Minimal barriers for geometric evolutions

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Abstract. We study some properties of De Giorgi's minimal barriers and local minimal barriers for geometric flows of subsets of $\mathbf{R}^n$. Concerning evolutions of the form $\frac{\partial u}{\partial t} + F(\nabla u, \nabla^2 u) = 0$, we prove a representation result for the minimal barrier $\mathcal{M}(E, \mathcal{F}_F)$ when F is not degenerate elliptic, namely we show that $\mathcal{M}(E, \mathcal{F}_F) = \mathcal{M}(E, \mathcal{F}_{F^+})$, where F^+ is the smallest degenerate elliptic function above F. We also characterize the disjoint sets property and the joint sets property in terms of the function F.

1. Introduction

Minimal barriers were introduced by De Giorgi in [10,9] in a general setting, in order to provide a notion of weak solution for various problems in differential equations. In the particular case of geometric flows of subsets of $\mathbf{R}^n$, the minimal barriers approach can be adapted to different situations, including the flow by mean curvature of manifolds of arbitrary codimension (see [9] and the paper of Ambrosio-Soner [1]), and gives raise to a unique global evolution. In the case of motion by mean curvature of hypersurfaces, we recall the paper of Ilmanen [13], where he introduced the set theoretic subsolutions, which are related to minimal barriers (see (7.10)); recently, also White [17] considered a similar approach for motion by mean curvature.

Concerning fully non linear geometric evolutions, it has been shown in [3] (see also [6]) that the minimal barriers recover the level set approach (defined through viscosity solutions, see Evans-Spruck [11], Chen-Giga-Goto [7], Giga-Goto-Ishii-Sato [12]) and that, in general, the minimal barrier selects the maximal viscosity subsolution (in the sense of the aforementioned papers) of the problem at hand. We recall that, to define a unique evolution of a set $E \subseteq \mathbf{R}^n$ by means of the barriers approach, no degenerate ellipticity condition is required and no assumption on E is needed.

The aim of this paper is to study general properties of minimal barriers for geometric evolutions of subsets of $\mathbf{R}^n$. We begin in Section 3 by defining the generalized evolution $\mathcal{M}(E, \mathcal{F}, \bar{t})(t)$ of any set $E \subseteq \mathbf{R}^n$ at time $t \geq \bar{t}$ (where $\mathcal{F}$

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is any family of set-valued maps) as the minimum in the class $\mathcal{B}(\mathcal{F})$ of all $\mathcal{F}$-barriers starting from E at $t = \bar{t}$, where the minimality is with respect to sets inclusion. Then $\mathcal{M}(E, \mathcal{F}, \bar{t})$ is unique, verifies the comparison principle and, under minor assumptions on $\mathcal{F}$, it satisfies the semigroup property. We remark that, if we choose $\mathcal{F} = \mathcal{F}_F$ as the family of all smooth local geometric (super) solutions of an equation of the form

$$\frac{\partial u}{\partial t} + F(\nabla u, \nabla^2 u) = 0, \tag{1.1}$$

and if f is an element of $\mathcal{F}_F$ mapping the time interval $[a, b] \subseteq [\bar{t}, +\infty[$ into the class $\mathcal{P}(\mathbf{R}^n)$ of all subsets of $\mathbf{R}^n$, then $\mathcal{M}(f(a), \mathcal{F}_F, a)(t) \supseteq f(t)$ for any $t \in [a, b]$. The equality holds true when F is smooth and uniformly elliptic but, in general, it does not hold for a not degenerate elliptic function F, when it happens that the elements of $\mathcal{F}_F$ are not necessarily $\mathcal{F}_F$-barriers. Related to this observation is Theorem 6.1, which is one of the main results of the present paper (see Section 6 for precise statements). *Assume that F is lower semicontinuous. Denote by $\mathcal{F}_F^{>}$ the family of all strict local geometric supersolutions of (1.1). Then*

$$\mathcal{B}(\mathcal{F}_F^{>}) = \mathcal{B}(\mathcal{F}_{F^+}^{>}), \tag{1.2}$$

where F^+ is the smallest degenerate elliptic function greater than or equal to F, that is

$$F^+(p, X) = \sup\{F(p, Y) : Y \geq X\}.$$

In particular we have $\mathcal{M}(E, \mathcal{F}_F^{>}, \bar{t}) = \mathcal{M}(E, \mathcal{F}_{F^+}^{>}, \bar{t})$.

This result shows that, in presence of a non degenerate elliptic function F, the generalized evolution of any set by (1.1) is governed by the parabolic equation in which F is replaced by F^+.

Given any set $E \subseteq \mathbf{R}^n$ and $\varrho > 0$, let $E_\varrho^- := \{x \in \mathbf{R}^n : \mathrm{dist}\,(x, \mathbf{R}^n \setminus E) > \varrho\}$, $E_\varrho^+ := \{x \in \mathbf{R}^n : \mathrm{dist}(x, E) < \varrho\}$, and define

$$\mathcal{M}_*(E, \mathcal{F}, \bar{t}) := \bigcup_{\varrho > 0} \mathcal{M}(E_\varrho^-, \mathcal{F}, \bar{t}), \qquad \mathcal{M}^*(E, \mathcal{F}, \bar{t}) := \bigcap_{\varrho > 0} \mathcal{M}(E_\varrho^+, \mathcal{F}, \bar{t}).$$

After having studied some properties of $\mathcal{M}_*(E, \mathcal{F}, \bar{t})$ and $\mathcal{M}^*(E, \mathcal{F}, \bar{t})$, such as stability with respect to topological closure and interior part, in Section 3.2 we introduce the disjoint sets property and the joint sets property with respect to $(\mathcal{F}, \mathcal{G})$, where $\mathcal{F}$, $\mathcal{G}$ are two arbitrary families of set-valued maps. Due to elementary counterexamples (to the joint sets property, for instance, in case of motion by curvature in two dimensions) we introduce the regularized versions of these two properties, which read as follows:

$$\begin{aligned} &E_1 \cap E_2 = \emptyset \;\Rightarrow\; \mathcal{M}_*(E_1, \mathcal{F}, \bar{t})(t) \cap \mathcal{M}^*(E_2, \mathcal{G}, \bar{t})(t) = \emptyset, \qquad t \geq \bar{t}, \\ &E_1 \cup E_2 = \mathbf{R}^n \;\Rightarrow\; \mathcal{M}_*(E_1, \mathcal{F}, \bar{t})(t) \cup \mathcal{M}^*(E_2, \mathcal{G}, \bar{t})(t) = \mathbf{R}^n, \qquad t \geq \bar{t}. \end{aligned} \tag{1.3}$$

These two properties play an important role, in general, in geometric evolutions of sets. In Theorems 7.1, 7.3 we characterize (1.3) for geometric evolutions of the form (1.1). More precisely, if we set $F_c(p, X) := -F(-p, -X)$, we have the following assertions.

(i) *Assume that* F, G *are lower semicontinuous. Then the regularized disjoint sets property with respect to* $(\mathcal{F}_F, \mathcal{F}_G)$ *holds if and only if* $(F^+)_c \geq G^+$.

(ii) *Assume that* F, G *are continuous,* $F^+ < +\infty$, $G^+ < +\infty$ *and* F^+, G^+ *are continuous. Then the regularized joint sets property with respect to* $\big(\mathcal{F}_F, \mathcal{F}_G\big)$ *holds if and only if* $(F^+)_c \leq G^+$.

We notice that, in general, the assertions referring to the joint sets property are more difficult to prove that the corresponding ones concerning the disjoint sets property.

As a consequence of Theorems 7.1, 7.3, the following result holds (see Corollary 7.1). *Assume that* F *is continuous,* F^+ *is continuous, and* $F^+ < +\infty$. *Then the regularized disjoint and joint sets properties with respect to* $(\mathcal{F}_F, \mathcal{F}_{F_c})$ *(resp. with respect to* $(\mathcal{F}_F, \mathcal{F}_F)$*) hold if and only if* F *is degenerate elliptic (resp.* F^+ *is odd).*

We remark that the disjoint and joint sets properties, and hence their characterization, are related to the so called fattening phenomenon (see Remark 7.1).

In Section 5 we study the connections between the barriers and the class $\mathcal{B}_{\rm loc}(\mathcal{F})$ of local (in space) barriers (see Definition 5.1). In particular, we prove that *if* F *is lower semicontinuous, then*

$$\mathcal{B}_{\rm loc}\big(\mathcal{F}_F^{\geq}\big) = \mathcal{B}\big(\mathcal{F}_F^{\geq}\big),$$

hence $\mathcal{M}_{\rm loc}\big(E, \mathcal{F}_F^{\geq}, \bar{t}\big) = \mathcal{M}\big(E, \mathcal{F}_F^{\geq}, \bar{t}\big)$, *where* $\mathcal{M}_{\rm loc}\big(E, \mathcal{F}_F^{\geq}, \bar{t}\big)$ *denotes the local minimal barrier.*

Finally, in Section 8 we show the connections between barriers and inner barriers.

The results of this paper have been announced in [4].

2. Some notation

In the following we let $I := [t_0, +\infty[$, for a fixed $t_0 \in \mathbf{R}$. For $n \geq 1$, $x \in \mathbf{R}^n$ and $R > 0$ we set $B_R(x) := \{y \in \mathbf{R}^n : |y - x| < R\}$. We denote by $\mathcal{P}(\mathbf{R}^n)$ (resp. $A(\mathbf{R}^n)$, $C(\mathbf{R}^n)$) the family of all subsets (resp. open subsets, closed subsets) of $\mathbf{R}^n$. Given a set $E \subseteq \mathbf{R}^n$, we denote by $\mathrm{int}(E)$, $\overline{E}$ and ∂E the interior part, the closure and the boundary of E, respectively; moreover, we set $\mathrm{dist}(\cdot, \emptyset) \equiv +\infty$, $d_E(x) := \mathrm{dist}(x, E) - \mathrm{dist}(x, \mathbf{R}^n \setminus E)$, and for any $\varrho > 0$

$$E_\varrho^- := \{x \in \mathbf{R}^n : \mathrm{dist}\,(x, \mathbf{R}^n \setminus E) > \varrho\}, \qquad E_\varrho^+ := \{x \in \mathbf{R}^n : \mathrm{dist}(x, E) < \varrho\}.$$

Given a map $\phi : L \to \mathcal{P}(\mathbf{R}^n)$, where $L \subseteq \mathbf{R}$ is a convex set, we let

$$d_\phi(t, x) := \mathrm{dist}(x, \phi(t)) - \mathrm{dist}(x, \mathbf{R}^n \setminus \phi(t)) = d_{\phi(t)}(x), \qquad (t, x) \in L \times \mathbf{R}^n.$$

By $\mathrm{int}(\phi)$ (resp. $\overline{\phi}$) we mean the map $t \in L \to \mathrm{int}(\phi(t))$ (resp. $t \in L \to \overline{\phi(t)}$). If $\phi_1, \phi_2 : L \to \mathcal{P}(\mathbf{R}^n)$, by $\phi_1 \subseteq \phi_2$ (resp. $\phi_1 = \phi_2$, $\phi_1 \cap \phi_2$, $\phi_1 \cup \phi_2$) we mean $\phi_1(t) \subseteq \phi_2(t)$ (resp. $\phi_1(t) = \phi_2(t)$, $\phi_1(t) \cap \phi_2(t)$, $\phi_1(t) \cup \phi_2(t)$) for any $t \in L$.

3. General definitions and preliminaries

In this section we recall the definition of barriers and minimal barriers, following [9], and we study some of their properties. In the particular case of geometric flows described by a function F as in (1.1), we include the case in which F is not degenerate elliptic.

Definition 3.1. *Let $\mathcal{F}$ be a family of functions with the following property: for any $f \in \mathcal{F}$ there exist $a, b \in \mathbf{R}$, $a < b$, such that $f : [a, b] \to \mathcal{P}(\mathbf{R}^n)$. A function ϕ is a barrier with respect to $\mathcal{F}$ if and only if ϕ maps a convex set $L \subseteq I$ into $\mathcal{P}(\mathbf{R}^n)$ and the following property holds: if $f : [a, b] \subseteq L \to \mathcal{P}(\mathbf{R}^n)$ belongs to $\mathcal{F}$ and $f(a) \subseteq \phi(a)$ then $f(b) \subseteq \phi(b)$. Given such a map ϕ, we shall write $\phi \in \mathcal{B}(\mathcal{F}, L)$. When $L = I$, we simply write $\phi \in \mathcal{B}(\mathcal{F})$.*

Definition 3.2. *Let $E \subseteq \mathbf{R}^n$ be a given set and let $\bar{t} \in I$. The minimal barrier $\mathcal{M}(E, \mathcal{F}, \bar{t}) : [\bar{t}, +\infty[\to \mathcal{P}(\mathbf{R}^n)$ (with origin in E at time $\bar{t}$) with respect to the family $\mathcal{F}$ at any time $t \geq \bar{t}$ is defined by*

$$\mathcal{M}(E, \mathcal{F}, \bar{t})(t) := \bigcap \Big\{ \phi(t) : \ \phi : [\,\bar{t}, +\infty[\to \mathcal{P}(\mathbf{R}^n), \ \phi \in \mathcal{B}\big(\mathcal{F}, [\bar{t}, +\infty[\big), \ \phi(\bar{t}) \supseteq E \Big\}.$$

Lemma 3.1. *The following properties hold.*

(1) $\mathcal{M}(E, \mathcal{F}, \bar{t}) \in \mathcal{B}(\mathcal{F}, [\bar{t}, +\infty[)$;

(2) $E_1 \subseteq E_2 \Rightarrow \mathcal{M}(E_1, \mathcal{F}, \bar{t}) \subseteq \mathcal{M}(E_2, \mathcal{F}, \bar{t})$;

(3) $\mathcal{M}(E, \mathcal{F}, \bar{t})(\bar{t}) = E$;

(4) if $f : [a, b] \subseteq [\bar{t}, +\infty[\to \mathcal{P}(\mathbf{R}^n)$, $f \in \mathcal{F}$, *then*
$$f(t) \subseteq \mathcal{M}(f(a), \mathcal{F}, a)(t), \qquad t \in [a, b]; \tag{3.1}$$

(5) $\mathcal{F} \subseteq \mathcal{G} \Rightarrow \mathcal{B}(\mathcal{F}, [\bar{t}, +\infty[) \supseteq \mathcal{B}(\mathcal{G}, [\bar{t}, +\infty[)$, *hence* $\mathcal{M}(E, \mathcal{F}, \bar{t}) \subseteq \mathcal{M}(E, \mathcal{G}, \bar{t})$;

(6) assume that the family $\mathcal{F}$ satisfies the following assumption: given $f : [a, b] \subseteq [\bar{t}, +\infty[\to \mathcal{P}(\mathbf{R}^n)$, $f \in \mathcal{F}$, $t \in]a, b[$, then $f_{|[a,t]}, f_{|[t,b]} \in \mathcal{F}$. Then $\mathcal{M}(E, \mathcal{F}, \bar{t})$ verifies the semigroup property, i.e.,
$$\mathcal{M}(E, \mathcal{F}, \bar{t})(t_2) = \mathcal{M}\big(\mathcal{M}(E, \mathcal{F}, \bar{t})(t_1), \mathcal{F}, t_1\big)(t_2), \qquad \bar{t} \leq t_1 \leq t_2.$$

Proof. Assertions (1),(2),(5) are immediate, and (4) is a consequence of (1). Using (1) and the fact that $\mathcal{M}(E, \mathcal{F}, \bar{t})(\bar{t}) \supseteq E$, we have that the map $\phi : [\bar{t}, +\infty[\to \mathcal{P}(\mathbf{R}^n)$ defined by $\phi(t) := E$ if $t = \bar{t}$, and $\phi(t) := \mathcal{M}(E, \mathcal{F}, \bar{t})(t)$ if $t > \bar{t}$, is a barrier on $[\bar{t}, +\infty[$, and (3) follows.

Let us prove (6). Let $\phi : [\bar{t}, +\infty[\to \mathcal{P}(\mathbf{R}^n)$ be defined by
$$\phi(t) := \begin{cases} \mathcal{M}(E, \mathcal{F}, \bar{t})(t) & \text{if } \bar{t} \leq t \leq t_1, \\ \mathcal{M}\big(\mathcal{M}(E, \mathcal{F}, \bar{t})(t_1), \mathcal{F}, t_1\big)(t) & \text{if } t \geq t_1. \end{cases}$$
Then $\phi(\bar{t}) = E$ by (3) and, using (1), (3), and the hypothesis on $\mathcal{F}$, we have $\phi \in \mathcal{B}(\mathcal{F}, [\bar{t}, +\infty[)$. Hence $\mathcal{M}(E, \mathcal{F}, \bar{t})(t_2) \subseteq \phi(t_2) = \mathcal{M}(\mathcal{M}(E, \mathcal{F}, \bar{t})(t_1), \mathcal{F}, t_1)(t_2)$. Conversely, since $\mathcal{M}(E, \mathcal{F}, \bar{t})$ is a barrier on $[t_1, +\infty[$ which coincides with $\phi(t_1)$ at $t = t_1$, we have $\mathcal{M}(E, \mathcal{F}, \bar{t})(t_2) \supseteq \mathcal{M}(\phi(t_1), \mathcal{F}, t_1)(t_2) = \phi(t_2)$, and property (6) is proved. □

Definition 3.3. *Let $E \subseteq \mathbf{R}^n$ and $\bar{t} \in I$. For any $t \in [\bar{t}, +\infty[$ we set*
$$\begin{aligned} \mathcal{M}_*(E, \mathcal{F}, \bar{t})(t) &:= \bigcup_{\varrho > 0} \mathcal{M}(E^-_\varrho, \mathcal{F}, \bar{t})(t), \\ \mathcal{M}^*(E, \mathcal{F}, \bar{t})(t) &:= \bigcap_{\varrho > 0} \mathcal{M}(E^+_\varrho, \mathcal{F}, \bar{t})(t). \end{aligned} \tag{3.2}$$

Clearly $\mathcal{M}^*(E, \mathcal{F}, \bar{t}) \in \mathcal{B}(\mathcal{F}, [\bar{t}, +\infty[)$,
$$\mathcal{M}_*(E, \mathcal{F}, \bar{t}) = \mathcal{M}_*(\mathrm{int}(E), \mathcal{F}, \bar{t}) \subseteq \mathcal{M}(E, \mathcal{F}, \bar{t}) \subseteq \mathcal{M}^*(E, \mathcal{F}, \bar{t}) = \mathcal{M}^*(\overline{E}, \mathcal{F}, \bar{t}),$$
and if $A, B \in \mathcal{P}(\mathbf{R}^n)$, $A \subseteq B$, $\mathrm{dist}(A, \mathbf{R}^n \setminus B) > 0$, then $\mathcal{M}^*(A, \mathcal{F}, \bar{t}) \subseteq \mathcal{M}_*(B, \mathcal{F}, \bar{t})$.

Unless otherwise specified, from now on we shall assume $\bar{t} = t_0$, and we often drop it in the notation of $\mathcal{M}, \mathcal{M}_*$ and $\mathcal{M}^*$.

3.1. Consequences of the translation invariance in space. Given a map $\phi : L \to \mathcal{P}(\mathbf{R}^n)$, $L \subseteq I$ a convex set, and $y \in \mathbf{R}^n$, by $\phi + y$ we mean the map $t \in L \to \phi(t) + y := \bigcup_{x \in \phi(t)} (x + y)$.

Definition 3.4. *We say that $\mathcal{F}$ is translation invariant (in space) if, given any $f : [a,b] \to \mathcal{P}(\mathbf{R}^n)$, $f \in \mathcal{F}$, and $y \in \mathbf{R}^n$, then $f + y \in \mathcal{F}$.*
We say that $\mathcal{F}$ is compact if, given $f : [a,b] \to \mathcal{P}(\mathbf{R}^n)$, $f \in \mathcal{F}$, the set $f(t)$ is closed and $\partial f(t)$ is compact for any $t \in [a,b]$.

Notice that if $\mathcal{F}$ is translation invariant, then $\phi \in \mathcal{B}(\mathcal{F})$ if and only if $\phi + y \in \mathcal{B}(\mathcal{F})$ for any $y \in \mathbf{R}^n$.

Many of the following results can be proved under weaker assumptions on $\mathcal{F}$; for instance, when $\mathcal{F} = \mathcal{F}_F$ (see Definition 4.1 below), instead of requiring the translation invariance of $\mathcal{F}_F$, one could allow F to depend explicitly on $(t,x) \in I \times \mathbf{R}^n$, provided that F is uniformly Lipschitz continuous with respect to x. For simplicity, we confine ourselves to the translation invariant case.

Property (3.3) of the next proposition is particularly useful.

Proposition 3.1. *Let $\mathcal{F}$ be translation invariant and $E \subseteq \mathbf{R}^n$. The following properties hold.*

(1) For any $y \in \mathbf{R}^n$ we have $\mathcal{M}(E+y,\mathcal{F}) = \mathcal{M}(E,\mathcal{F}) + y$;

(2) for any $\varrho > 0$ and any $t \in I$ we have

$$\mathcal{M}(E^+_\varrho, \mathcal{F})(t) \supseteq \Big(\mathcal{M}(E,\mathcal{F})(t)\Big)^+_\varrho; \tag{3.3}$$

(3) for any $t \in I$ we have

$$\mathcal{M}_*(E,\mathcal{F})(t) \in A(\mathbf{R}^n), \qquad \mathcal{M}^*(E,\mathcal{F})(t) \in C(\mathbf{R}^n); \tag{3.4}$$

(4) if $\mathcal{F}$ is compact then $\phi \in \mathcal{B}(\mathcal{F}) \Rightarrow int(\phi) \in \mathcal{B}(\mathcal{F})$;

(5) set $\mathcal{F}^c := \{f : [a,b] \to \mathcal{P}(\mathbf{R}^n), f \in \mathcal{F}, f(t)$ is compact for any $t \in [a,b]\}$. Then $\mathcal{M}_*(E,\mathcal{F}^c) \in \mathcal{B}(\mathcal{F}^c)$ and

$$E \in A(\mathbf{R}^n) \Rightarrow \mathcal{M}_*(E,\mathcal{F}^c)(t) = \mathcal{M}(E,\mathcal{F}^c)(t) \in A(\mathbf{R}^n), \qquad t \in I. \tag{3.5}$$

Proof. Letting $\psi(t) = \phi(t) + y$ we have

$$\begin{aligned}\mathcal{M}(E,\mathcal{F})(t) &= \bigcap\{\psi(t) - y : \ \psi : I \to \mathcal{P}(\mathbf{R}^n), \psi \in \mathcal{B}(\mathcal{F}), \psi(t_0) \supseteq E + y\} \\ &= \mathcal{M}(E+y,\mathcal{F})(t) - y,\end{aligned}$$

which is property (1). Therefore, if $\varrho > 0$,

$$\bigcup_{y \in B_\varrho(0)} \mathcal{M}(E+y,\mathcal{F})(t) = \mathcal{M}(E,\mathcal{F})(t) + B_\varrho(0) = \Big(\mathcal{M}(E,\mathcal{F})(t)\Big)^+_\varrho. \tag{3.6}$$

Now $E + y \subseteq E + B_\varrho(0) = E^+_\varrho$ for any $y \in B_\varrho(0)$, hence (3.3) follows from (3.6) and Lemma 3.1 (2). By (3.3) applied with E^+_ϱ replaced by E^-_σ, $\sigma > 0$, and using the fact that $E^-_\sigma \supseteq (E^-_{\sigma+\varrho})^+_\varrho$, we have

$$\mathcal{M}(E^-_\sigma, \mathcal{F}) \supseteq \bigcup_{\varrho > 0} \left(\mathcal{M}(E^-_{\sigma+\varrho}, \mathcal{F})\right)^+_\varrho. \tag{3.7}$$

As the right hand side of (3.7) is an open set, we get

$$\mathrm{int}(\mathcal{M}(E^-_\sigma,\mathcal{F})) \supseteq \bigcup_{\varrho>0} \left(\mathcal{M}(E^-_{\sigma+\varrho},\mathcal{F})\right)^+_\varrho \supseteq \bigcup_{\varrho>0} \mathcal{M}(E^-_{\sigma+\varrho},\mathcal{F}).$$

It follows that $\bigcup_{\sigma>0}\mathrm{int}(\mathcal{M}(E^-_\sigma,\mathcal{F})) \supseteq \bigcup_{\varrho,\sigma>0}\mathcal{M}(E^-_{\sigma+\varrho},\mathcal{F}) = \mathcal{M}_*(E,\mathcal{F})$, which yields

$$\mathcal{M}_*(E,\mathcal{F})(t) = \bigcup_{\varrho>0} \mathrm{int}\Big(\mathcal{M}(E^-_\varrho,\mathcal{F})(t)\Big) \in A(\mathbf{R}^n), \qquad t\in I. \tag{3.8}$$

By (3.3) we have

$$\mathcal{M}(E^+_{2\varrho},\mathcal{F}) \supseteq \overline{\Big(\mathcal{M}(E^+_\varrho,\mathcal{F})\Big)^+_{\varrho/2}} \supseteq \overline{\mathcal{M}(E^+_\varrho,\mathcal{F})}.$$

Hence

$$\mathcal{M}^*(E,\mathcal{F}) = \bigcap_{\varrho>0}\mathcal{M}(E^+_{2\varrho},\mathcal{F}) \supseteq \bigcap_{\varrho>0}\overline{\mathcal{M}(E^+_\varrho,\mathcal{F})},$$

so that $\mathcal{M}^*(E,\mathcal{F})(t) = \bigcap_{\varrho>0}\overline{\mathcal{M}(E^+_\varrho,\mathcal{F})(t)} \in C(\mathbf{R}^n)$ for any $t\in I$. The proof of (3.4) is complete.

Let us prove (4). Let $\phi\in\mathcal{B}(\mathcal{F})$, $f:[a,b]\subseteq I\to\mathcal{P}(\mathbf{R}^n)$, $f\in\mathcal{F}$ and $f(a)\subseteq \mathrm{int}(\phi(a))$. Set $\eta(t) := \mathrm{dist}(f(t),\mathbf{R}^n\setminus\mathrm{int}(\phi(t)))$, $t\in[a,b]$. As $\partial f(a)$ is compact we have $\eta(a)>0$. Since $\mathcal{F}$ is translation invariant and $\phi\in\mathcal{B}(\mathcal{F})$ we have that $\eta(\cdot)$ is non decreasing on $[a,b]$, which implies $f(b)\subseteq\mathrm{int}(\phi(b))$.

It remains to prove (5). Let $f:[a,b]\subseteq I\to\mathcal{P}(\mathbf{R}^n)$, $f\in\mathcal{F}^c$, $f(a)\subseteq \mathcal{M}_*(E,\mathcal{F}^c)(a)$. By (3.8) applied with $\mathcal{F}$ replaced by $\mathcal{F}^c$ there exists $\varrho_1>0$ so that

$$f(a)\subseteq\mathrm{int}\big(\mathcal{M}(E^-_{\varrho_1},\mathcal{F}^c)(a)\big)\subseteq\mathcal{M}(E^-_{\varrho_1},\mathcal{F}^c)(a).$$

Then $f(b)\subseteq\mathcal{M}(E^-_{\varrho_1},\mathcal{F}^c)(b)\subseteq\mathcal{M}_*(E,\mathcal{F}^c)(b)$. Hence $\mathcal{M}_*(E,\mathcal{F}^c)\in\mathcal{B}(\mathcal{F}^c)$.
If $E\in A(\mathbf{R}^n)$ we have $E=\bigcup_{\varrho>0}E^-_\varrho$, so that $\mathcal{M}_*(E,\mathcal{F}^c)(t_0)=E$. Therefore, as $\mathcal{M}_*(E,\mathcal{F})\in\mathcal{B}(\mathcal{F}^c)$ we have $\mathcal{M}_*(E,\mathcal{F}^c)\supseteq\mathcal{M}(E,\mathcal{F}^c)$, which, together with (3.4), concludes the proof of (3.5). □

3.2. The disjoint and joint sets properties.

Definition 3.5. *We say that the disjoint sets property (resp. the regularized disjoint sets property) with respect to* $(\mathcal{F},\mathcal{G})$ *holds if for any* $E_1,E_2\subseteq\mathbf{R}^n$ *and* $\bar t\in I$

$$E_1\cap E_2=\emptyset \;\Rightarrow\; \mathcal{M}(E_1,\mathcal{F},\bar t)\cap\mathcal{M}(E_2,\mathcal{G},\bar t)=\emptyset \tag{3.9}$$

$$\Big(\textit{resp.}\quad E_1\cap E_2=\emptyset \;\Rightarrow\; \mathcal{M}_*(E_1,\mathcal{F},\bar t)\cap\mathcal{M}^*(E_2,\mathcal{G},\bar t)=\emptyset\Big). \tag{3.10}$$

We say that the joint sets property (resp. the regularized joint sets property) with respect to $(\mathcal{F},\mathcal{G})$ *holds if for any* $E_1,E_2\subseteq\mathbf{R}^n$ *and* $\bar t\in I$

$$E_1\cup E_2=\mathbf{R}^n \;\Rightarrow\; \mathcal{M}(E_1,\mathcal{F},\bar t)\cup\mathcal{M}(E_2,\mathcal{G},\bar t)=\mathbf{R}^n \tag{3.11}$$

$$\Big(\textit{resp.}\quad E_1\cup E_2=\mathbf{R}^n \;\Rightarrow\; \mathcal{M}_*(E_1,\mathcal{F},\bar t)\cup\mathcal{M}^*(E_2,\mathcal{G},\bar t)=\mathbf{R}^n\Big). \tag{3.12}$$

Notice that if (3.10) holds then $\mathcal{M}^*(E_1,\mathcal{F},\bar{t})\cap\mathcal{M}_*(E_2,\mathcal{G},\bar{t})=\emptyset$, and conversely. Indeed, if (3.10) holds we have

$$\begin{aligned}\mathcal{M}^*(E_1,\mathcal{F},\bar{t}) &= \bigcap_{\varrho>0}\mathcal{M}_*\big((E_1)^+_\varrho,\mathcal{F},\bar{t}\big)\subseteq\bigcap_{\varrho>0}\big[\mathbf{R}^n\setminus\mathcal{M}_*((E_2)^-_\varrho,\mathcal{G},\bar{t})\big]\\ &= \mathbf{R}^n\setminus\bigcup_{\varrho>0}\mathcal{M}_*\big((E_2)^-_\varrho,\mathcal{G},\bar{t}\big)=\mathbf{R}^n\setminus\mathcal{M}_*(E_2,\mathcal{G},\bar{t}).\end{aligned}\tag{3.13}$$

Similarly, if (3.12) holds then $\mathcal{M}^*(E_1,\mathcal{F},\bar{t})\cup\mathcal{M}_*(E_2,\mathcal{G},\bar{t})=\mathbf{R}^n$, and conversely (it is enough to replace $\subseteq$ with $\supseteq$ in (3.13)).

Lemma 3.2. *The following properties hold.*

(1) If (3.9) (resp. (3.10)) holds, then it holds if $(\mathcal{F},\mathcal{G})$ is replaced by $\big(\widetilde{\mathcal{F}},\widetilde{\mathcal{G}}\big)$, with $\widetilde{\mathcal{F}}\subseteq\mathcal{F}$, $\widetilde{\mathcal{G}}\subseteq\mathcal{G}$;

(2) if (3.11) (resp. (3.12)) holds, then it holds if $(\mathcal{F},\mathcal{G})$ is replaced by $\big(\widetilde{\mathcal{F}},\widetilde{\mathcal{G}}\big)$, with $\widetilde{\mathcal{F}}\supseteq\mathcal{F}$, $\widetilde{\mathcal{G}}\supseteq\mathcal{G}$;

(3) if for any $E\subseteq\mathbf{R}^n$ we have

$$\mathbf{R}^n\setminus\mathcal{M}(E,\mathcal{F},\bar{t})\in\mathcal{B}(\mathcal{G},[\bar{t},+\infty[),\tag{3.14}$$

then (3.9) holds;

(4) if $\mathcal{F}$ satisfies the assumption of Lemma 3.1 (6) and (3.9) holds, then (3.14) is satisfied;

(5) if $\mathcal{F}$ (or equivalently $\mathcal{G}$) satisfies the assumption of Lemma 3.1 (6), then (3.9) implies (3.10), and (3.11) implies (3.12).

Proof. (1), (2) follow from Lemma 3.1 (5). Assume (3.14); using Lemma 3.1 (3), we then have $\mathbf{R}^n\setminus\mathcal{M}(E,\mathcal{F},\bar{t})\supseteq\mathcal{M}(\mathbf{R}^n\setminus E,\mathcal{G},\bar{t})$, which is equivalent to (3.9).

Let us prove (4). Assume (3.9) and let $g:[a,b]\subseteq[\bar{t},+\infty[\to\mathcal{P}(\mathbf{R}^n)$, $g\in\mathcal{G}$, $g(a)\subseteq\mathbf{R}^n\setminus\mathcal{M}(E,\mathcal{F},\bar{t})(a)$. Then, by (3.9) and Lemma 3.1 (6) and (4), we have

$$\begin{aligned}\emptyset &= \mathcal{M}(g(a),\mathcal{G},a)(b)\cap\mathcal{M}(\mathcal{M}(E,\mathcal{F},\bar{t})(a),\mathcal{F},a)(b)\\ &= \mathcal{M}(g(a),\mathcal{G},a)(b)\cap\mathcal{M}(E,\mathcal{F},\bar{t})(b)\supseteq g(b)\cap\mathcal{M}(E,\mathcal{F},\bar{t})(b),\end{aligned}$$

so that $g(b)\subseteq\mathbf{R}^n\setminus\mathcal{M}(E,\mathcal{F},\bar{t})(b)$, and (3.14) is proved.

It remains to show (5). Assume that (3.9) holds; hence by (4), relation (3.14) holds. As $\mathbf{R}^n\setminus E^+_{2\varrho}\subseteq(\mathbf{R}^n\setminus E)^-_\varrho\subseteq\mathbf{R}^n\setminus E^+_{\frac{\varrho}{2}}$, by (3.14) we have

$$\begin{aligned}\mathcal{M}_*\left(\mathbf{R}^n\setminus E,\mathcal{F},\bar{t}\right) &= \bigcup_{\varrho>0}\mathcal{M}\left(\mathbf{R}^n\setminus E^+_\varrho,\mathcal{F},\bar{t}\right)\subseteq\bigcup_{\varrho>0}\left[\mathbf{R}^n\setminus\mathcal{M}\left(E^+_\varrho,\mathcal{G},\bar{t}\right)\right]\\ &= \mathbf{R}^n\setminus\bigcap_{\varrho>0}\mathcal{M}\left(E^+_\varrho,\mathcal{G},\bar{t}\right)=\mathbf{R}^n\setminus\mathcal{M}^*(E,\mathcal{G},\bar{t}),\end{aligned}$$

which is equivalent to (3.10). A similar proof (replacing $\subseteq$ with $\supseteq$) holds for the joint sets property. □

Note that, in the case of motion by mean curvature (with the correct choice of the family $\mathcal{F}_F$, see Definition 4.1 below), (3.9) and (3.12) hold, but in general (3.11) does not hold, see [6].

4. Definitions and preliminary results on $\mathcal{B}(\mathcal{F}_F)$

In this section we study some properties of barriers for evolutions of sets specified by a suitable function F, which will be useful in the next sections. Let us introduce some notation: $\mathrm{Sym}(n)$ is the space of all symmetric real $(n \times n)$-matrices, endowed with the norm $|X|^2 = \sum_{i,j} X_{ij}^2$; we set $J_0 := (\mathbf{R}^n \setminus \{0\}) \times \mathrm{Sym}(n)$ and $P_p := \mathrm{Id} - p \otimes p/|p|^2$, for $p \in \mathbf{R}^n \setminus \{0\}$.

Let $F : J_0 \to \mathbf{R}$ be a given function. We recall that F is *geometric* (see [7, (1.2)]) if $F(\lambda p, \lambda X + \sigma p \otimes p) = \lambda F(p, X)$ for any $\lambda > 0$, $\sigma \in \mathbf{R}$, $(p, X) \in J_0$, and that F is *degenerate elliptic* if

$$F(p, X) \geq F(p, Y), \qquad (p, X) \in J_0, \ Y \in \mathrm{Sym}(n), Y \geq X. \tag{4.1}$$

We say that F is locally Lipschitz in X if for any $p \in \mathbf{R}^n \setminus \{0\}$ the function $F(p, \cdot)$ is locally Lipschitz.

In the sequel we will always assume that F is geometric. If $(p, X) \in J_0$ we set

$$\begin{aligned} F_c(p, X) &:= -F(-p, -X), \\ F^+(p, X) &:= \sup\{F(p, Y) : \ Y \in \mathrm{Sym}(n), Y \geq X\}, \\ F^-(p, X) &:= \inf\{F(p, Y) : \ Y \in \mathrm{Sym}(n), Y \leq X\}. \end{aligned} \tag{4.2}$$

If F is degenerate elliptic then F_c is degenerate elliptic; moreover $(F_c)^+ = (F^-)_c$.

Let us define the families of local smooth geometric supersolutions (resp. strict supersolutions, subsolutions) of (1.1).

Definition 4.1. *We write $f \in \mathcal{F}_F$ (resp. $f \in \mathcal{F}_F^{>}$, $f \in \mathcal{F}_F^{\leq}$) if and only if there exist $a, b \in \mathbf{R}$, $a < b$, such that $f : [a, b] \to \mathcal{P}(\mathbf{R}^n)$, and the following properties hold: $f(t)$ is closed and $\partial f(t)$ is compact for any $t \in [a, b]$, there exists an open set $A \subseteq \mathbf{R}^n$ such that $d_f \in \mathcal{C}^\infty([a, b] \times A)$, $\partial f(t) \subseteq A$ for any $t \in [a, b]$, and*

$$\frac{\partial d_f}{\partial t} + F(\nabla d_f, \nabla^2 d_f) \geq 0, \qquad t \in]a, b[, \ x \in \partial f(t) \tag{4.3}$$

$$\Big(\textit{resp.} \ \frac{\partial d_f}{\partial t} + F(\nabla d_f, \nabla^2 d_f) > 0, \qquad t \in]a, b[, \ x \in \partial f(t),$$

$$\frac{\partial d_f}{\partial t} + F(\nabla d_f, \nabla^2 d_f) \leq 0, \qquad t \in]a, b[, \ x \in \partial f(t)\Big). \tag{4.4}$$

Clearly $\mathcal{F}_F$, $\mathcal{F}_F^{>}$ and $\mathcal{F}_F^{\leq}$ are translation invariant and satisfy the assumption of Lemma 3.1 (6). Moreover

$$F_1 \leq F_2 \Rightarrow \mathcal{F}_{F_1} \subseteq \mathcal{F}_{F_2} \Rightarrow \mathcal{B}(\mathcal{F}_{F_1}) \supseteq \mathcal{B}(\mathcal{F}_{F_2}) \Rightarrow \mathcal{M}(E, \mathcal{F}_{F_1}) \subseteq \mathcal{M}(E, \mathcal{F}_{F_2}), \tag{4.5}$$

and $\mathcal{M}(E, \mathcal{F}_F) \supseteq \mathcal{M}(E, \mathcal{F}_F^{>})$.

Assume that $F : J_0 \to \mathbf{R}$ is bounded below on compact subsets of J_0; denote by $h : [0, +\infty[\to]0, +\infty[$ a strictly increasing $\mathcal{C}^\infty$ function such that $h(R) > \sup\{-F(p, X) : |p| = 1, |X| \leq R\}$ for any $R \geq 0$. For any $\varrho > 0$ define $H(\varrho) := \displaystyle\int_0^\varrho \frac{1}{h(\sqrt{n-1}/r)} \, dr$. Then $H : [0, +\infty[\to [0, +\infty[$ is strictly increasing, surjective, $H(0) = 0$, $H \in \mathcal{C}^0([0, +\infty[) \cap \mathcal{C}^\infty(]0, +\infty[)$. Let

$$\varrho_F := H^{-1}.$$

One consequence of the following lemma is that, given $f \in \mathcal{F}_F$, we can suppose, if needed, that $f(t)$ is compact, so that Proposition 3.1 is appliable.

Lemma 4.1. *Assume that F is bounded below on compact subsets of J_0. Then*

$$\mathcal{B}(\mathcal{F}_F^c) = \mathcal{B}(\mathcal{F}_F), \tag{4.6}$$

where $\mathcal{F}_F^c := (\mathcal{F}_F)^c$, see Proposition 3.1 (5). Moreover, given $\phi \in \mathcal{B}(\mathcal{F}_F)$ and $t \in I$ we have

$$\{x \in \mathbf{R}^n : dist(x, \mathbf{R}^n \setminus \phi(t)) > \varrho_F(s-t)\} \subseteq int(\phi(s)), \qquad s \geq t. \tag{4.7}$$

Proof. To prove (4.6) it is enough to show that $\mathcal{B}(\mathcal{F}_F^c) \subseteq \mathcal{B}(\mathcal{F}_F)$. Let $\phi \in \mathcal{B}(\mathcal{F}_F^c)$, $f : [a,b] \subseteq I \to \mathcal{P}(\mathbf{R}^n)$, $f \in \mathcal{F}_F$, $f(a) \subseteq \phi(a)$. We have to show that $f(b) \subseteq \phi(b)$. We can assume that $\overline{\mathbf{R}^n \setminus f(t)}$ is compact for any $t \in [a,b]$. Pick $R > 0$ so that $\partial f(t) \subseteq B_R(0)$ for any $t \in [a,b]$. Given $\epsilon > 0$, one can check that the map $t \in [a,b] \to \overline{B_{\varrho_F(\epsilon+b-t)}(0)}$ belongs to $\mathcal{F}_F^{>}$. For any $m \in \mathbf{N}$, let $t_m := \varrho_F^{-1}(mR)$ and

$$\tilde{f}_m(t) := f(t) \cap \overline{B_{\varrho_F(t_m+b-t)}(0)}, \qquad t \in [a,b].$$

For m sufficiently large we can assume that $\partial f(t) \cap \overline{B_{\varrho_F(t_m+b-t)}(0)} = \emptyset$ for any $t \in [a,b]$. As $\phi \in \mathcal{B}(\mathcal{F}_F^c)$, $\tilde{f} \in \mathcal{B}(\mathcal{F}_F^c)$, and $\tilde{f}_m(a) \subseteq \phi(a)$, we have $f(b) \cap B_{mR}(0) \subseteq \tilde{f}_m(b) \subseteq \phi(b)$. Letting m to $+\infty$, we get (4.6).

Let us prove (4.7). Let $x \in \phi(t)$, $s > t$ and $\epsilon > 0$ be such that $\mathrm{dist}(x, \mathbf{R}^n \setminus \phi(t)) > \varrho_F(\epsilon + s - t) > \varrho(s-t)$. The map $\sigma \in [t,s] \to \overline{B_{\varrho_F(\epsilon+s-\sigma)}(x)}$ belongs to $\mathcal{F}_F$ hence $B_{\varrho_F(\epsilon)}(x) \subseteq \phi(s)$. Therefore $x \in \mathrm{int}(\phi(s))$. □

Proposition 4.1. *Assume that F is bounded below on compact subsets of J_0 and is locally Lipschitz in X. Then, for any $E \subseteq \mathbf{R}^n$ we have*

$$\mathcal{M}_*(E, \mathcal{F}_F) = \mathcal{M}_*(E, \mathcal{F}_F^{>}), \qquad \mathcal{M}^*(E, \mathcal{F}_F) = \mathcal{M}^*(E, \mathcal{F}_F^{>}). \tag{4.8}$$

Moreover let $\phi \in \mathcal{B}(\mathcal{F}_F^{>})$, $f : [a,b] \subseteq I \to \mathcal{P}(\mathbf{R}^n)$, $f \in \mathcal{F}_F$, $int(f(a)) \subseteq \phi(a)$. Then $int(f(b)) \subseteq \phi(b)$.

Proof. In view of (3.2), equalities (4.8) are proved if we show

$$E \in \mathcal{A}(\mathbf{R}^n) \Rightarrow \mathcal{M}(E, \mathcal{F}_F) = \mathcal{M}(E, \mathcal{F}_F^{>}). \tag{4.9}$$

Let $E \in \mathcal{A}(\mathbf{R}^n)$; to prove (4.9) we need to show that $\mathcal{M}(E, \mathcal{F}_F^{>}) \in \mathcal{B}(\mathcal{F}_F)$. Let $f : [a,b] \subseteq I \to \mathcal{P}(\mathbf{R}^n)$, $f \in \mathcal{F}_F$, $f(a) \subseteq \mathcal{M}(E, \mathcal{F}_F^{>})(a) =: A$. By Lemma 4.1, we can replace $\mathcal{F}_F$ with $\mathcal{F}_F^c$, hence by (3.5) the set A is open. We have to show that $f(b) \subseteq \mathcal{M}(E, \mathcal{F}_F^{>})(b)$. For any $t \in [a,b]$ we can find a bounded tubular neighbourhood $(\partial f(t))^+_{c(t)}$ of $\partial f(t)$, of thickness $c(t)$, each point of which has a unique projection on $\partial f(t)$; we set $2c := \inf\{c(t), t \in [a,b]\}$, which is positive. Let L be the Lipschitz constant of $F(\nabla d_f, \nabla^2 d_f)$ and M be the supremum of $|\nabla^2 d_f|^2$ when $t \in [a,b]$ and $x \in (\partial f(t))^+_c$. Pick a $\mathcal{C}^\infty$ function $\varrho : [a,b] \to]0,+\infty[$ such that $\varrho(a) < \min(c, \mathrm{dist}(\partial f(a), \mathbf{R}^n \setminus A))$ and $\dot{\varrho} + 2ML\varrho < 0$. The map $g : [a,b] \to \mathcal{P}(\mathbf{R}^n)$, $g(t) := \overline{f^+_{\varrho(t)}(t)} = \{x \in \mathbf{R}^n : \mathrm{dist}(x, f(t)) \leq \varrho(t)\}$ is of class $\mathcal{C}^\infty$, and each point $y \in \partial g(t)$ is of the form $y = x + \varrho(t)\nabla d_f(t,x)$ for a unique $x \in \partial f(t)$. Moreover

$g \in \mathcal{F}_F^{\gtrless}$. Indeed for any $t \in]a,b[$ and any $y \in \partial g(t)$, $y = x + \varrho(t)\nabla d_f(t,x)$, $x \in \partial f(t)$, we have $\nabla^2 d_g(t,y) = \nabla^2 d_f(t,x)(\mathrm{Id} - \varrho(t)\nabla^2 d_f(t,x))^{-1}$, so that

$$|\nabla^2 d_g(t,y) - \nabla^2 d_f(t,x)| \le 2M\varrho(t).$$

Therefore, recalling that $f \in \mathcal{F}_F$, we have

$$\begin{aligned}
&-\frac{\partial d_g}{\partial t}(t,y) = -\frac{\partial d_f}{\partial t}(t,x) + \dot\varrho(t) \\
&\le F(\nabla d_f(t,x), \nabla^2 d_f(t,x)) + \dot\varrho(t) = F(\nabla d_g(t,y), \nabla^2 d_f(t,x)) + \dot\varrho(t) \\
&\le F(\nabla d_g(t,y), \nabla^2 d_g(t,y)) + 2LM\varrho(t) + \dot\varrho(t) < F(\nabla d_g(t,y), \nabla^2 d_g(t,y)),
\end{aligned} \tag{4.10}$$

so that $g \in \mathcal{F}_F^{\gtrless}$. Hence $f(b) \subseteq g(b) \subseteq \mathcal{M}(E, \mathcal{F}_F^{\gtrless})(b)$.

Let $\phi \in \mathcal{B}(\mathcal{F}_F^{\gtrless})$ and f be as in the statement. Let c, L, M be defined as before. Let $r : [a,b] \to]0,+\infty[$ be a $\mathcal{C}^\infty$ function such that $r(b) < c$ and $\dot r - 2MLr > 0$. Then the map $h : [a,b] \to \mathcal{P}(\mathbf{R}^n)$, $h(t) := \overline{f^-_{r(t)}(t)} = \{x \in \mathbf{R}^n : \mathrm{dist}(x, \mathbf{R}^n \setminus f(t)) \ge r(t)\}$ belongs to $\mathcal{F}_F^{\gtrless}$. As $h(a) \subseteq \mathrm{int}(f(a)) \subseteq \phi(a)$, we have $h(b) \subseteq \phi(b)$.

Choose $r(t) := r(a)e^{3ML(t-a)} \le r(a)e^{3ML(b-a)} < c$; letting $r(a) \to 0^+$, we have $r(b) \to 0^+$, therefore $\mathrm{int}(f(b)) = \bigcup_{r(a) \in\ ]0,c[} h(b) \subseteq \phi(b)$. □

Remark 4.1. *If F is continuous and degenerate elliptic then (4.8) holds for any $E \subseteq \mathbf{R}^n$ (see (7.8) below).*

5. Comparison between barriers and local barriers

In this section we compare barriers with local barriers; we basically prove that these two classes coincide (Theorem 5.1). Some arguments in the proof of Theorem 5.1 will be used to prove (1.2). Let us introduce the notion of local (in space) barrier.

Definition 5.1. *A function ϕ is a local barrier with respect to $\mathcal{F}$ if and only if ϕ maps a convex set $L \subseteq I$ into $\mathcal{P}(\mathbf{R}^n)$ and the following property holds: for any $x \in \mathbf{R}^n$ there exists $R > 0$ (depending on ϕ and x) so that if $f : [a,b] \subseteq L \to \mathcal{P}(\mathbf{R}^n)$ belongs to $\mathcal{F}$ and $f(a) \subseteq \phi(a) \cap B_R(x)$, then $f(b) \subseteq \phi(b)$. Given such a map ϕ, we shall write $\phi \in \mathcal{B}_{\mathrm{loc}}(\mathcal{F}, L)$. When $L = I$ we simply write $\phi \in \mathcal{B}_{\mathrm{loc}}(\mathcal{F})$.*

The definition of local minimal barrier reads as follows.

Definition 5.2. *Let $E \subseteq \mathbf{R}^n$ be a given set and $\bar t \in I$. We set*

$$\mathcal{M}_{\mathrm{loc}}(E, \mathcal{F}, \bar t)(t) := \bigcap \Big\{ \phi(t) : \ \phi : [\bar t, +\infty[\to \mathcal{P}(\mathbf{R}^n), \phi \in \mathcal{B}_{\mathrm{loc}}(\mathcal{F}, [\bar t, +\infty[), \phi(\bar t) \supseteq E \Big\}.$$

As $\mathcal{B}_{\mathrm{loc}}(\mathcal{F}, [\bar t, +\infty[) \supseteq \mathcal{B}(\mathcal{F}, [\bar t, +\infty[)$, we have $\mathcal{M}_{\mathrm{loc}}(E, \mathcal{F}, \bar t) \subseteq \mathcal{M}\left(E, \mathcal{F}, \bar t\right)$.

We shall assume for simplicity that $\bar t = t_0$, and we shall omit t_0 in the notation of the local minimal barrier.

Theorem 5.1. *Let $F : J_0 \to \mathbf{R}$ be a lower semicontinuous function. Then*

$$\mathcal{B}_{\mathrm{loc}}(\mathcal{F}_F^{\gtrless}) = \mathcal{B}(\mathcal{F}_F^{\gtrless}).$$

In particular, for any $E \subseteq \mathbf{R}^n$ we have $\mathcal{M}(E, \mathcal{F}_F^{\gtrless}) = \mathcal{M}_{\mathrm{loc}}(E, \mathcal{F}_F^{\gtrless})$.

To prove the theorem we need several preliminary observations.

Lemma 5.1. *Assume that F is bounded below on compact subsets of J_0 and is locally Lipschitz in X. Let $\phi \in \mathcal{B}_{\rm loc}(\mathcal{F}_F^{\gtrsim})$, $x \in \mathbf{R}^n$, $R = R(\phi, x)$ be given by Definition 5.1, $f : [a,b] \subseteq I \to \mathcal{P}(\mathbf{R}^n)$, $f \in \mathcal{F}_F$, and $int(f(a)) \subseteq \phi(a) \cap B_R(x)$. Then $int(f(b)) \subseteq \phi(b)$.*

Proof. It is enough to repeat the arguments of the second part of the proof of Proposition 4.1. □

Lemma 5.2. *Assume that $F : J_0 \to \mathbf{R}$ is lower semicontinuous. Let $L \subseteq \mathbf{R}^n$ be a closed set with smooth boundary. Let $\overline{x} \in \partial L$ and $\alpha \in \mathbf{R}$ be such that*

$$\alpha + F\big(\nabla d_L(\overline{x}), \nabla^2 d_L(\overline{x})\big) > 0. \tag{5.1}$$

Then for any $R > 0$ there exist $\tau > 0$, $f : [a, a+\tau] \to \mathcal{P}(\mathbf{R}^n)$ and $\sigma > 0$ such that

$$f(a) \subseteq L, \quad \partial f(a) \cap B_\sigma(\overline{x}) = \partial L \cap B_\sigma(\overline{x}), \quad \alpha = \frac{\partial d_f}{\partial t}(a, \overline{x}), \tag{5.2}$$

$$f \in \mathcal{F}_F^{\gtrsim}, \qquad f(t) \subseteq B_R(\overline{x}), \qquad t \in [a, a+\tau]. \tag{5.3}$$

Moreover, τ depends in a continuous way on small perturbations of ∂L around $\overline{x}$ in the $\mathcal{C}^\infty$ norm.

Proof. See [3, Lemma 4.1, Remarks 4.1, 4.2]. □

Definition 5.3. *Let $f : [a,b] \subseteq I \to \mathcal{P}(\mathbf{R}^n)$. We say that f is a smooth compact flow if and only if $\partial f(t)$ is compact for any $t \in [a,b]$ and there exists an open set $A \subseteq \mathbf{R}^n$ such that $d_f \in \mathcal{C}^\infty([a,b] \times A)$ and $\partial f(t) \subseteq A$ for any $t \in [a,b]$.*

Lemma 5.3. *Let $f, g : [a,b] \subseteq I \to \mathcal{P}(\mathbf{R}^n)$ be two smooth compact flows, $x \in \mathbf{R}^n$ and $\varrho > 0$. Assume that*

$$\begin{aligned} &\{x\} = \partial f(a) \cap \partial g(a) \cap \overline{B_\varrho(x)}, \\ &\big(g(a) \setminus \{x\}\big) \cap \overline{B_\varrho(x)} \subseteq int(f(a)) \cap \overline{B_\varrho(x)}, \\ &\frac{\partial d_f}{\partial t}(a, x) < \frac{\partial d_g}{\partial t}(a, x). \end{aligned}$$

Then there exists $0 < \tau \leq b - a$ such that

$$g(t) \cap \overline{B_\varrho(x)} \subseteq int(f(t)) \cap \overline{B_\varrho(x)}, \qquad t \in \,]a, a+\tau]. \tag{5.4}$$

Moreover, τ depends in a continuous way on small perturbations of f and g in the $\mathcal{C}^2$ norm.

Proof. See [3, Lemma 5.1]. □

Proof of Theorem 5.1. It is sufficient to prove that $\mathcal{B}_{\rm loc}(\mathcal{F}_F^{\gtrsim}) \subseteq \mathcal{B}(\mathcal{F}_F^{\gtrsim})$. Let $\phi \in \mathcal{B}_{\rm loc}(\mathcal{F}_F^{\gtrsim})$, $g : [a,b] \subseteq I \to \mathcal{P}(\mathbf{R}^n)$, $g \in \mathcal{F}_F^{\gtrsim}$, $g(a) \subseteq \phi(a)$; we have to show that $g(b) \subseteq \phi(b)$. As F is lower semicontinuous and $g \in \mathcal{F}_F^{\gtrsim}$, we can suppose that the function F is locally Lipschitz in X. In fact, it is enough to choose a function $G \leq F$, G lower semicontinuous and locally Lipschitz in X, such that $g \in \mathcal{F}_G^{\gtrsim}$, and to notice that $\mathcal{B}_{\rm loc}(\mathcal{F}_F^{\gtrsim}) \subseteq \mathcal{B}_{\rm loc}(\mathcal{F}_G^{\gtrsim})$.

We preliminarly prove that

$$\mathrm{int}(g(b)) \subseteq \phi(b). \tag{5.5}$$

Suppose by contradiction that there exists $g : [a,b] \subseteq I \to \mathcal{P}(\mathbf{R}^n)$, $g \in \mathcal{F}_F^{\geq}$, $g(a) \subseteq \phi(a)$, such that $\mathrm{int}(g(b))$ is not contained in $\phi(b)$. Set

$$t^* := \sup \{ t \in [a,b] : \mathrm{int}(g(s)) \subseteq \phi(s), \quad s \in [a,t] \}. \tag{5.6}$$

STEP 1. We have $\mathrm{int}(g(t^*)) \subseteq \phi(t^*)$, so that $t^* < b$.
We can assume $t^* > a$. If by contradiction there exists $x \in \mathrm{int}(g(t^*)) \setminus \phi(t^*)$, as g is a smooth flow, we can find $0 < \tau_1 < t^* - a$ and $R > 0$ so that $B_R(x) \subseteq \mathrm{int}(g(t))$ for any $t \in [t^* - \tau_1, t^*]$. Therefore $B_R(x) \subseteq \phi(t)$ for any $t \in [t^* - \tau_1, t^*[$ and $x \notin \phi(t^*)$, a contradiction since ϕ is a local barrier.

STEP 2. There exist $x^* \in \partial g(t^*) \cap \partial \phi(t^*)$, a decreasing sequence $\{t_m\}$ of points of $]t^*, b]$ and a sequence $\{R_m\}$ of positive numbers, with $\lim_{m\to+\infty} t_m = t^*$, $\lim_{m\to+\infty} R_m = 0$, such that for any $m \in \mathbf{N}$

$$\Big(\mathrm{int}(g(t_m)) \setminus \phi(t_m)\Big) \cap B_{R_m}(x^*) \neq \emptyset. \tag{5.7}$$

Let us first prove that $\partial g(t^*) \cap \partial \phi(t^*) \neq \emptyset$. Assume by contradiction that $\partial g(t^*) \cap \partial \phi(t^*) = \emptyset$, and set $\eta(t) := \mathrm{dist}(g(t), \mathbf{R}^n \setminus \phi(t))$ for $t \in [a,b]$. As $\partial g(t^*)$ is compact, we have $\eta(t^*) > 0$. Let us prove that $\eta(t^*) \leq \liminf_{s \downarrow t} \eta(s)$. Indeed, if not, there exists a sequence $\{s_m\}$, $s_m > t^*$, $s_m \downarrow t^*$, such that $\lim_{m\to+\infty} \eta(s_m) < \eta(t^*)$. Then $\eta(s_m) = |y_m - p_m|$, for some $y_m \in g(s_m)$, $p_m \in \overline{\mathbf{R}^n \setminus \phi(s_m)}$; possibly passing to a subsequence, we have $y_m \to y \in g(t^*)$, $p_m \to p \notin \overline{\mathbf{R}^n \setminus \phi(t^*)}$ as $m \to +\infty$. Let $\varrho > 0$ be such that $B_\varrho(p) \subseteq \mathrm{int}(\phi(t^*))$. Then $B_{\varrho/2}(p) \cap \overline{(\mathbf{R}^n \setminus \phi(s_m))} \neq \emptyset$ definitively in m, which is impossible since ϕ is a local barrier. Then $0 < \eta(t^*) \leq \liminf_{s\downarrow t^*} \eta(s) = 0$, a contradiction. Then $K := \partial g(t^*) \cap \partial \phi(t^*) \neq \emptyset$.

Assume now by contradiction that for any $x \in K$ there exists $R(x) > 0$ and $0 < t(x) < b - t^*$ so that

$$\Big(\mathrm{int}(g(s)) \setminus \phi(s)\Big) \cap B_R(x) = \emptyset, \qquad R \in \,]0, R(x)], \; s \in \,]t^*, t^* + t(x)]. \tag{5.8}$$

As K is compact, we can find $x_1, \ldots, x_h \in K$ (and corresponding $t(x_1), \ldots, t(x_h)$) so that each $R(x_i)$ satisfies (5.8) and $\bigcup_{i=1}^h B_{R(x_i)} \supseteq K$. Let $\overline{R} > 0$ be such that $H := \bigcup_{x\in K} B_{\overline{R}}(x) \subseteq \bigcup_{i=1}^h B_{R(x_i)}$, and let $\overline{t} := \min_{i=1,\ldots,h} t(x_i)$. Then for any $x \in K$ we have

$$\Big(\mathrm{int}(g(s)) \setminus \phi(s)\Big) \cap B_{\overline{R}}(x) = \emptyset, \qquad s \in \,]t^*, t^* + \overline{t}].$$

Let $c > 0$ be such that $\mathrm{dist}(g(t^*) \setminus H, \mathbf{R}^n \setminus \phi(t^*)) \geq c$. Then using (4.7) and the fact that g is a smooth flow, we contradict the definition of t^*.

STEP 3. Let x^* be as in STEP 2. We can assume that

$$\{x^*\} = \partial g(t^*) \cap \partial \phi(t^*), \qquad g(t^*) \setminus \{x^*\} \subseteq \mathrm{int}\,(\phi(t^*)). \tag{5.9}$$

Indeed, let $0 < \tau_1 < b - t^*$ be such that each point $x \in \partial g(t)$ has a unique smooth projection $\pi(t,x)$ on $\partial g(t^*)$ for any $t \in [t^*, t^* + \tau_1]$. Choose a function $\varrho : \partial g(t^*) \to [0, +\infty[$ of class $\mathcal{C}^\infty$ verifying the following properties:

(i) $\varrho(x) = 0$ if and only if $x = x^*$;

(ii) the map $t \in [t^*, t^* + \tau_1] \to \zeta(t)$ belongs to $\mathcal{F}_F^{\geq}$, where $\zeta(t) := \overline{g^-_{\varrho(\cdot)}(t)} \subseteq g(t)$ and $\partial\zeta(t) := \{y \in \mathbf{R}^n : y = x - \varrho(\pi(t,x))\nabla d_g(t,x), x \in \partial g(t)\}$;

(iii) ϕ is not a barrier for ζ on $[t^*, t^* + \tau_1]$.

Property (ii) can be achieved by taking $\varrho(\cdot)$ sufficiently small in the $\mathcal{C}^2$ norm, since there exists $c > 0$ so that $\frac{\partial d_g}{\partial t} + F(\nabla d_g, \nabla^2 d_g) \geq c$ for any $x \in \partial g(t)$, $t \in]a, b[$, and F is lower semicontinuous.

Property (iii) can be achieved by observing that, by (5.7), for any $m \in \mathbf{N}$ there exist a point $x_m \in \mathrm{int}(g(t_m)) \setminus \phi(t_m)$ and $\sigma_m > 0$ such that $B_{\sigma_m}(x_m) \subseteq \mathrm{int}(g(t_m)) \cap B_{R_m}(x^*)$. Then, if we impose $\varrho(x) < \sigma_m$ for any $x \in \partial g(t^*)$ such that $|\pi^{-1}(t_m, x) - x^*| < R_m$, we get $x_m \in \mathrm{int}(\zeta(t_m))$. Therefore, possibly replacing g by ζ, we can assume that (5.9) holds, and the proof of STEP 3 is concluded.

Pick now a closed set L with smooth compact boundary such that $g(t^*) \subseteq L \subseteq \phi(t^*)$, $\partial L \cap \partial g(t^*) \cap \partial\phi(t^*) = \{x^*\}$, $g(t^*) \setminus \{x^*\} \subseteq \mathrm{int}(L)$, $L \setminus \{x^*\} \subseteq \mathrm{int}(\phi(t^*))$, and $\nabla^2 d_L(x^*) = \nabla^2 d_g(t^*, x^*)$.

Let $R = R(\phi, x^*) > 0$ be given by Definition 5.1. We apply Lemma 5.2 with $a := t^*$, $\overline{x} := x^*$ and $\alpha \in \mathbf{R}$ such that

$$\alpha < \frac{\partial d_g}{\partial t}(t^*, x^*), \qquad \alpha + F\big(\nabla d_g(t^*, x^*), \nabla^2 d_g(t^*, x^*)\big) > 0.$$

Therefore, there exist $0 < \tau < b - t^*$, $f : [t^*, t^* + \tau] \to \mathcal{P}(\mathbf{R}^n)$, and $\sigma > 0$ such that (5.2), (5.3) hold. We then have

$$\begin{aligned} &\{x^*\} = \partial f(t^*) \cap \partial g(t^*) \cap \overline{B_\sigma(x^*)}, \\ &\big(g(t^*) \setminus \{x^*\}\big) \cap \overline{B_\sigma(x^*)} \subseteq \mathrm{int}(f(t^*)) \cap \overline{B_\sigma(x^*)}, \\ &\big(f(t^*) \setminus \{x^*\}\big) \cap \overline{B_\sigma(x^*)} \subseteq \mathrm{int}(\phi(t^*)) \cap \overline{B_\sigma(x^*)}. \end{aligned}$$

Using Lemma 5.1 we have

$$\mathrm{int}(f(t)) \subseteq \phi(t), \qquad t \in [t^*, t^* + \tau]. \tag{5.10}$$

By Lemma 5.3 there exists $0 < \tau_2 < b - t^*$ such that

$$g(t) \cap \overline{B_\sigma(x^*)} \subseteq f(t) \cap \overline{B_\sigma(x^*)}, \qquad t \in [t^*, t^* + \tau_2]. \tag{5.11}$$

By (5.11) and (5.10) we get

$$\mathrm{int}(g(t)) \cap \overline{B_\sigma(x^*)} \subseteq \phi(t), \qquad t \in [t^*, t^* + \min(\tau, \tau_2)]$$

which contradicts (5.7). It follows that (5.5) is proved.

To complete the proof, it remains to show that $g(b) \subseteq \phi(b)$. Let $k > 0$ be such that $\frac{\partial d_g}{\partial t} + F(\nabla d_g, \nabla^2 d_g) \geq 2k$ for any $x \in \partial g(t)$ and $t \in]a, b[$. Pick a $\mathcal{C}^\infty$ function $\varrho : [a, b] \to [0, +\infty[$ such that $\varrho(a) = 0$, $\varrho(b) < c$ and $0 < \dot\varrho < k(1 + 2ML(b-a))^{-1}$, where c, L and M are as in the proof of Proposition 4.1 (with f replaced by g). Then $\dot\varrho + 2ML\varrho - 2k < \frac{k}{1+2ML(b-a)} + \frac{2MLk(b-a)}{1+2ML(b-a)} - 2k < 0$, so that, reasoning as in (4.10), it follows that the map taking $t \in [a, b]$ into $\overline{g^+_{\varrho(t)}(t)} = \{x \in \mathbf{R}^n : \mathrm{dist}(x, g(t)) \leq \varrho(t)\}$ belongs to $\mathcal{F}_F^{\geq}$. Therefore, from (5.5) (applied with $\overline{g^+_{\varrho(\cdot)}}$ in place of f) we have

$$g(b) \subseteq \mathrm{int}\Big(\overline{g^+_{\varrho(b)}(b)}\Big) \subseteq \phi(b),$$

and this concludes the proof. □

6. Representation of $\mathcal{M}(E, \mathcal{F}_F)$ for a not degenerate elliptic F

The aim of this section is to prove that the minimal barrier with respect to $\mathcal{F}_F$ coincides with the minimal barrier with respect to $\mathcal{F}_{F^+}$. More precisely, we will prove the following result.

Theorem 6.1. *Assume that $F : J_0 \to \mathbf{R}$ is lower semicontinuous. Let F^+ be defined as in (4.2). Then*

$$\mathcal{B}(\mathcal{F}_F^{\gtrless}) = \mathcal{B}(\mathcal{F}_{F^+}^{\gtrless}).$$

In particular, for any $E \subseteq \mathbf{R}^n$ we have $\mathcal{M}(E, \mathcal{F}_F^{\gtrless}) = \mathcal{M}(E, \mathcal{F}_{F^+}^{\gtrless})$.

To prove the theorem, we need the following lemma, whose proof is similar to that of Lemma 5.3.

Lemma 6.1. *Let $f, g : [a,b] \subseteq I \to \mathcal{P}(\mathbf{R}^n)$ be two smooth compact flows, $x \in \mathbf{R}^n$ and $\varrho > 0$. Assume that*

$$\begin{aligned} & x \in \partial f(a) \cap \partial g(a), \\ & f(a) \cap \overline{B_\varrho(x)} \subseteq g(a) \cap \overline{B_\varrho(x)}, \\ & \frac{\partial d_f}{\partial t}(a,x) < \frac{\partial d_g}{\partial t}(a,x). \end{aligned}$$

Let $0 < \delta < b - a$ be such that each point of $\partial g(t)$ has a unique smooth projection $\pi(t, \cdot)$ on $\partial g(a)$ for any $t \in [a, a+\delta]$. Set $x(t) := \pi^{-1}(t,x)$. Then there exists $0 < \tau \le \delta$ such that the following holds: for any $t \in]a, a+\tau]$ there exists $\varrho(t) > 0$ such that

$$g(t) \cap \overline{B_{\varrho(t)}(x(t))} \subseteq int(f(t)) \cap \overline{B_{\varrho(t)}(x(t))}.$$

Moreover, τ depends in a continuous way on small perturbations of f, g in the $\mathcal{C}^2$ norm.

Proof of Theorem 6.1. It is sufficient to prove that $\mathcal{B}(\mathcal{F}_F^{\gtrless}) \subseteq \mathcal{B}(\mathcal{F}_{F^+}^{\gtrless})$. Let $\phi \in \mathcal{B}(\mathcal{F}_F^{\gtrless})$, $g : [a,b] \subseteq I \to \mathcal{P}(\mathbf{R}^n)$, $g \in \mathcal{F}_{F^+}^{\gtrless}$ and $g(a) \subseteq \phi(a)$. We have to show that $g(b) \subseteq \phi(b)$. Reasoning as in the proof of Theorem 5.1, it is enough to show (5.5), under the further assumption that F is locally Lipschitz in X. Suppose by contradiction that there exists $g : [a,b] \subseteq I \to \mathcal{P}(\mathbf{R}^n)$, $g \in \mathcal{F}_{F^+}^{\gtrless}$, $g(a) \subseteq \phi(a)$ such that $\mathrm{int}(g(b))$ is not contained in $\phi(b)$. Following STEPS 1,2,3 of the proof of Theorem 5.1, defining t^* as in (5.6), we can assume that $t^* < b$, and that there exist $x^* \in \partial g(t^*) \cap \partial \phi(t^*)$, a decreasing sequence $\{t_m\}$ of points of $]t^*, b]$, a sequence $\{R_m\}$ of positive numbers, with $\lim_{m \to +\infty} t_m = t^*$, $\lim_{m \to +\infty} R_m = 0$, so that (5.7) holds for any $m \in \mathbf{N}$ and (5.9) holds.

Set

$$\overline{p} := \nabla d_g(t^*, x^*), \qquad \overline{X} := \nabla^2 d_g(t^*, x^*).$$

Let $0 < \tau < b - t^*$ be such that each point $y \in \partial g(t)$ has a unique smooth projection $\pi(t,y)$ on $\partial g(t^*)$ for any $t \in [t^*, t^* + \tau]$. For any $x \in \partial g(t^*)$ let $x(t) := \pi^{-1}(t,x)$.

As $g \in \mathcal{F}_{F^+}^{\gtrless}$, there exists a constant $c \in]0, +\infty[$ such that

$$\frac{\partial d_g}{\partial t}(t^*, x^*) + F^+(\overline{p}, \overline{X}) > c.$$

As F^+ is lower semicontinuous, we can find $R_1 > 0$ such that

$$\frac{\partial d_g}{\partial t}(t^*, x) + F^+(p(x), X(x)) \geq c, \qquad x \in \Gamma := \partial g(t^*) \cap \overline{B_{R_1}(x^*)}, \tag{6.1}$$

where, for any $x \in \Gamma$, we set $p(x) := \nabla d_g(t^*, x)$, $X(x) := \nabla^2 d_g(t^*, x)$. Recalling the definition of F^+, for any $x \in \Gamma$ there exists $Y(x) \in \mathrm{Sym}(n)$ such that $Y(x) \geq X(x)$ and

$$\begin{aligned} &\max\left\{\frac{\partial d_g}{\partial t}(t^*, x) - \frac{c}{2}, \frac{c}{4} - F^+(p(x), Y(x))\right\} + F(p(x), Y(x)) > 0, \\ &F^+(p(x), X(x)) = F(p(x), Y(x)). \end{aligned} \tag{6.2}$$

Set $\alpha(x) := \max\left\{\frac{\partial d_g}{\partial t}(t^*, x) - \frac{c}{2}, \frac{c}{4} - F^+(p(x), Y(x))\right\}$. Given $x \in \Gamma$, let L_x be a closed set with smooth compact boundary such that $L_x \subseteq g(t^*)$, $L_x \cap g(t^*) = \{x\}$, $L_x \setminus \{x\} \subseteq \mathrm{int}(g(t^*))$, and $\nabla^2 d_{L_x}(x) = Y(x)$. For any $x \in \Gamma$, applying Lemma 5.2 there exist $0 < \tau(x) < \tau$ and $f_x : [t^*, t^* + \tau(x)] \to \mathcal{P}(\mathbf{R}^n)$ such that $f_x \in \mathcal{F}_F^{>}$,

$$\nabla d_{f_x}(t^*, x) = p(x), \quad \nabla^2 d_{f_x}(t^*, x) = Y(x), \qquad \frac{\partial d_{f_x}}{\partial t}(t^*, x) = \alpha(x),$$

and we can also assume $\{x\} = \partial g(t^*) \cap \partial f_x(t^*)$, $f_x(t^*) \setminus \{x\} \subseteq \mathrm{int}(g(t^*))$. Therefore, using Lemma 5.1, for any $x \in \Gamma$ we have

$$\mathrm{int}(f_x(t)) \subseteq \phi(t), \qquad t \in [t^*, t^* + \tau(x)]. \tag{6.3}$$

Moreover, given $x \in \Gamma$, we have $\frac{\partial d_g}{\partial t}(t^*, x) > \frac{\partial d_{f_x}}{\partial t}(t^*, x)$. Indeed, if $\alpha(x) = \frac{\partial d_g}{\partial t}(t^*, x) - \frac{c}{2}$, the inequality is obvious, and if $\alpha(x) = \frac{c}{4} - F^+(p(x), Y(x))$, by (6.1) and (6.2) we have

$$\begin{aligned} \frac{\partial d_g}{\partial t}(t^*, x) &\geq c - F^+(p(x), X(x)) = c - F(p(x), Y(x)) \\ &> c - \frac{c}{4} - F^+(p(x), Y(x)) = \frac{c}{2} + \frac{\partial d_{f_x}}{\partial t}(t^*, x) > \frac{\partial d_{f_x}}{\partial t}(t^*, x). \end{aligned}$$

By Lemma 6.1 it follows that, given $x \in \Gamma$, possibly reducing $\tau(x)$, for any $t \in]t^*, t^* + \tau(x)]$ there exists $\varrho = \varrho(t, x) > 0$ such that

$$g(t) \cap \overline{B_{\varrho(t,x)}(x(t))} \subseteq \mathrm{int}(f_x(t)).$$

Since x varies on the compact set Γ and $\tau(x)$ depends in a continuous way from $x \in \Gamma$ we have $\tau^* := \min\{\tau(x) : x \in \Gamma\} > 0$. Possibly reducing τ^* and using (6.3), we deduce that

$$\partial g(t) \cap \overline{B_{\frac{R_1}{2}}(x^*)} \subseteq \bigcup_{x \in \Gamma} \mathrm{int}(f_x(t)) \subseteq \phi(t), \qquad t \in]t^*, t^* + \tau^*].$$

Furthermore, we can find $\delta > 0$ so that

$$(\partial g(t))^+_\delta \cap g(t) \cap \overline{B_{\frac{R_1}{2}}(x^*)} \subseteq \bigcup_{x \in \Gamma} \mathrm{int}(f_x(t)) \subseteq \phi(t), \qquad t \in]t^*, t^* + \tau^*].$$

Using (4.7) and setting $\tau' := \min(\tau^*, \varrho_F^{-1}(\delta))$, we then have

$$g(t) \cap \overline{B_{\frac{R_1}{2}}(x^*)} \subseteq \phi(t), \qquad t \in]t^*, t^* + \tau'].$$

Moreover there exists $\tau'' > 0$ such that

$$g(t) \setminus B_{\frac{R_1}{2}}(x^*) \subseteq \phi(t), \qquad t \in [t^*, t^* + \tau''].$$

Hence for any $t \in]t^*, t^* + \min(\tau', \tau'')]$ we have $g(t) \subseteq \phi(t)$, which contradicts (5.7). □

7. Characterizations of the disjoint and joint sets properties

The main results of this section are Theorems 7.1 and 7.3, where we characterize the regularized disjoint and joint sets properties with respect to $(\mathcal{F}_F, \mathcal{F}_G)$.

Definition 7.1. *Let $F : J_0 \to \mathbf{R}$ be a given function. We say that F is compatible from above (resp. from below) if there exists an odd degenerate elliptic function $F_1 : J_0 \to \mathbf{R}$ such that $F_1 \geq F$ (resp. $F_1 \leq F$).*

Lemma 7.1. *F is compatible from above (resp. below) if and only if*

$$(F^+)_c \geq F^+ \qquad \Big(\text{resp. } (F^-)_c \leq F^-\Big). \tag{7.1}$$

Proof. If (7.1) holds, then the function $F_1 := (F^+ + (F^+)_c)/2$ is odd, degenerate elliptic and $F_1 \geq F$. Conversely, let $F_1 \geq F$ be odd and degenerate elliptic. Given $(p, X) \in J_0$, $Y \in \mathrm{Sym}(n)$, $Y \geq X$, we have

$$F(p, Y) \leq F_1(p, Y) = -F_1(-p, -Y) \leq -F_1(-p, -X) \leq -F(-p, -X). \tag{7.2}$$

Recalling the definition of F^+, we have

$$F^+(-p, -X) + F^+(p, X) = \sup\{F(-p, Z) + F(p, Y) : Z \geq -X, Y \geq X\}. \tag{7.3}$$

Given $Z \geq -X$, $Y \geq X$, we have $Y \geq -Z$, and so $F(-p, Z) + F(p, Y) \leq 0$ by (7.2). Passing to the supremum with respect to Z and Y and using (7.3) we get (7.1).

The case concerning F^- is similar. □

Lemma 7.2. *Let $F : J_0 \to \mathbf{R}$ be degenerate elliptic. Let $f, g : [a, b] \subseteq I \to \mathcal{P}(\mathbf{R}^n)$, $f \in \mathcal{F}_F$, $g \in \mathcal{F}_F^{\lessgtr}$. Then $f(a) \subseteq g(a) \Rightarrow f(b) \subseteq g(b)$.*

Proof. For any $t \in [a, b]$ set $\eta(t) := \mathrm{dist}(f(t), \mathbf{R}^n \setminus g(t)) - \max_{x \in \partial f(t)} \mathrm{dist}(x, g(t))$. For any $t \in [a, b[$ one can check that

$$\liminf_{\tau \to 0^+} \frac{\eta(t + \tau) - \eta(t)}{\tau} = \frac{\partial d_f}{\partial t}(t, x) - \frac{\partial d_g}{\partial t}(t, y),$$

for two suitable points $x \in \partial g(t)$, $y \in \partial f(t)$, with $|x - y| = |\eta(t)|$ (note that $\nabla d_f(t, x) = \nabla d_g(t, y)$ and $\nabla^2 d_f(t, x) \geq \nabla^2 d_g(t, y)$). Recalling that $f \in \mathcal{F}_F$, $g \in \mathcal{F}_F^{\lessgtr}$, and F is degenerate elliptic, we have

$$\liminf_{\tau \to 0^+} \frac{\eta(t + \tau) - \eta(t)}{\tau} \geq F(\nabla d_g(t, x), \nabla^2 d_g(t, x)) - F(\nabla d_f(t, y), \nabla^2 d_f(t, y)) \geq 0.$$

We deduce that η is non decreasing, and the assertion follows. □

The following theorem characterizes the disjoint sets property in terms of the functions F, G describing the evolution.

Theorem 7.1. *Assume that $F, G : J_0 \to \mathbf{R}$ are lower semicontinuous. Then the disjoint sets property (equivalently, the regularized disjoint sets property) with respect to $(\mathcal{F}_F, \mathcal{F}_G)$ holds if and only if $(F^+)_c \geq G^+$. In particular*

(i) the (regularized) disjoint sets property with respect to $\big(\mathcal{F}_F, \mathcal{F}_{F_c}\big)$ holds if and only if F is degenerate elliptic;

(ii) the (regularized) disjoint sets property with respect to $\big(\mathcal{F}_F, \mathcal{F}_F\big)$ holds if and only if F is compatible from above.

Proof. Assume that $(F^+)_c \geq G^+$. Let $E \subseteq \mathbf{R}^n$; we shall prove that

$$\mathbf{R}^n \setminus \mathcal{M}(E, \mathcal{F}_F) \in \mathcal{B}(\mathcal{F}_G), \tag{7.4}$$

which implies the disjoint sets property (hence the regularized disjoint sets property, see Lemma 3.2 (5)) with respect to $(\mathcal{F}_F, \mathcal{F}_G)$. As $\mathcal{B}(\mathcal{F}_G) \supseteq \mathcal{B}\big(\mathcal{F}_{(F^+)_c}\big)$, to prove (7.4) it is enough to show that $\mathbf{R}^n \setminus \mathcal{M}(E, \mathcal{F}_F) \in \mathcal{B}\big(\mathcal{F}_{(F^+)_c}\big)$. We first show that if $f, h : [a, b] \subseteq I \to \mathcal{P}(\mathbf{R}^n)$, $f \in \mathcal{F}_{(F^+)_c}$, $h \in \mathcal{F}_F$, then

$$h(a) \subseteq \mathbf{R}^n \setminus f(a) \ \Rightarrow\ h(b) \subseteq \mathbf{R}^n \setminus f(b). \tag{7.5}$$

Set $g := \overline{\mathbf{R}^n \setminus f}$; as $f \in \mathcal{F}_{(F^+)_c}$ we deduce $\frac{\partial d_g}{\partial t} + F^+(\nabla d_g, \nabla^2 d_g) \leq 0$. As $h \in \mathcal{F}_F \subseteq \mathcal{F}_{F^+}$ we also have $\frac{\partial d_h}{\partial t} + F^+(\nabla d_h, \nabla^2 d_h) \geq 0$. By Lemma 7.2, if $h(a) \subseteq g(a)$, then $h(b) \subseteq g(b) = \overline{\mathbf{R}^n \setminus f(b)}$. Then (7.5) follows from the translation invariance of $\mathcal{F}_F$ and the compactness of $\partial h(t)$, $t \in [a, b]$ (see Proposition 3.1 (4)).

Assume by contradiction that there exists a function $f : [a, b] \subseteq I \to \mathcal{P}(\mathbf{R}^n)$, $f \in \mathcal{F}_{(F^+)_c}$, with $f(a) \subseteq \mathbf{R}^n \setminus \mathcal{M}(E, \mathcal{F}_F)(a)$ and $\mathcal{M}(E, \mathcal{F}_F)(b)$ is not contained in $\mathbf{R}^n \setminus f(b)$. Let us define

$$\phi(t) := \begin{cases} \mathcal{M}(E, \mathcal{F}_F)(t) \cap (\mathbf{R}^n \setminus f(t)) & \text{if } t \in [a, b], \\ \mathcal{M}(E, \mathcal{F}_F)(t) & \text{if } t \in I \setminus [a, b]. \end{cases}$$

Since $\phi(b)$ is strictly contained in $\mathcal{M}(E, \mathcal{F}_F)(b)$, to have a contradiction it is enough to show that $\phi \in \mathcal{B}(\mathcal{F}_F)$, which follows from Lemma 3.1 (1), (6), and (7.5).

Assume now that the regularized disjoint sets property with respect to $\big(\mathcal{F}_F, \mathcal{F}_G\big)$ holds. Suppose by contradiction that $F(-p, -X) + G(p, Y) = 2c > 0$ for some $(p, X) \in J_0$, $|p| = 1$, $X, Y \in \mathrm{Sym}(n)$, $Y \geq X$. Let $x \in \mathbf{R}^n$ and $\alpha, \alpha' \in \mathbf{R}$ be such that

$$0 < \alpha + G(p, Y) < c, \qquad 0 < \alpha' + F(-p, -X) < c.$$

Note that $\alpha + \alpha' < 0$. By Lemma 5.2, there exist $\tau > 0$, $f, h : [0, \tau] \to \mathcal{P}(\mathbf{R}^n)$, $f \in \mathcal{F}_G^{\gtrless}$, $h \in \mathcal{F}_F^{\gtrless}$, such that $f(0) \cap h(0) = \{x\}$,

$$\begin{aligned} &x \in \partial f(0), \quad p = \nabla d_f(0, x), \quad Y = \nabla^2 d_f(0, x), \quad \alpha = \frac{\partial d_f}{\partial t}(0, x), \\ &x \in \partial h(0), \quad -p = \nabla d_h(0, x), \quad -X = \nabla^2 d_h(0, x), \quad \alpha' = \frac{\partial d_h}{\partial t}(0, x). \end{aligned}$$

Let $g := \overline{\mathbf{R}^n \setminus h}$. Then $\alpha = \frac{\partial d_f}{\partial t}(0, x) < -\frac{\partial d_h}{\partial t}(0, x) = -\alpha' = \frac{\partial d_g}{\partial t}(0, x)$. By Lemma 6.1 there exists $0 < \tau_1 < \tau$ such that

$$\mathrm{int}(f(t)) \cap \mathrm{int}(h(t)) \neq \emptyset, \qquad t \in\,]0, \tau_1]. \tag{7.6}$$

As F and G are lower semicontinuous and $f \in \mathcal{F}_G^{\gtrless}$, $h \in \mathcal{F}_F^{\gtrless}$, there exists $\varrho > 0$ such that the map $t \in [0,\tau] \to \overline{f_\varrho^-(t)}$ belongs to $\mathcal{F}_G^{\gtrless}$, and the map $t \in [0,\tau] \to \overline{h_\varrho^-(t)}$ belongs to $\mathcal{F}_F^{\gtrless}$. Then, recalling (3.1), for $t \in [0,\tau]$ we have

$$\mathcal{M}_*(\mathrm{int}(f(0)), \mathcal{F}_G^{\gtrless})(t) \supseteq \mathcal{M}\left(\overline{f_\varrho^-(0)}, \mathcal{F}_G^{\gtrless}\right)(t) \supseteq \overline{f_\varrho^-(t)},$$

hence $\mathcal{M}_*(\mathrm{int}(f(0)), \mathcal{F}_G^{\gtrless})(t) \supseteq \mathrm{int}(f(t))$. Similarly we have $\mathcal{M}^*(\mathrm{int}(h(0)), \mathcal{F}_F^{\gtrless})(t) \supseteq \mathrm{int}(h(t))$ for $t \in [0,\tau]$. Hence, using the regularized disjoint sets property with respect to $(\mathcal{F}_F^{\gtrless}, \mathcal{F}_G^{\gtrless})$ (see Lemma 3.2 (1)), from $\mathrm{int}(f(0)) \cap \mathrm{int}(h(0)) = \emptyset$ we get

$$\mathrm{int}(f(t)) \cap \mathrm{int}(h(t)) \subseteq \mathcal{M}_*\big(\mathrm{int}(f(0)), \mathcal{F}_G^{\gtrless}\big)(t) \cap \mathcal{M}^*\big(\mathrm{int}(h(0)), \mathcal{F}_F^{\gtrless}\big)(t) = \emptyset,$$

for $t \in [0,\tau]$, which contradicts (7.6).

Let us prove assertions (i), (ii). The disjoint sets property with respect to $(\mathcal{F}_F, \mathcal{F}_{F_c})$ is equivalent to $(F_c)^- = (F^+)_c \geq (F_c)^+$. This is equivalent to say that F_c is degenerate elliptic, hence to say that $F = (F_c)_c$ is degenerate elliptic. Finally, assertion (ii) follows from Lemma 7.1. □

The following theorem will be used to characterize the regularized joint sets property.

Theorem 7.2. *Assume that $F : J_0 \to \mathbf{R}$ is continuous and degenerate elliptic. Then, for any $E \subseteq \mathbf{R}^n$ we have*

$$\begin{aligned} \mathcal{M}_*(E, \mathcal{F}_F) &= \mathbf{R}^n \setminus \mathcal{M}^*(\mathbf{R}^n \setminus E, \mathcal{F}_{F_c}), \\ \mathcal{M}^*(E, \mathcal{F}_F) &= \mathbf{R}^n \setminus \mathcal{M}_*(\mathbf{R}^n \setminus E, \mathcal{F}_{F_c}). \end{aligned} \tag{7.7}$$

Proof. From Corollary 6.1 and Remark 6.6 in [3], for any $t \in I$ we have

$$\begin{aligned} \mathcal{M}_*(E, \mathcal{F}_F)(t) &= \mathcal{M}_*(E, \mathcal{F}_F^{\gtrless})(t) = \{x \in \mathbf{R}^n : v_{E,F}(t,x) < 0\}, \\ \mathcal{M}^*(E, \mathcal{F}_F)(t) &= \mathcal{M}^*(E, \mathcal{F}_F^{\gtrless})(t) = \{x \in \mathbf{R}^n : v_{E,F}(t,x) \leq 0\}, \end{aligned} \tag{7.8}$$

where $v_{E,F}$ is the unique continuous viscosity solution (in the sense of [15]) of (1.1), with $v_{E,F}(t_0,x) = d_E(x)$. From the uniqueness of the viscosity solution we have $-v_{E,F} = v_{\mathbf{R}^n \setminus E, F_c}$, therefore (7.7) follows from (7.8). □

The following theorem characterizes the regularized joint sets property in terms of the functions F, G describing the evolution.

Theorem 7.3. *Assume that $F, G : J_0 \to \mathbf{R}$ are continuous, $F^+ < +\infty$, $G^+ < +\infty$ and F^+, G^+ are continuous. Then the regularized joint sets property with respect to $\big(\mathcal{F}_F, \mathcal{F}_G\big)$ holds if and only if $(F^+)_c \leq G^+$. In particular*

(i) the regularized joint sets property with respect to $\big(\mathcal{F}_F, \mathcal{F}_{F_c}\big)$ holds for any function F satisfying the hypotheses listed above;

(ii) the regularized joint sets property with respect to $\big(\mathcal{F}_F, \mathcal{F}_F\big)$ holds if and only if F^+ is compatible from below.

Proof. Assume that $(F^+)_c \leq G^+$. We have to show that $\mathcal{M}_*(E, \mathcal{F}_F) \supseteq \mathbf{R}^n \setminus \mathcal{M}^*(\mathbf{R}^n \setminus E, \mathcal{F}_G)$ for any $E \subseteq \mathbf{R}^n$. Let us first prove that

$$\mathcal{M}_*(E, \mathcal{F}_{F^+}) \supseteq \mathbf{R}^n \setminus \mathcal{M}^*(\mathbf{R}^n \setminus E, \mathcal{F}_{G^+}). \tag{7.9}$$

As $(F^+)_c \le G^+$, to prove (7.9) it is enough to show (see (4.5))

$$\mathcal{M}_*(E, \mathcal{F}_{F^+}) \supseteq \mathbf{R}^n \setminus \mathcal{M}^*(\mathbf{R}^n \setminus E, \mathcal{F}_{(F^+)_c}),$$

which follows from (7.7).
Using (7.8), from (7.9) we deduce $\mathcal{M}_*(E, \mathcal{F}^{>}_{F^+}) \supseteq \mathbf{R}^n \setminus \mathcal{M}^*(\mathbf{R}^n \setminus E, \mathcal{F}^{>}_{G^+})$, which, from Theorem 6.1 is equivalent to $\mathcal{M}_*(E, \mathcal{F}^{>}_{F}) \supseteq \mathbf{R}^n \setminus \mathcal{M}^*(\mathbf{R}^n \setminus E, \mathcal{F}^{>}_{G})$. Hence the assertion follows from Lemma 3.2 (2).

Assume now that the regularized joint sets property with respect to $\big(\mathcal{F}_F, \mathcal{F}_G\big)$ holds; then, by Lemma 3.2 (2), it holds with respect to $\big(\mathcal{F}_{F^+}, \mathcal{F}_{G^+}\big)$. Fix $E \subseteq \mathbf{R}^n$; using (7.7) with F replaced by F^+, our hypothesis becomes

$$\begin{aligned}\mathbf{R}^n &= \big[\mathbf{R}^n \setminus \mathcal{M}^*(\mathbf{R}^n \setminus E, \mathcal{F}_{(F^+)_c})\big] \cup \big[\mathbf{R}^n \setminus \mathcal{M}_*(E, \mathcal{F}_{(G^+)_c})\big] \\ &= \mathbf{R}^n \setminus \big(\mathcal{M}^*(\mathbf{R}^n \setminus E, \mathcal{F}_{(F^+)_c}) \cap \mathcal{M}_*(E, \mathcal{F}_{(G^+)_c})\big),\end{aligned}$$

which is equivalent to $\mathcal{M}^*(\mathbf{R}^n \setminus E, \mathcal{F}_{(F^+)_c}) \cap \mathcal{M}_*(E, \mathcal{F}_{(G^+)_c}) = \emptyset$. Therefore the regularized disjoint sets property with respect to $(\mathcal{F}_{(F^+)_c}, \mathcal{F}_{(G^+)_c})$ holds. Applying Theorem 7.1 to $(F^+)_c$,$(G^+)_c$, we get $F^+ \ge (G^+)_c$, which is equivalent to $(F^+)_c \le G^+$.

Let us prove assertions (i), (ii). The regularized disjoint sets property with respect to $(\mathcal{F}_F, \mathcal{F}_{F_c})$ (resp. $(\mathcal{F}_F, \mathcal{F}_F)$) is equivalent to $(F_c)^- = (F^+)_c \le (F_c)^+$ (resp. to $(F^+)_c = ((F^+)^-)_c \le F^+ = (F^+)^-$) which is always satisfied (resp. which is satisfied if and only if F^+ is compatible from below). $\square$

Example 7.1. Consider motion by mean curvature in codimension $k \ge 1$, i.e., $F(p, X) = -\sum_{i=1}^{n-k} \lambda_i$, where $\lambda_1 \le \ldots \le \lambda_{n-1}$ are the eigenvalues of the matrix $P_p X P_p$ which correspond to eigenvectors orthogonal to p. The function F is degenerate elliptic and is not compatible from above, hence the regularized disjoint sets property with respect to $(\mathcal{F}_F, \mathcal{F}_F)$ does not hold, whereas the regularized joint sets property with respect to $(\mathcal{F}_F, \mathcal{F}_F)$ holds.

Corollary 7.1. *Assume that $F : J_0 \to \mathbf{R}$ is continuous, $F^+ < +\infty$ and F^+ is continuous. Then the regularized disjoint sets property and the regularized joint sets property with respect to $(\mathcal{F}_F, \mathcal{F}_{F_c})$ (resp. with respect to $(\mathcal{F}_F, \mathcal{F}_F)$) hold if and only if F is degenerate elliptic (resp. if and only if F^+ is odd).*

Remark 7.1. *The disjoint and joint sets properties, and hence Theorems 7.1 and 7.3, are related to the n-dimensional fattening phenomenon (with respect to $\mathcal{F}$), [11,2,14,16,8,3], that is when, for some $t_1 \ge \bar{t}$,*

$$\begin{aligned}&\mathcal{H}^n\Big(\mathcal{M}^*(E, \mathcal{F}, \bar{t})(t) \setminus \mathcal{M}_*(E, \mathcal{F}, \bar{t})(t)\Big) = 0 \qquad \text{for } t \in [\bar{t}, t_1],\\ &\mathcal{H}^n\Big(\mathcal{M}^*(E, \mathcal{F}, \bar{t})(t) \setminus \mathcal{M}_*(E, \mathcal{F}, \bar{t})(t)\Big) > 0 \qquad \text{for some } t \in\,]t_1, +\infty[\ ,\end{aligned}$$

where $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure. For instance, in [5] it is exhibited a two dimensional example of fattening (with respect to $\mathcal{F}_F$) for curvature flow with a constant forcing term (starting from the union of two disjoint closed balls) and the singularity, in this specific case, is due to the fact that the disjoint sets property with respect to $(\mathcal{F}_F, \mathcal{F}_F)$ is violated. In this case $F = F^+$ and F^+ is not odd.

Remark 7.2. *Assume that $F : J_0 \to \mathbf{R}$ is continuous, odd and degenerate elliptic. Let $A, B \in \mathcal{P}(\mathbf{R}^n)$ be such that $int(B) \subseteq \overline{A}$. Then*

$$\mathcal{M}^*(A \setminus B, \mathcal{F}_F) = \mathcal{M}^*(A, \mathcal{F}_F) \setminus \mathcal{M}_*(B, \mathcal{F}_F).$$

In particular, for any $E \subseteq \mathbf{R}^n$ we have

$$\mathcal{M}^*(\partial E, \mathcal{F}_F) = \mathcal{M}^*(E, \mathcal{F}_F) \setminus \mathcal{M}_*(E, \mathcal{F}_F). \tag{7.10}$$

Proof. From Theorem 7.2 and from the fact that $F = F_c$, (7.10) is equivalent to

$$\mathcal{M}_*\big(\mathrm{int}(B) \cup \mathrm{int}(\mathbf{R}^n \setminus A), \mathcal{F}_F\big) = \mathcal{M}_*\big(\mathrm{int}(B), \mathcal{F}_F\big) \cup \mathcal{M}_*\big(\mathrm{int}(\mathbf{R}^n \setminus A), \mathcal{F}_F\big),$$

which follows from the disjoint sets property with respect to $(\mathcal{F}_F, \mathcal{F}_F)$. □

8. The maximal inner barrier

Beside all barriers introduced in the previous sections we can consider also the definition of inner barriers.

Definition 8.1. *Let $E \subseteq \mathbf{R}^n$ be a given set and $\bar{t} \in I$. The maximal inner barrier $\mathcal{N}(E, \mathcal{F}, \bar{t}) : [\bar{t}, +\infty[\to \mathcal{P}(\mathbf{R}^n)$ (with origin in E at time $\bar{t}$) with respect to the family $\mathcal{F}$ at any time $t \geq \bar{t}$ is defined by*

$$\mathcal{N}(E, \mathcal{F}, \bar{t})(t) := \bigcup \Big\{\phi(t) : \ \phi : [\bar{t}, +\infty[\to \mathcal{P}(\mathbf{R}^n), \ \phi \in \widetilde{\mathcal{B}}(\mathcal{F}, [\bar{t}, +\infty[), \ \phi(\bar{t}) \subseteq E\Big\},$$

where $\widetilde{\mathcal{B}}(\mathcal{F}, [\bar{t}, +\infty[)$ is the family of all functions $\phi : [\bar{t}, +\infty[\to \mathcal{P}(\mathbf{R}^n)$ such that the following property holds: if $f : [a, b] \subseteq [\bar{t}, +\infty[\to \mathcal{P}(\mathbf{R}^n)$ belongs to $\mathcal{F}$ and $\phi(a) \subseteq int\big(f(a)\big)$ then $\phi(b) \subseteq int\big(f(b)\big)$. Similarly to (3.2), we can define $\mathcal{N}_*(E, \mathcal{F}, \bar{t})$ and $\mathcal{N}^*(E, \mathcal{F}, \bar{t})$.

Note that $\phi \in \mathcal{B}(\mathcal{F}_F)$ if and only if $\mathbf{R}^n \setminus \phi \in \widetilde{\mathcal{B}}(\mathcal{F}^{\leq}_{F_c})$. Consequently, for any $E \subseteq \mathbf{R}^n$ we have $\mathcal{N}(E, \mathcal{F}^{\leq}_{F}) = \mathbf{R}^n \setminus \mathcal{M}(\mathbf{R}^n \setminus E, \mathcal{F}_{F_c})$, hence

$$\begin{aligned} \mathcal{N}_*(E, \mathcal{F}^{\leq}_{F}) &= \mathbf{R}^n \setminus \mathcal{M}^*(\mathbf{R}^n \setminus E, \mathcal{F}_{F_c}), \\ \mathcal{N}^*(E, \mathcal{F}^{\leq}_{F}) &= \mathbf{R}^n \setminus \mathcal{M}_*(\mathbf{R}^n \setminus E, \mathcal{F}_{F_c}). \end{aligned} \tag{8.1}$$

The following theorem shows the connection between the minimal barrier and the maximal inner barrier.

Theorem 8.1. *Assume that $F : J_0 \to \mathbf{R}$ is continuous and degenerate elliptic. Then, for any $E \subseteq \mathbf{R}^n$ we have*

$$\mathcal{M}_*(E, \mathcal{F}_F) = \mathcal{N}_*(E, \mathcal{F}^{\leq}_{F}), \qquad \mathcal{M}^*(E, \mathcal{F}_F) = \mathcal{N}^*(E, \mathcal{F}^{\leq}_{F}).$$

Proof. The assertions follow from (7.7) and (8.1). □

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