, $E_-(\bar{t}) \cup E_+(\bar{t})$ is the union of the two circles centered at $(0, -c, 0)$ and $(0, c, 0)$, of radius c, and lying respectively in the (x_2, x_3)-plane and in the (x_1, x_2)-plane.
For notational simplicity, we set

$$M(t) := \mathcal{M}^*\big(E_-(\bar{t}) \cup E_+(\bar{t}), \mathcal{F}, \bar{t}\big)(t), \qquad t \geq \bar{t}.$$

Observe that, by the semigroup property of $\mathcal{M}^*$ (see (ii) of Remark 2.6) there holds

$$M(t) = \mathcal{M}^*(E, \mathcal{F})(t), \qquad t \geq \bar{t} \tag{4.1}$$

(recall that by $\mathcal{M}^*(\cdot, \mathcal{F})(t)$ we mean $\mathcal{M}^*(\cdot, \mathcal{F}, 0)(t)$). Therefore, in order to prove the theorem we need to show that

$$\mathcal{H}^3\big(M(t)\big) > 0, \tag{4.2}$$

for times $t > \bar{t}$ arbitrarily close to $\bar{t}$.

Step 1. Construction of the immersed evolving curve $\gamma(t)$ in the (x_1, x_2)-plane.

Let $\gamma(0)$ be the curve, lying in the (x_1, x_2)-plane, whose trace $(\gamma(0))$ is the union of the two circles $\gamma^-(0)$, $\gamma^+(0)$ of radius c, tangent at the origin,

$$\gamma^-(0) = \{(x_1, x_2) : x_1^2 + (x_2 + c)^2 = c^2\}, \ \ \gamma^+(0) = \{(x_1, x_2) : x_1^2 + (x_2 - c)^2 = c^2\}.$$

As $\gamma(0)$ is an immersed curve of class $\mathbf{C}^{1,1}$, there exists the evolution by curvature starting from $\gamma(0)$, in the sense of smooth planar *immersed* curves (see [5], [6]), which we shall denote by $\gamma(t)$. Such an evolution is smooth for $t \in]0,T[$, and has the shape of a figure eight curve; more precisely, by symmetry, its trace $(\gamma(t))$ is the union of two parts $(\gamma^-(t))$, $(\gamma^+(t))$, where $(\gamma^-(t)) \cap (\gamma^+(t)) = \{(0,0)\}$, and $(\gamma^-(t))$ (resp. $(\gamma^+(t))$) lies in the half plane $\{(x_1,x_2) : x_2 \leq 0\}$ (resp. $\{(x_1,x_2) : x_2 \geq 0\}$). Notice that, at the origin, $\gamma(t)$ has a transverse intersection, i.e., the two branches of $\gamma(t)$ meet with an angle less than π (this is a consequence of the strong maximum principle governing the curvature equation, see for instance [26]). Notice also that $\gamma^-(t)$ (resp. $\gamma^+(t)$) coincides with the Dirichlet curvature flow of $\gamma^-(0)$ (resp. of $\gamma^+(0)$) keeping the origin as fixed point of the evolution.

Step 2. Definition of the evolving space curves $G(t)$.

For any $t \in [0,T[$ denote by $G(t)$ the space curve whose trace is $r(\gamma^-(t)) \cup (\gamma^+(t))$, where $r(\gamma^-(t))$ is the rotation of $(\gamma^-(t))$ around its symmetry axis x_2 of $\pi/2$; $G(t)$ is then simply constructed by the figure eight curve $\gamma(t)$ by rotating one of its two parts by $\pi/2$ around the x_2-axis (see Figure 1). Clearly, $(G(t))$ contains the origin of $\mathbb{R}^3$ for any $t \in [0,T[$.
Notice that $M(\bar{t}) = G(0)$; moreover, $G(t)$ satisfies all assumptions of Lemma 3.2 (take z as the origin of $\mathbb{R}^3$), hence we have $G \in \widetilde{\mathcal{B}}(\mathcal{F}^{\leq}, [0,T[)$ by (3.2). Therefore, since by the semigroup property of $\mathcal{M}^*$ and (2.8) we have $M(t) = \mathcal{M}^*\big(M(\bar{t}), \mathcal{F}, \bar{t}\big)(t) = \mathcal{N}^*\big(M(\bar{t}), \mathcal{F}^{\leq}, \bar{t}\big)(t)$ for $t \geq \bar{t}$, we deduce

$$M(\bar{t}+t) = \bigcap_{\rho>0} \mathcal{M}\big(M(\bar{t})^+_\rho, \mathcal{F}, \bar{t}\big)(\bar{t}+t) \supseteq G(t) \qquad \text{for any } t \in [0,T[. \tag{4.3}$$

Step 3. Developping of two dimensional fattening.

Given $\sigma \in [0,T[$, let us denote by $\alpha^-_\sigma(t)$ (resp. $\alpha^+_\sigma(t)$), for $t \in [\sigma,T[$, the evolution by curvature starting from the Lipschitz continuous planar curve whose trace is $r(\gamma^-(\sigma))$ (resp. $(\gamma^+(\sigma))$).
The evolutions $\alpha^\pm_\sigma(t)$ belong to the family $\mathcal{G}$ just after their starting time σ, i.e., for $t \in]\sigma,T[$; therefore if $\phi \in \mathcal{B}(\mathcal{G})$,

$$\alpha^\pm_\sigma(\sigma) \subseteq \operatorname{int}\big(\phi(\sigma)\big) \Rightarrow \alpha^\pm_\sigma(t) \subseteq \phi(t), \qquad \text{for } t \in [\sigma,T[\tag{4.4}$$

(here we use well-known semicontinuity properties in time of the barriers, see [12]). Moreover, recalling that $\mathcal{M}\big(M(\bar{t})^+_\rho, \mathcal{F}, \bar{t}\big)(\bar{t}+t)$ is open for any $\rho > 0$ and any $t \geq 0$ (see Remark 2.3), using (3.1) we deduce that $\mathcal{M}\big(M(\bar{t})^+_\rho, \mathcal{F}, \bar{t}\big) \in \mathcal{B}(\mathcal{G})$. Hence, by (4.3) and (4.4), for any $\rho > 0$ we find

$$\mathcal{M}\big(M(\bar{t})^+_\rho, \mathcal{F}, \bar{t}\big)(\bar{t}+t) \supseteq \alpha^-_\sigma(t) \cup \alpha^+_\sigma(t), \qquad t \in [0,T[, \ \sigma \in [0,t], \tag{4.5}$$

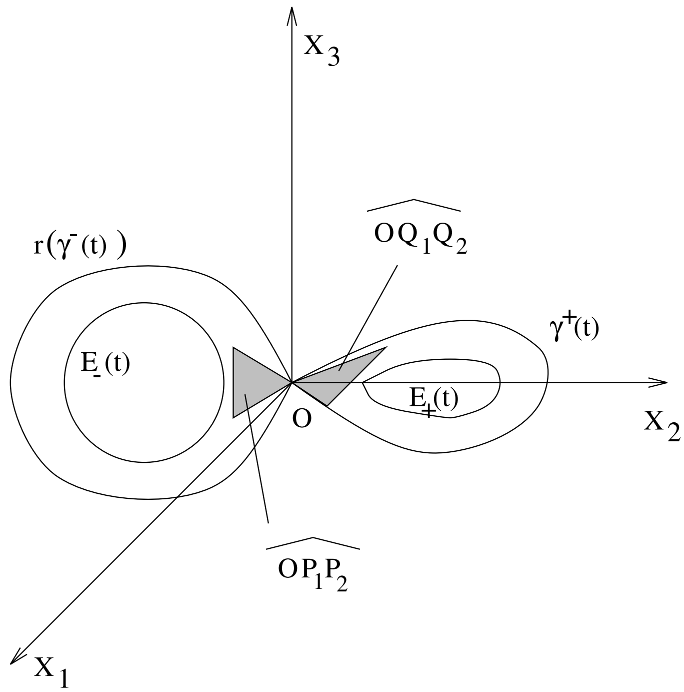

Figure 1: The space curve $G(t)$ and the two dimensional fat region

which implies

$$M(\bar{t}+t) \supseteq \bigcup_{0\leq\sigma\leq t} \big(\alpha_\sigma^-(t) \ \cup \ \alpha_\sigma^+(t)\big), \qquad t \in [0,T[. \tag{4.6}$$

Notice that the right hand side of (4.6) is the union of two planar sets having the shape of "rings" (see Figure 1): the first set lies in the (x_2, x_3)-plane and is bounded by $G(t)$ and $E_-(t) = \alpha_0^-(t)$; the other set lies in the (x_1, x_2)-plane and is bounded by $G(t)$ and $E_+(t) = \alpha_0^+(t)$. Thus, by (4.6),

$$\mathcal{H}^2(M(\bar{t}+t)) > 0, \qquad t \in]0,T[,$$

hence E develops 2-dimensional fattening for times arbitrarily close to $\bar{t}$, and the fat region lies partly in the (x_2, x_3)-plane and partly in the (x_1, x_2)-plane.

Step 4. Conclusion of the proof of (4.2).

Thanks to step 3, for any t_1, t_2 sufficiently small with $0 < t_1 < t_2 < T$, there exists $l > 0$ such that the set

$$\mathcal{K} := \big\{(0, x_2, x_3) \colon -l \leq x_2 \leq 0, |x_3| \leq -x_2\big\} \bigcup \big\{(x_1, x_2, 0) \colon |x_1| \leq x_2, 0 \leq x_2 \leq l\big\}$$

is contained in $M(\bar{t}+t)$ for any $t \in [t_1, t_2]$. The set $\mathcal{K}$ is the union of two triangles, having a common vertex at the origin O of $\mathbb{R}^3$; the first one, which

we denote by $\widehat{OP_1P_2}$, lies in the (x_2, x_3)-plane, and the second one, denoted by $\widehat{OQ_1Q_2}$, lies in the (x_1, x_2)-plane (see Figure 1).
Let P be a point belonging to the segment P_1P_2. Lemma 3.3 implies that there exists a planar Dirichlet evolution (contained in the plane OPQ_1) starting from the curve which is the union of the two segments PO and OQ_1, with fixed extremes in P and Q_1. Since $P, Q_1 \in M(\bar{t}+t)$ for $t \in [t_1, t_2]$, applying Lemma 3.4 (with Q_1 in place of Q), and starting the evolution of the broken line $PO \cup OQ_1$ at any time $t \in [t_1, t_2]$, we obtain that there exist $\rho > 0$ and $\tilde{t} \in]t_1, t_2[$ such that $M(\bar{t}+\tilde{t})$ contains the planar set $B_\rho(O) \cap \widehat{OPQ_1}$.
Since ρ and $\tilde{t}$ can be chosen independently of $P \in P_1P_2$, letting P vary in P_1P_2, we obtain that $M(\bar{t}+\tilde{t})$ contains the set $B_\rho(O) \cap T(O, P_1, P_2, Q_1)$, where $T(O, P_1, P_2, Q_1)$ denotes the tethraedron having vertices in O, P_1, P_2, Q_1. This concludes the proof. □

Remark 4.2. *With an argument similar to the one given in the proof of Theorem 4.1 we can show that three dimensional fattening occurs if the initial set E consists of the disjoint union of two planar convex closed linked curves, if neither one vanishes at the first collision time.*

Notice that our proof is not appliable in the case of two closed planar linked curves, due to the fact that assumption (iii) of Lemma 3.2 is not, in general, satisfied.
We conclude the paper pointing out some open problems, which should deserve further investigation.
Problem. We expect that three dimensional fattening occurs if the initial set E consists of the union of two disjoint closed smooth curves, which are close in the $\mathbf{C}^2$-norm to the set E of Theorem 4.1. If this is the case, then the singularity is stable, with respect to small perturbations in the $\mathbf{C}^2$-norm of the initial datum E.
Problem. "Almost every" curve in $\mathbb{R}^n$, with $n \geq 4$, do not develop fattening; this is a particular case of a conjecture given by De Giorgi.
Problem. If a smooth embedded curve in $\mathbb{R}^3$ develops fattening under curvature flow, then the fattening is necessarily three dimensional.

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