Calcolo delle Variazioni. — *Barriers for a class of geometric evolution problems.* Nota di Giovanni Bellettini e Matteo Novaga.

Abstract. — We present some general results on minimal barriers in the sense of De Giorgi for geometric evolution problems. We also compare minimal barriers with viscosity solutions for fully nonlinear parabolic geometric problems of the form $u_t + F(t, x, \nabla u, \nabla^2 u) = 0$. If F is not degenerate elliptic, it turns out that we obtain the same minimal barriers if we replace F with F^+, which is defined as the smallest degenerate elliptic function above F.

Key words: Barriers; Nonlinear partial differential equations of parabolic type; Mean curvature flow; Viscosity solutions.

Riassunto. — *Barriere per una classe di problemi geometrici di evoluzione.* Vengono presentati alcuni risultati di carattere generale sulle minime barriere nel senso di De Giorgi per evoluzioni geometriche di insiemi. Vengono anche confrontate le minime barriere con le evoluzioni ottenute usando le soluzioni nel senso della viscosità, per problemi geometrici parabolici completamente non lineari della forma $u_t + F(t, x, \nabla u, \nabla^2 u) = 0$. Se F non è ellittica degenere, si osserva che si ottengono le stesse minime barriere se, al posto di F, si considera la funzione F^+, definita come la più piccola funzione ellittica degenere maggiore o uguale a F.

0. Introduction

In [9] De Giorgi introduced a notion of weak solution, called minimal barrier, for a wide class of evolution problems. An interesting example that falls within this general definition is the mean curvature flow; in this case, since singularities may appear at a finite time even starting from smooth compact data, it is particularly important to have a (possibly unique) notion of weak evolution. In the literature there are many different generalized approaches to geometric evolutions; in particular we mention the pioneristic work of Brakke in the context of geometric measure theory, the viscosity approach of Evans-Spruck [10], Chen-Giga-Goto [5], Giga-Goto-Ishii-Sato [11], the method of the distance function of Soner [16], the variational approach of Almgren-Taylor-Wang [1] and its generalization by means of the minimizing movements of De Giorgi [7], the elliptic regularization method [14] and the set-theoretic subsolutions of Ilmanen [13], the minimal barriers [9] and the penalization method on higher derivatives of De Giorgi [8].

The aim of this note is twofold. Firstly in Section 3 we present some general properties of minimal barriers for geometric evolutions: in particular, concerning geometric fully nonlinear parabolic problems of the form

$$\frac{\partial u}{\partial t} + F(t, x, \nabla u, \nabla^2 u) = 0, \tag{0.1}$$

we study under which conditions on F the disjoint sets property and the joint sets property hold (see Definition 4.1). Moreover, denoting by $\mathcal{F}_F$ the family of all smooth local geometric supersolutions of (0.1) (see Definition 2.8), and denoting by $\mathcal{M}(E, \mathcal{F}_F)$ the minimal barrier starting from an open set $E \subseteq \mathbf{R}^n$ (see Definition 2.2), we observe (Theorem 3.2) that

$$\mathcal{M}(E, \mathcal{F}_F) = \mathcal{M}(E, \mathcal{F}_{F^+}),$$

where F^+ is defined as the smallest degenerate elliptic function greater than or equal to F (see (1.1)). Secondly, in Section 5 we show (see Theorems 5.2, 5.3) that minimal barriers are equivalent to viscosity solutions for geometric problems of the form (0.1) under the assumptions on F made by Giga-Goto-Ishii-Sato in [11]. More generally, it turns out that the minimal barrier is the maximal between all viscosity subsolutions assuming a given initial datum (Corollary 5.3). All proofs will appear in [2,3].

1. Notation

In the following we let $I := [t_0, +\infty[$, for a fixed $t_0 \in \mathbf{R}$; in Section 5 we will take $t_0 = 0$. We denote by $\mathcal{P}(\mathbf{R}^n)$ the family of all subsets of $\mathbf{R}^n$, $n \geq 1$. If C is a subset of $\mathbf{R}^n$ such that $C \neq \mathbf{R}^n$ and $C \neq \emptyset$, we set $d_C(x) := \text{dist}(x, C) - \text{dist}(x, \mathbf{R}^n \setminus C)$, and for any $\varrho > 0$

$$C_\varrho^- := \{x \in \mathbf{R}^n : \text{dist}\,(x, \mathbf{R}^n \setminus C) > \varrho\}, \qquad C_\varrho^+ := \{x \in \mathbf{R}^n : \text{dist}(x, C) < \varrho\}.$$

Given a map $\phi : J \to \mathcal{P}(\mathbf{R}^n)$, where $J \subseteq \mathbf{R}$ is a convex set, if $\phi(t) \neq \mathbf{R}^n$ and $\phi(t) \neq \emptyset$ for any $t \in J$ we let $d_\phi : J \times \mathbf{R}^n \to \mathbf{R}$ be the function defined as

$$d_\phi(t, x) := \text{dist}(x, \phi(t)) - \text{dist}(x, \mathbf{R}^n \setminus \phi(t)) = d_{\phi(t)}(x).$$

Given $\phi_1, \phi_2 : J \to \mathcal{P}(\mathbf{R}^n)$, by $\phi_1 \subseteq \phi_2$ (resp. $\phi_1 = \phi_2$) we mean $\phi_1(t) \subseteq \phi_2(t)$ (resp. $\phi_1(t) = \phi_2(t)$) for any $t \in J$.
Given a function $v : J \times \mathbf{R}^n \to \mathbf{R}$ we denote by v_* (resp. v^*) the lower (resp. upper) semicontinuous envelope of v.
For $x \in \mathbf{R}^n$ and $R > 0$ we set $B_R(x) := \{y \in \mathbf{R}^n : |y - x| < R\}$ and $\mathbf{S}^{n-1} := \{x \in \mathbf{R}^n : |x| = 1\}$. If $c_1, c_2 \in \mathbf{R}$, we let $c_1 \wedge c_2 = \min(c_1, c_2)$ and $c_1 \vee c_2 = \max(c_1, c_2)$. We denote by $\text{Sym}(n)$ the space of all symmetric real $(n \times n)$-matrices. Given $p \in \mathbf{R}^n \setminus \{0\}$, we set $P_p := \text{Id} - p \otimes p/|p|^2$. We also set $J_0 := I \times \mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\}) \times \text{Sym}(n)$.

Given a function $F : J_0 \to \mathbf{R}$, we denote by F_* (resp. F^*) the lower (resp. upper) semicontinuous envelope of F.

For any $(t, x, p, X) \in J_0$, we define

$$\begin{aligned} F_c(t, x, p, X) &:= -F(t, x, -p, -X), \\ F^+(t, x, p, X) &:= \sup\{F(t, x, p, Y) : Y \geq X\}, \\ F^-(t, x, p, X) &:= \inf\{F(t, x, p, Y) : Y \leq X\}. \end{aligned} \tag{1.1}$$

We say that F is locally Lipschitz in X if for any $(t, x, p) \in I \times \mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\})$ the function $F(t, x, p, \cdot)$ is locally Lipschitz.

We recall that F is geometric [5, (1.2)] if $F(t, x, \lambda p, \lambda X + \sigma p \otimes p) = \lambda F(t, x, p, X)$, for any $\lambda > 0$, $\sigma \in \mathbf{R}$, $(t, x, p, X) \in J_0$.
For all definitions and results concerning viscosity solutions we refer to [6] and references therein. In the appendix we list some assumptions used in the paper, following the notation of [11].

2. Definitions of barriers and minimal barriers.

Definitions 2.1, 2.2 are a particular case of the definitions proposed in [9].

Definition 2.1 (barriers). *Let $\mathcal{F}$ be a family of functions with the following property: for any $f \in \mathcal{F}$ there exist $a, b \in \mathbf{R}$, $a < b$, such that $f : [a,b] \to \mathcal{P}(\mathbf{R}^n)$. A function ϕ is a barrier with respect to $\mathcal{F}$ if and only if there exists a convex set $L \subseteq I$ such that $\phi : L \to \mathcal{P}(\mathbf{R}^n)$ and the following property holds: if $f : [a,b] \subseteq L \to \mathcal{P}(\mathbf{R}^n)$ belongs to $\mathcal{F}$ and $f(a) \subseteq \phi(a)$ then $f(b) \subseteq \phi(b)$. We denote by $\mathcal{B}(\mathcal{F})$ the family of all barriers ϕ such that $L = I$ (that is, barriers on the whole of I).*

Definition 2.2 (minimal barrier). *Let $E \subseteq \mathbf{R}^n$ be a given set. The minimal barrier $\mathcal{M}(E, \mathcal{F}, t_0) : I \to \mathcal{P}(\mathbf{R}^n)$ (with origin in E at time t_0) with respect to the family $\mathcal{F}$ at any time $t \in I$ is defined by*

$$\mathcal{M}(E, \mathcal{F}, t_0)(t) := \bigcap \Big\{ \phi(t) : \quad \phi : I \to \mathcal{P}(\mathbf{R}^n),\ \phi \in \mathcal{B}(\mathcal{F}),\ \phi(t_0) \supseteq E \Big\}. \tag{2.1}$$

Definition 2.3 (maximal inner barrier). *Let $E \subseteq \mathbf{R}^n$ be a given set. The maximal inner barrier $\mathcal{N}(E, \mathcal{F}, t_0) : I \to \mathcal{P}(\mathbf{R}^n)$ (with origin in E at time t_0) with respect to the family $\mathcal{F}$ at any time $t \in I$ is defined by*

$$\mathcal{N}(E, \mathcal{F}, t_0)(t) := \bigcup \Big\{ \psi(t) : \quad \psi : I \to \mathcal{P}(\mathbf{R}^n),\ \psi \in \widetilde{\mathcal{B}}(\mathcal{F}),\ \psi(t_0) \subseteq E \Big\}, \tag{2.2}$$

where $\widetilde{\mathcal{B}}(\mathcal{F})$ is as in Definition 2.1 with the set inclusion $\subseteq$ replaced by $\supseteq$.

The connections between $\mathcal{M}(E, \mathcal{F}, t_0)$ and $\mathcal{N}(E, \mathcal{F}, t_0)$ are explained in Theorem 4.3.

The following regularization was introduced in [4] and turns out to be very useful.

Definition 2.4 (regularizations of barriers). *Let $E \subseteq \mathbf{R}^n$. If $t \in I$ we set*

$$\mathcal{M}_*(E, \mathcal{F}, t_0)(t) := \bigcup_{\varrho > 0} \mathcal{M}(E_\varrho^-, \mathcal{F}, t_0)(t), \quad \mathcal{M}^*(E, \mathcal{F}, t_0)(t) := \bigcap_{\varrho > 0} \mathcal{M}(E_\varrho^+, \mathcal{F}, t_0)(t),$$

$$\mathcal{N}_*(E, \mathcal{F}, t_0)(t) := \bigcup_{\varrho > 0} \mathcal{N}(E_\varrho^-, \mathcal{F}, t_0)(t), \quad \mathcal{N}^*(E, \mathcal{F}, t_0)(t) := \bigcap_{\varrho > 0} \mathcal{N}(E_\varrho^+, \mathcal{F}, t_0)(t).$$

Once we have a unique evolution of any subset E of $\mathbf{R}^n$, we have a unique evolution of any initial function u_0.

Definition 2.5. *Let $u_0 : \mathbf{R}^n \to \mathbf{R}$ be a given function. The two functions $\mathcal{M}_{u_0,\mathcal{F}}, \overline{\mathcal{M}}_{u_0,\mathcal{F}} : I \times \mathbf{R}^n \to \mathbf{R} \cup \{\pm\infty\}$ are defined by*

$$\begin{aligned} \mathcal{M}_{u_0,\mathcal{F}}(t,x) &:= \inf\{\lambda \in \mathbf{R} : \mathcal{M}(\{u_0 < \lambda\}, \mathcal{F}, t_0)(t) \ni x\}, \\ \overline{\mathcal{M}}_{u_0,\mathcal{F}}(t,x) &:= \inf\{\lambda \in \mathbf{R} : \mathcal{M}_*(\{u_0 < \lambda\}, \mathcal{F}, t_0)(t) \ni x\}. \end{aligned} \tag{2.3}$$

Besides the concept of barrier, we can also consider the concept of local barrier.

Definition 2.6 (local barriers). *A function ϕ is a local barrier with respect to $\mathcal{F}$ if and only if there exists a convex set $L \subseteq I$ such that $\phi : L \to \mathcal{P}(\mathbf{R}^n)$ and the following property holds: for any $x \in \mathbf{R}^n$ there exists $R > 0$ (depending on ϕ and x) so that if $f : [a,b] \subseteq L \to \mathcal{P}(\mathbf{R}^n)$ belongs to $\mathcal{F}$ and $f(a) \subseteq \phi(a) \cap B_R(x)$, then $f(b) \subseteq \phi(b)$. We denote by $\mathcal{B}_{\mathrm{loc}}(\mathcal{F})$ the family of all local barriers ϕ such that $L = I$ (that is, local barriers on the whole of I).*

Definition 2.7 (local minimal barrier). *Let $E \subseteq \mathbf{R}^n$ be a given set. The local minimal barrier $\mathcal{M}_{\mathrm{loc}}(E, \mathcal{F}, t_0) : I \to \mathcal{P}(\mathbf{R}^n)$ (with origin in E at time t_0) with respect to the family $\mathcal{F}$ at any time $t \in I$ is defined by*

$$\mathcal{M}_{\mathrm{loc}}(E, \mathcal{F}, t_0)(t) := \bigcap \Big\{ \phi(t) : \quad \phi : I \to \mathcal{P}(\mathbf{R}^n), \; \phi \in \mathcal{B}_{\mathrm{loc}}(\mathcal{F}), \; \phi(t_0) \supseteq E \Big\}.$$

Note that a similar definition to Definition (2.6) can be given by localizing also with respect to time. The connection between barriers and local barriers is explained in Theorem 3.1.

The definitions of barriers for geometric evolutions described by a function F for problems of the form (0.1) are a particular case of the previous definitions, by choosing a suitable family $\mathcal{F}_F$, and read as follows. Let $F : J_0 \to \mathbf{R}$ be an arbitrary function.

Definition 2.8. *Let $a, b \in \mathbf{R}$, $a < b$, $[a,b] \subseteq I$ and let $f : [a,b] \to \mathcal{P}(\mathbf{R}^n)$. We write $f \in \mathcal{F}_F$ (and we say that f is a smooth local geometric supersolution of (0.1)) if and only if the following conditions hold: $f(t)$ is closed and $\partial f(t)$ is compact for any $t \in [a,b]$, there exists an open set $A \subseteq \mathbf{R}^n$ such that $d_f \in \mathcal{C}^\infty([a,b] \times A)$, $\partial f(t) \subseteq A$ for any $t \in [a,b]$, and*

$$\frac{\partial d_f}{\partial t}(t,x) + F(t, x, \nabla d_f(t,x), \nabla^2 d_f(t,x)) \geq 0 \qquad t \in]a,b[, \; x \in \partial f(t). \tag{2.4}$$

We write $f \in \mathcal{F}_F^{>}$ (resp. $f \in \mathcal{F}_F^{\leq}$, $f \in \mathcal{F}_F^{=}$) if the strict inequality (resp. the inequality $\leq$, the equality) holds in (2.4).

It turns out that, if F is bounded on compact subsets of J_0, then $\mathcal{B}(\mathcal{F}_F)$ coincides with the class of all barriers with respect to the subfamily of $\mathcal{F}_F$ consisting of all $f : [a,b] \to \mathcal{P}(\mathbf{R}^n)$ such that $f \in \mathcal{F}_F$ and $f(t)$ is compact for any $t \in [a,b]$. Notice also that $\mathbf{R}^n \setminus \mathcal{M}_*(E, \mathcal{F}_F, t_0) = \mathcal{N}^*(\mathbf{R}^n \setminus E, \mathcal{F}_{F_c}^{\leq}, t_0)$, and $\mathbf{R}^n \setminus \mathcal{M}^*(E, \mathcal{F}_F, t_0) = \mathcal{N}_*(\mathbf{R}^n \setminus E, \mathcal{F}_{F_c}^{\leq}, t_0)$.

3. General results on barriers.

The following lemma shows some general properties of the minimal barrier, such as comparison and semigroup property. If $\tau \in \mathbf{R}$, by $\mathcal{M}(E, \mathcal{F}, \tau)$ we mean the minimal barrier constructed by taking barriers on the interval $[\tau, +\infty[$ containing E at the time τ.

Lemma 3.1. *Let $E \subseteq \mathbf{R}^n$. Then the following properties hold.*

(1) $\mathcal{M}(E, \mathcal{F}, t_0)$ *exists and is unique;*
(2) $\mathcal{M}(E, \mathcal{F}, t_0) \in \mathcal{B}(\mathcal{F})$;
(3) $E_1 \subseteq E_2 \Rightarrow \mathcal{M}(E_1, \mathcal{F}, t_0) \subseteq \mathcal{M}(E_2, \mathcal{F}, t_0)$;
(4) $\mathcal{M}(E, \mathcal{F}, t_0)(t_0) = E$;
(5) if $f : [a,b] \subseteq I \to \mathcal{P}(\mathbf{R}^n)$, $f \in \mathcal{F}$, *then*

$$f(t) \subseteq \mathcal{M}(f(a), \mathcal{F}, a)(t), \qquad t \in [a,b]; \tag{3.1}$$

(6) $\mathcal{F} \subseteq \mathcal{G} \Rightarrow \mathcal{M}(E, \mathcal{F}, t_0) \subseteq \mathcal{M}(E, \mathcal{G}, t_0)$;

(7) *assume that the family* $\mathcal{F}$ *satisfies the following property: given* $f : [a,b] \subseteq I \to \mathcal{P}(\mathbf{R}^n)$, $f \in \mathcal{F}$, $t \in]a,b[$, *then* $f\big|_{[a,t]}, f\big|_{[t,b]} \in \mathcal{F}$. *Then* $\mathcal{M}(E, \mathcal{F}, t_0)$ *verifies the semigroup property, i.e.,*

$$\mathcal{M}(E, \mathcal{F}, t_0)(t_2) = \mathcal{M}(\mathcal{M}(E, \mathcal{F}, t_0)(t_1), \mathcal{F}, t_1)(t_2) \qquad \text{if } t_0 \le t_1 \le t_2.$$

The following proposition shows in particular that the minimal barrier coincides with the smooth evolution of (0.1) whenever the latter exists (see (3.2)).

Proposition 3.1. *Assume that* F *does not depend on* x, *is geometric, uniformly elliptic and of class* $\mathcal{C}^\infty$. *Then for any* $E \subseteq \mathbf{R}^n$ *we have* $\mathcal{M}(E, \mathcal{F}_F^{=}) = \mathcal{M}(E, \mathcal{F}_F)$. *Moreover for any* $f : [a,b] \subseteq I \to \mathcal{P}(\mathbf{R}^n)$, $f \in \mathcal{F}_F^{=}$, *we have*

$$f(t) = \mathcal{M}(f(a), \mathcal{F}_F, a)(t), \qquad t \in [a,b]. \tag{3.2}$$

For simplicity of notation, from now on we drop the dependence on t_0 of the minimal barrier.

Under suitable assumptions on F, the families $\mathcal{F}_F$ and $\mathcal{F}_F^{\gtrless}$ give raise to the same minimal barriers, as explained in the following useful remark.

Remark 3.1. *Assume that* $F : (\mathbf{R}^n \setminus \{0\}) \times Sym(n) \to \mathbf{R}$ *is continuous and locally Lipschitz in* X. *Then for any* $E \subseteq \mathbf{R}^n$ *we have*

$$\mathcal{M}_*(E, \mathcal{F}_F^{\gtrless}) = \mathcal{M}_*(E, \mathcal{F}_F), \qquad \mathcal{M}^*(E, \mathcal{F}_F^{\gtrless}) = \mathcal{M}^*(E, \mathcal{F}_F),$$

and the same holds for local minimal barriers.

The following result shows the connection between barriers and local barriers.

Theorem 3.1. *Assume that* $F : (\mathbf{R}^n \setminus \{0\}) \times Sym(n) \to \mathbf{R}$ *is continuous and locally Lipschitz in* X. *Then*

$$\mathcal{B}_{\mathrm{loc}}(\mathcal{F}_F^{\gtrless}) = \mathcal{B}(\mathcal{F}_F^{\gtrless}).$$

In particular, for any $E \subseteq \mathbf{R}^n$ *we have* $\mathcal{M}(E, \mathcal{F}_F^{\gtrless}) = \mathcal{M}_{\mathrm{loc}}(E, \mathcal{F}_F^{\gtrless})$.

The following theorem provides a sort of canonical representation for minimal barriers when F is not degenerate elliptic (i.e., for evolutions without comparison), and it is one of the main results of this note.

Theorem 3.2. *Assume that* $F : (\mathbf{R}^n \setminus \{0\}) \times Sym(n) \to \mathbf{R}$ *is continuous, locally Lipschitz in* X *and* $F^+ < +\infty$ *in* $(\mathbf{R}^n \setminus \{0\}) \times Sym(n)$. *Then*

$$\mathcal{B}(\mathcal{F}_F^{\gtrless}) = \mathcal{B}(\mathcal{F}_{F^+}^{\gtrless}).$$

In particular, for any $E \subseteq \mathbf{R}^n$ *we have* $\mathcal{M}(E, \mathcal{F}_F^{\gtrless}) = \mathcal{M}(E, \mathcal{F}_{F^+}^{\gtrless})$.

4. The disjoint sets property and the joint sets property. The following properties play an important rôle in the theory of minimal barriers.

Definition 4.1. *Let $F, G : (\mathbf{R}^n \setminus \{0\}) \times Sym(n) \to \mathbf{R}$ be two functions, and let $\mathcal{F}_F$, $\mathcal{F}_G$ be the corresponding families of smooth local geometric supersolutions. We say that the disjoint sets property with respect to $(\mathcal{F}_F, \mathcal{F}_G)$ holds if, for any $E \subseteq \mathbf{R}^n$, we have*

$$\mathcal{M}_*(E, \mathcal{F}_F) \cap \mathcal{M}^*(\mathbf{R}^n \setminus E, \mathcal{F}_G) = \emptyset.$$

We say that the joint sets property with respect to $(\mathcal{F}_F, \mathcal{F}_G)$ holds if, for any $E \subseteq \mathbf{R}^n$, we have

$$\mathcal{M}_*(E, \mathcal{F}_F) \cup \mathcal{M}^*(\mathbf{R}^n \setminus E, \mathcal{F}_G) = \mathbf{R}^n.$$

The following theorems characterizes the disjoint sets property and the joint sets property in terms of the functions F and G describing the evolution.

Theorem 4.1. *Assume that F, $G : (\mathbf{R}^n \setminus \{0\}) \times Sym(n) \to \mathbf{R}$ are continuous and locally Lipschitz in X. Assume that $F^+ < +\infty$ and $G^+ < +\infty$ in $(\mathbf{R}^n \setminus \{0\}) \times Sym(n)$ and that F^+, G^+ are continuous. The following two statements hold.*

(i) The disjoint sets property with respect to $(\mathcal{F}_F, \mathcal{F}_G)$ holds if and only if $G^+ \leq (F^+)_c$.

(ii) The joint sets property with respect to $(\mathcal{F}_F, \mathcal{F}_G)$ holds if and only if $G^+ \geq (F^+)_c$.

The following theorem was proved in [4] in the case of driven motion by mean curvature.

Theorem 4.2. *Assume that $F : (\mathbf{R}^n \setminus \{0\}) \times Sym(n) \to \mathbf{R}$ is continuous and degenerate elliptic. Then, for any $E \subseteq \mathbf{R}^n$ we have*

$$\mathcal{M}_*(E, \mathcal{F}_F) = \mathbf{R}^n \setminus \mathcal{M}^*(\mathbf{R}^n \setminus E, \mathcal{F}_{F_c}),$$
$$\mathcal{M}^*(E, \mathcal{F}_F) = \mathbf{R}^n \setminus \mathcal{M}_*(\mathbf{R}^n \setminus E, \mathcal{F}_{F_c}).$$

The following result shows the connection between minimal barriers and maximal inner barriers.

Theorem 4.3. *Assume that $F : (\mathbf{R}^n \setminus \{0\}) \times Sym(n) \to \mathbf{R}$ is continuous and degenerate elliptic. Then for any $E \subseteq \mathbf{R}^n$ we have*

$$\mathcal{N}_*(E, \mathcal{F}_F) = \mathcal{M}_*(E, \mathcal{F}_F), \qquad \mathcal{N}^*(E, \mathcal{F}_F) = \mathcal{M}^*(E, \mathcal{F}_F).$$

5. Comparison between the minimal barriers and the level set flow. From now on we take $I = [0, +\infty[$ (i.e., $t_0 = 0$) and all barriers we consider are barriers on $[0, +\infty[$. The following theorem is proved in [11, Theorem 4.9].

Theorem 5.1. *Assume that $F : J_0 \to \mathbf{R}$ is geometric and satisfies either (F1)-(F4), (F8), or (F1), (F3), (F4), (F9), (F10) (see the Appendix). Let $v_0 : \mathbf{R}^n \to \mathbf{R}$ be a continuous function which is constant outside a bounded subset of $\mathbf{R}^n$. Then there exists a unique continuous viscosity solution (constant outside a bounded subset of $\mathbf{R}^n$) of (0.1) with $v(0, x) = v_0(x)$.*

Theorems 5.2 and 5.3 clarify the relations between minimal barriers and viscosity subsolutions for geometric evolutions.

Theorem 5.2. *Assume that $F : J_0 \to \mathbf{R}$ is geometric and satisfies (F1), (F3), (F4), (F6'), (F7), (F9), (F10). Let u and v be, respectively, a viscosity sub- and supersolution of (0.1) in $]0,+\infty[\times\mathbf{R}^n$. Then for any $\lambda \in \mathbf{R}$ we have*

$$\{x \in \mathbf{R}^n : u^*(\cdot, x) < \lambda\} \in \mathcal{B}(\mathcal{F}_F), \tag{5.1}$$

$$\{x \in \mathbf{R}^n : u^*(\cdot, x) \leq \lambda\} \in \mathcal{B}(\mathcal{F}_F). \tag{5.2}$$

$$\{x \in \mathbf{R}^n : v_*(\cdot, x) > \lambda\} \in \mathcal{B}(\mathcal{F}_{F_c}), \tag{5.3}$$

$$\{x \in \mathbf{R}^n : v_*(\cdot, x) \geq \lambda\} \in \mathcal{B}(\mathcal{F}_{F_c}). \tag{5.4}$$

Moreover, if we take F^+ in place of F in all previous assumptions, then (5.1), (5.2) still hold.

The next theorem is a sort of converse of Theorem 5.2.

Theorem 5.3. *Let $u, v : [0,+\infty[\times\mathbf{R}^n \to \mathbf{R}$ be functions such that $u^* < +\infty$ (resp. $v_* > -\infty$) in $[0,+\infty[\times\mathbf{R}^n$. Assume that $F : J_0 \to \mathbf{R}$ is geometric, lower (resp. upper) semicontinuous and satisfies (F4). Suppose that for any $\lambda \in \mathbf{R}$*

$$\{x \in \mathbf{R}^n : u^*(\cdot, x) < \lambda\} \in \mathcal{B}(\mathcal{F}_F^{\gtrless}) \tag{5.5}$$

$$\Big(\textit{resp.}\quad \{x \in \mathbf{R}^n : v_*(\cdot, x) > \lambda\} \in \mathcal{B}(\mathcal{F}_{F_c}^{\gtrless})\Big). \tag{5.6}$$

If F satisfies (F2), (F8') then u (resp. v) is viscosity subsolution (resp. supersolution) of (0.1) in $]0,+\infty[\times\mathbf{R}^n$. If F^+ (resp. F^-) satisfies (F4), (F8') then u (resp. v) is viscosity subsolution (resp. supersolution) of

$$\frac{\partial u}{\partial t} + F^+(t, x, \nabla u, \nabla^2 u) = 0 \tag{5.7}$$

$$\Big(\textit{resp.}\ \frac{\partial u}{\partial t} + F^-(t, x, \nabla u, \nabla^2 u) = 0\Big) \tag{5.8}$$

in $]0,+\infty[\times\mathbf{R}^n$.

The following result shows the connection between minimal barriers and the continuous viscosity solution whenever the latter exists and is unique, and generalizes a result of [4].

Corollary 5.1. *Assume that $F : J_0 \to \mathbf{R}$ is geometric and satisfies (F1), (F3), (F4), (F6'), (F7), (F9), (F10). Let $E \subseteq \mathbf{R}^n$ be a bounded set and denote with $v : [0,+\infty[\times\mathbf{R}^n \to \mathbf{R}$ the unique uniformly continuous viscosity solution of (0.1) with $v(0,x) = v_0(x) := (-1) \vee d_E(x) \wedge 1$. Then for any $t \in [0,+\infty[$ we have*

$$\mathcal{M}_*(E, \mathcal{F}_F^{\gtrless})(t) = \mathcal{M}_*(E, \mathcal{F}_F)(t) = \{x \in \mathbf{R}^n : v(t,x) < 0\}, \tag{5.9}$$

$$\mathcal{M}^*(E, \mathcal{F}_F^{\gtrless})(t) = \mathcal{M}^*(E, \mathcal{F}_F)(t) = \{x \in \mathbf{R}^n : v(t,x) \leq 0\}, \tag{5.10}$$

hence $\mathcal{M}_{v_0,\mathcal{F}_F} = v$. Moreover if $F = F_c$ then $\mathcal{M}^*(E, \mathcal{F}_F) \setminus \mathcal{M}_*(E, \mathcal{F}_F) \in \mathcal{B}(\mathcal{F}_F)$.

The following results generalize Corollary 5.1.

Corollary 5.2. *Assume that $F : J_0 \to \mathbf{R}$ is geometric, lower semicontinuous and satisfies (F4). Assume that F^+ satisfies (F1), (F3), (F4), (F6'), (F7), (F9), (F10). Then for any bounded set $E \subseteq \mathbf{R}^n$ and any $t \in [0, +\infty[$ we have*

$$\mathcal{M}_*(E, \mathcal{F}_F)(t) = \mathcal{M}_*(E, \mathcal{F}_F^{\geq})(t) = \{x \in \mathbf{R}^n : v(t,x) < 0\},$$

$$\mathcal{M}^*(E, \mathcal{F}_F)(t) = \mathcal{M}^*(E, \mathcal{F}_F^{\geq})(t) = \{x \in \mathbf{R}^n : v(t,x) \leq 0\},$$

where v is the unique uniformly continuous viscosity solution of (5.7) and $v(0,x) = v_0(x) := (-1) \vee d_E(x) \wedge 1$. In particular, thanks to Corollary 5.1, we have

$$\mathcal{M}_*(E, \mathcal{F}_F) = \mathcal{M}_*(E, \mathcal{F}_{F^+}), \qquad \mathcal{M}^*(E, \mathcal{F}_F) = \mathcal{M}^*(E, \mathcal{F}_{F^+})$$

(compare Theorem 3.2).

Corollary 5.3. *Assume that $F : J_0 \to \mathbf{R}$ is geometric and satisfies (F1), (F3), (F4), (F6'), (F7), (F9), (F10). Let $u_0 : \mathbf{R}^n \to \mathbf{R}$ be a given function such that $u_0^* < +\infty$ in $\mathbf{R}^n$. Define*

$$S_{u_0} := \{v : v \text{ is a viscosity subsolution of } (0.1) \text{ in }]0, +\infty[\, \times \mathbf{R}^n, v^*(0,x) = u_0^*(x)\}.$$

If u_0 is upper semicontinuous then $\mathcal{M}_{u_0, \mathcal{F}_F} = \mathcal{M}_{u_0, \mathcal{F}_F^{\geq}} = \sup\{v : v \in S_{u_0}\}$. In the general case we have $\overline{\mathcal{M}}_{u_0, \mathcal{F}_F} = \overline{\mathcal{M}}_{u_0, \mathcal{F}_F^{\geq}} = \sup\{v : v \in S_{u_0}\}$.

Remark 5.1. *A similar assertion of Corollary 5.3 (under the same hypotheses) holds for supersolutions. Precisely, if u_0 is lower semicontinuous (resp. arbitrary) such that $u_{0*} > -\infty$ in $\mathbf{R}^n$ we have that, for any $(t,x) \in [0, +\infty[\times \mathbf{R}^n$, the function*

$$\sup\{\mu : \mathcal{M}(\{u_0 > \mu\}, \mathcal{F}_F)(t) \ni x\}$$

$$(\textit{resp. } \sup\{\mu : \mathcal{M}_*(\{u_0 > \mu\}, \mathcal{F}_F)(t) \ni x\})$$

coincides with the infimum of $u(t,x)$, where u varies over all viscosity supersolutions of (0.1) in $]0, +\infty[\, \times \mathbf{R}^n$ such that $u_*(0,x) = u_0(x)$ (resp. $u_*(0,x) = u_{0*}(x)$) and same assertions with $\mathcal{F}_F$ replaced by $\mathcal{F}_F^{\geq}$.

The following remark shows the connections between the minimal barrier and the viscosity evolution without growth conditions on F (see [15,12]) and for unbounded sets E.

Remark 5.2. *Assume that $F : (\mathbf{R}^n \setminus \{0\}) \times Sym(n) \to \mathbf{R}$ is geometric and satisfies (F1), (F2). Let u and v be, respectively, a viscosity sub- and supersolution of*

$$\frac{\partial u}{\partial t} + F(\nabla u, \nabla^2 u) = 0 \tag{5.11}$$

in $]0, +\infty[\, \times \mathbf{R}^n$, in the sense of [15, Definition 1.2]. Then (5.1)-(5.4) hold. Moreover, if $u : [0, +\infty[\times \mathbf{R}^n \to \mathbf{R}$ is a function such that $u^* < +\infty$ in $[0, +\infty[\times \mathbf{R}^n$ and satisfies (5.5) for any $\lambda \in \mathbf{R}$, then u is a viscosity subsolution of (5.11) in $]0, +\infty[\, \times \mathbf{R}^n$. Finally, Corollary 5.1 still holds, even if E is unbounded.

In particular we have the following result.

Corollary 5.4. *Assume that $F : (\mathbf{R}^n \setminus \{0\}) \times Sym(n) \to \mathbf{R}$ is geometric and satisfies (F1),(F2). Let $E \subseteq \mathbf{R}^n$ and let $v : [0, +\infty[\times \mathbf{R}^n \to \mathbf{R}$ be the unique uniformly continuous viscosity solution of (5.11) with $v(0,x) = v_0(x) := d_E(x)$ (in the sense of [15, Definition 1.2]). Then for any $t \in [0, +\infty[$ we have (5.9) and (5.10). In particular $\mathcal{M}^*(E, \mathcal{F}_F)(t) \setminus \mathcal{M}_*(E, \mathcal{F}_F)(t) = \{x \in \mathbf{R}^n : v(t,x) = 0\}$ and $\mathcal{M}_{v_0, \mathcal{F}_F} = v$.*

6. Appendix.

We list here some assumptions used in this note. We follow the notation of [11, pp. 462-463]; we omit those properties in [11] which are not useful in our context.

(F1) $F : J_0 \to \mathbf{R}$ is continuous;

(F2) F is degenerate elliptic, i.e., $F(t,x,p,X) \geq F(t,x,p,Y)$ for any $(t,x,p,X) \in J_0$, $Y \in \mathrm{Sym}(n)$, $Y \geq X$;

(F3) $-\infty < F_*(t,x,0,0) = F^*(t,x,0,0) < +\infty$ for all $t \in [0,+\infty[$, $x \in \mathbf{R}^n$;

(F4) for every $R > 0$, $\sup\{|F(t,x,p,X)| : |p|, |X| \leq R, (t,x,p,X) \in J_0\} < +\infty$;

(F6) for every $R > \varrho > 0$ there is a constant $c = c_{R,\varrho}$ such that

$$|F(t,x,p,X) - F(t,x,q,Y)| \leq c(|p-q| + |X-Y|)$$

for all $t \in [0,+\infty[$, $x \in \mathbf{R}^n$, $\varrho \leq |p|, |q| \leq R$, $|X|, |Y| \leq R$;

(F6') for every $R > \varrho > 0$ there is a constant $c = c_{R,\varrho}$ such that

$$|F(t,x,p,X) - F(t,x,q,X)| \leq c|p-q|$$

for any $t \in [0,+\infty[$, $x \in \mathbf{R}^n$, $\varrho \leq |p|, |q| \leq R$, $|X| \leq R$;

(F7) there are $\varrho_0 > 0$ and a modulus σ_1 such that

$$F^*(t,x,p,X) - F^*(t,x,0,0) \leq \sigma_1(|p| + |X|),$$
$$F_*(t,x,p,X) - F_*(t,x,0,0) \geq -\sigma_1(|p| + |X|),$$

provided $t \in [0,+\infty[$, $x \in \mathbf{R}^n$, $|p|, |X| \leq \varrho_0$;

(F8) there is a modulus σ_2 such that

$$|F(t,x,p,X) - F(t,y,p,X)| \leq |x-y||p|\sigma_2(1 + |x-y|)$$

for $y \in \mathbf{R}^n$, $(t,x,p,X) \in J_0$;

(F8') for any $R \geq 0$ there is a modulus σ_R such that

$$|F(t,x,p,X) - F(t,y,p,X)| \leq |x-y||p|\sigma_R(1 + |x-y|)$$

for $y \in \mathbf{R}^n$, $(t,x,p,X) \in J_0$, $|X| \leq R$;

(F9) there is a modulus σ_2 such that $F_*(t,x,0,0) - F^*(t,y,0,0) \geq -\sigma_2(|x-y|)$ for any $t \in [0,+\infty[$, $x, y \in \mathbf{R}^n$;

(F10) suppose that $-\mu \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & \mathrm{Id} \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq \nu \begin{pmatrix} \mathrm{Id} & -\mathrm{Id} \\ -\mathrm{Id} & \mathrm{Id} \end{pmatrix}$ with $\mu, \nu \geq 0$. Let $R \geq 2\nu \vee \mu$ and let $\varrho > 0$; then

$$F_*(t,x,p,X) - F^*(t,y,p,-Y) \geq -|x-y||p|\overline{\sigma}(1 + |x-y| + \nu|x-y|^2)$$

for $(t,x) \in [0,+\infty[\times\mathbf{R}^n$, $\varrho \leq |p| \leq R$, with some modulus $\overline{\sigma} = \overline{\sigma}_{R,\varrho}$ independent of t, x, y, X, Y, μ, ν.

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