ARMA manuscript No.
(will be inserted by the editor)

On a crystalline variational problem, part I: first variation and global L^∞–regularity

G. BELLETTINI, M. NOVAGA, M. PAOLINI

Abstract

Let $\phi : \mathbb{R}^n \to [0, +\infty[$ be a given positively one-homogeneous convex function, and let $\mathcal{W}_\phi := \{\phi \le 1\}$. Motivated by motion by crystalline mean curvature in three dimensions, we introduce and study the class $\mathcal{R}_\phi(\mathbb{R}^n)$ of "smooth" boundaries in the relative geometry induced by the ambient Banach space $(\mathbb{R}^n, \phi)$. One can realize that, even when $\mathcal{W}_\phi$ is a polytope, $\mathcal{R}_\phi(\mathbb{R}^n)$ cannot be reduced to the class of polyhedral boundaries (locally resembling $\partial\mathcal{W}_\phi$). Curved portions must be necessarily included and this fact (as well as the nonsmoothness of $\partial\mathcal{W}_\phi$) is source of several technical difficulties, related to the geometry of Lipschitz manifolds. Given a boundary ∂E in the class $\mathcal{R}_\phi(\mathbb{R}^n)$, we rigorously compute the first variation of the corresponding anisotropic perimeter, which leads to a variational problem on vector fields defined on ∂E. It turns out that the minimizers have a uniquely determined (intrinsic) tangential divergence on ∂E. We define such a divergence to be the ϕ-mean curvature κ_ϕ of ∂E; the function κ_ϕ is expected to be the initial velocity of ∂E, whenever ∂E is considered as the initial datum for the corresponding anisotropic mean curvature flow. We prove that κ_ϕ is bounded on ∂E and that its sublevel sets are characterized through a variational inequality.

1. Introduction

Motion by crystalline mean curvature describes the interface evolution obtained as gradient flow of a surface energy functional P_ϕ having a crystalline density with respect to the usual perimeter. This means that, denoting by $t \to \partial E(t)$ the evolving front, the flow tries to reduce as fast as possible the value of $P_\phi(E(t))$, where

$$P_\phi(E(t)) := \int_{\partial E(t)} \phi^o(\nu^{E(t)})\, d\mathcal{H}^{n-1}.$$

Here $\nu^{E(t)}$ is the outward euclidean unit normal to $\partial E(t)$ and ϕ^o (the surface tension) is a positively one homogeneous function such that $\mathcal{W}_\phi^o := \{\phi^o \leq 1\}$ is a convex *polytope* of $\mathbb{R}^n$ containing the origin in its interior. The term crystalline refers to the fact that the convex body $\mathcal{W}_\phi^o$ is faceted.

This evolution provides a simplified model for describing several phenomena in material science and crystal growth, see for instance [12], [13], [28], [23], [32]. It also represents an extreme example of anisotropic geometric flow, being the set $\mathcal{W}_\phi^o$ neither strictly convex nor of class C^2. The mathematical analysis of this problem begun with the works of J. Taylor [28], [27], [30], [31] and received a certain attention in the last few years [2], [11], [1], [20], [17], [33]. In comparison with more familiar geometric evolutions (such as motion by mean curvature) it presents additional difficulties, due to the lack of regularity both of the involved operators and of the flowing interfaces. These obstructions, which at a first sight are of technical nature, reveal that the study of the geometric properties of hypersurfaces in a finite dimensional Banach space endowed with a crystalline metric cannot easily be reduced to more regular situations.

The simplest (even if not realistic) model is in $n = 2$ space dimensions, when the interface is a closed curve. In this case several results have been proved: in particular, the class of curves which are admissible as "regular" initial data for the evolution is characterized. The structure of a curve in this class is the following: if we denote by $\mathcal{W}_\phi$ the Wulff shape (that is $\mathcal{W}_\phi := \{\phi \leq 1\}$, where ϕ is the dual function of ϕ^o), a closed Lipschitz curve is admissible if it is a sequence of segments (with a precise order) which are parallel to some edge of $\partial\mathcal{W}_\phi = \{\phi = 1\}$ and of segments or arcs which correspond to vertices of $\partial\mathcal{W}_\phi$ [31], [25], [21], [26], [22], [18], [19], [4]. In addition, a local in time existence theorem for "regular" evolutions holds: each segment corresponding to an edge of $\partial\mathcal{W}_\phi$ translates parallely to itself (with a suitable velocity), while the remaining segments or arcs have zero velocity. It is important to remark that the admissibility properties of the curve, in relation with the geometry of $\partial\mathcal{W}_\phi$, remain unchanged during this evolution. Finally, a comparison principle holds, and therefore the flow is uniquely determined [19].

The physical case is however in three space dimensions, where the situation is much more complicated. In this case, one of the crucial mathematical problems, which to the best knowledge of the authors is still open, is the short time existence of a "smooth" flow. Some examples can enlighten the difficulties related to such a kind of result. In [6] two explicit crystalline evolutions of surfaces are constructed. In both the examples, the initial surfaces are polyhedral and their facets correspond to facets of the Wulff shape. The first example is completely rigorous, and shows that, at the initial time of the evolution, some facet can subdivide into smaller facets (facet-breaking phenomenon) and therefore new facets may appear. By means of a comparison argument [5], one can also show that the computed evolution is the unique crystalline evolution of the given initial surface. The second example, which is not completely rigorous, but is confirmed by several numerical simulations performed with different methods [24], [29] suggests that some facets can instantly curve. This unexpected phenomenon has some interesting consequences, which influence the approach to the crystalline evolution problem in 3D. For instance, it shows that the preferred set of directions, corresponding to the orientation of the facets of $\partial\mathcal{W}_\phi$, is not preserved during the evolution: indeed, new directions outside of the preferred set may appear. In addition, one infers that the class of polyhedral surfaces (compatible with the geometry of $\partial\mathcal{W}_\phi$) is a too restricted class where looking for an existence result for crystalline evolutions. We remark that, in any case, the evolutions of the two examples should be regarded as "regular" crystalline evolutions. Summing up, in order to study crystalline motion by mean curvature, it seems that some preliminary steps are necessary. A first step is to have a reasonable definition of the class of crystalline "regular" surfaces. Given a set E in this class, another step is to understand which is the initial velocity field on ∂E, in particular which are its singularities (which could be interpreted as the breaking regions). Finally, it is natural to investigate whether the class of regular surfaces is stable or not under motion by crystalline mean curvature.

The above arguments are the motivations for the *stationary* problem studied in the present paper. We first introduce the class $\mathcal{R}_\phi(\mathbb{R}^n)$ of compact admissible interfaces, called Lipschitz ϕ-regular sets, which should be considered as the analog of smooth boundaries in the case of crystalline geometry. The boundary ∂E of a Lipschitz ϕ-regular set E may be polyhedral (with a structure locally resembling the structure of $\partial\mathcal{W}_\phi$) but may also have curve portions. In addition, we impose the esistence of a family of normal convex cones $x \to K(x)$ (with varying dimensions, in connection with the geometry of the Wulff shape) which contains a *continuous* section on ∂E. The precise requirement is that ∂E admits a vector field $n_\phi : \partial E \to \mathbb{R}^n$ which is a *Lipschitz* selection in this family of cones, i.e. $n_\phi \in \mathrm{Lip}(\partial E; \mathbb{R}^n)$ and $n_\phi(x) \in K(x)$ for any $x \in \partial E$. If $\partial\mathcal{W}_\phi$ were smooth and strictly convex, the vector field n_ϕ would be uniquely determined (since each cone reduces to a vector at any point of ∂E) and is usually called the Cahn-Hoffman vector field. A similar class (the so–called ϕ-regular sets) has been introduced in [5], [6]. Such a class may be, in principle, larger than $\mathcal{R}_\phi(\mathbb{R}^n)$, since in that case the selection n_ϕ is required only to be bounded with bounded divergence. Even if the local existence theorem is still missing, the class of ϕ-regular sets supports a uniqueness result, which is obtained as a by-product of an Allen-Cahn type approximation argument [5]. Several technical reasons in the present paper impose to require the selection n_ϕ to be Lipschitz: one of these is related to the weak definition of a suitable divergence operator (see (1)) on a Lipschitz manifold (Remark 42). We stress once again that the definition of Lipschitz ϕ-regularity is a regularity property in the relative sense of the geometry induced on $\mathbb{R}^n$ by the Finsler metric ϕ. For instance, it is easy to see that, in $n = 2$ space dimensions, if we choose $\phi(\xi) := |\xi_1| + |\xi_2|$, the euclidean ball $B_1(0) := \{x \in \mathbb{R}^2 : |x| \leq 1\}$ does not admit any vector field $\eta : \partial B_1(0) \to \mathbb{R}^2$ such that $(B_1(0), \eta)$ becomes Lipschitz ϕ-regular.

Fix now a Lipschitz ϕ-regular set $(E, n_\phi) \in \mathcal{R}_\phi(\mathbb{R}^n)$. We introduce a variational problem on vector fields defined on ∂E, whose solution gives the definition of ϕ-mean curvature of ∂E and is expected to describe the initial velocity field of the evolution problem having ∂E as initial datum. More precisely, we propose to study the following minimum problem:

$$\inf\left\{\int_{\partial E}(\mathrm{div}_{\phi,n_\phi,\tau}\, N)^2\ \phi^o(\nu^E)\ d\mathcal{H}^{n-1} : N\in H(\partial E;\mathbb{R}^n)\right\}. \tag{1}$$

The symbol $H(\partial E;\mathbb{R}^n)$ denotes the class of all vector fields $N\in L^2(\partial E;\mathbb{R}^n)$ such that $\mathrm{div}_{\phi,n_\phi,\tau}N\in L^2(\partial E)$ and which satisfy the constraint $N(x)\in K(x)$ for $\mathcal{H}^{n-1}$–almost every $x\in\partial E$. We refer to Section 3 for the precise definition of the tangential divergence operator appearing in (1), which must be intended in a weak sense and in the relative geometry induced on ∂E by ϕ. Any solution of problem (1) has the same divergence: if $N_{\min}$ denotes a solution of (1), we are therefore lead in a natural way to define the ϕ-mean curvature κ_ϕ of ∂E as $\kappa_\phi := \mathrm{div}_{\phi,n_\phi,\tau}\, N_{\min}$. Our conjecture is that κ_ϕ represents the effective velocity of the initial set E (recall that there is uniqueness of the evolution). To substantiate this conjecture we consider the following arguments. It is well known (see for instance [10]) that the solution of a parabolic partial differential inclusion of the form $u_t\in Au$ (where A is a maximal monotone multivalued operator) selects, at each time, a particular element (the so-called canonical element) which minimizes $\|Au\|_{L^2}$, i.e. u actually solves $u_t=A_m u$, where $A_m x$ realizes the minimum in $\min\{\|y\|^2_{L^2} : y\in Ax\}$. This idea has been applied to crystalline evolutions of graphs by T. Fukui and Y. Giga in [16]. It is not difficult to see [24] that the analog of this minimum property in our geometric framework is given by (1).

Another remarkable argument which leads to consider problem (1) comes from the expression of the first variation of the functional P_ϕ. This computation shows that the intrinsic perimeter P_ϕ is reduced with maximal speed when the variation is performed along the field $-\kappa_\phi\, N_{\min}$.

If κ_ϕ gives really the initial velocity, its jump set should correspond to the "fractures" along which new facets appear in the subsequent evolution; moreover, the regions where κ_ϕ is continuous but not constant should represent the regions of ∂E where curving is expected. This is in accordance with the examples computed in [6].

The plan of the paper is the following. In Section 2 we give some notation, we recall the main properties of the duality mappings and we define what we mean by a facet of ∂E. In Section 3 we introduce the class $\mathcal{R}_\phi(\mathbb{R}^n)$ of Lipschitz ϕ-regular sets (Definition 31). Relations between different definitions of ϕ-regular sets are briefly illustrated in Remark 32. Given $(E,n_\phi)\in\mathcal{R}_\phi(\mathbb{R}^n)$, one of the first technical difficulties is that we need to extend Lipschitz vector fields, originally defined on ∂E, in an open neighbourhood of ∂E, in a Lipschitz way. This is the content of Lemma 33. The properties of the level sets of the intrinsic oriented distance function from ∂E and the connection with n_ϕ are considered in Lemma 34. These preliminary results are used to define the tangential divergence of a vector field defined on ∂E with respect to ϕ (Definition 41), and to exploit some of its properties. The weak operator of ϕ-tangential divergence is denoted by $\mathrm{div}_{\phi,n_\phi,\tau}$. In principle, it may depend on the choice of the field n_ϕ, but it turns out that, for all vector fields of interest in this paper, this is not the case. It must be noted that, on flat portions of ∂E, this divergence coincides with the usual (weak) tangential divergence div_τ. In Proposition 46 and Corollary 47 we prove that, even if for the same set E there are infinitely many possible choices of the Cahn-Hoffman vector field n_ϕ so that (E, n_ϕ) becomes Lipschitz ϕ-regular, the definition of $\mathrm{div}_{\phi,n_\phi,\tau}$ as operator on ϕ-normal vector fields is intrinsic, i.e. does not depend on the choice of n_ϕ. In Section 5 we compute the first variation of the functional P_ϕ, thus relating the minimum problem (1) with motion by crystalline mean curvature. The first variation of P_ϕ turns out also to be strictly related to the operator $\mathrm{div}_{\phi,n_\phi,\tau}$. In Section 6 we are concerned with a minimum problem of the form (1), with an additional term depending on a given function g. In the evolution problem g plays the rôle of the forcing term. In Lemma 61 we prove the existence of minimizers; it is easily seen that their ϕ-tangential divergence is uniquely determined. The corresponding Euler-Lagrange inequality is computed in (63). We denote by $d_{\min}$ the tangential divergence of a minimizer ($d_{\min}$ reduces to κ_ϕ in the case $g = 0$). The crucial result proved in Theorem 65 is a reformulation of the Euler-Lagrange inequality for $d_{\min}$, by means of an inequality on each of its level sets. A direct consequence of Theorem 65 is that $d_{\min}$ is a bounded function (Theorem 67). In particular Lipschitz ϕ-regular sets have bounded ϕ-mean curvature.

In a forthcoming paper [7] we continue the analysis on the structure of Lipschitz ϕ-regular sets and, using the result $\kappa_\phi \in L^\infty(\partial E)$, we prove that κ_ϕ has bounded variation on suitable facets F. We also investigate further properties of the level sets of κ_ϕ on F, in connection with the geometry of the facet of $\mathcal{W}_\phi$ parallel to F.

2. Notation

In the following we denote by $\cdot$ the standard euclidean scalar product in $\mathbb{R}^n$ and by $|\cdot|$ the euclidean norm of $\mathbb{R}^n$, $n \geq 2$. If $\rho > 0$ and $x \in \mathbb{R}^n$, we set $B_\rho(x) := \{y \in \mathbb{R}^n : |y - x| < \rho\}$.

Given two vectors $a, b \in \mathbb{R}^n$ we let $a \otimes b$ be the matrix whose entries are $(a \otimes b)_{ij} = a_i b_j$. If M is a $(n \times n)$ matrix, by Ma (resp. aM) we denote the vector of components $(Ma)_i = M_{ij} a_j$ (resp. $(aM)_i = M_{ji} a_j$), and $\mathrm{tr} M$ is the trace of M. Note that $a \otimes b\, c = ab \cdot c$ for any $c \in \mathbb{R}^n$. With the notation $A \Subset B$ we mean that the set A is compactly contained in B.

The symbol $\mathcal{H}^k$ denotes the k-dimensional Hausdorff measure in $\mathbb{R}^n$, $k \in [0, n]$.

If $E \subset \mathbb{R}^n$, we denote by 1_E the characteristic function of E and by ∂E the topological boundary of E. By $\mathrm{Lip}(\partial E)$ (resp. $\mathrm{Lip}(\partial E; \mathbb{R}^n)$) we denote the class of all Lipschitz functions (resp. vector fields) defined on ∂E.

We say that E is Lipschitz if E is open and, for any $x \in \partial E$, there exists $\rho > 0$ such that $B_\rho(x) \cap \partial E$ is the graph of a Lipschitz function f and $B_\rho(x) \cap E$ is the subgraph of f (with respect to a suitable orthogonal coordinate system). If E is Lipschitz, for $\mathcal{H}^{n-1}$-almost every $x \in \partial E$ we denote by $\nu^E(x)$ the outward unit euclidean normal to ∂E at x, and by $T_x\partial E$ the tangent hyperplane to ∂E at x.

Finsler metrics on $\mathbb{R}^n$. We indicate by $\phi : \mathbb{R}^n \to [0, +\infty[$ a convex function satisfying the properties

$$\phi(\xi) \geq \Lambda|\xi|, \qquad \phi(a\xi) = a\phi(\xi), \qquad \xi \in \mathbb{R}^n, \ a \geq 0, \tag{2}$$

for a suitable constant $\Lambda \in]0, +\infty[$. The function $\phi^o : \mathbb{R}^n \to [0, +\infty[$ is defined as

$$\phi^o(\xi^*) := \sup \{\xi^* \cdot \xi \ : \ \phi(\xi) \leq 1\},$$

and is the dual of ϕ. ϕ and ϕ^o are sometimes called Finsler metrics on $\mathbb{R}^n$. We set

$$\mathcal{W}^o_\phi := \{\xi^* \in \mathbb{R}^n : \phi^o(\xi^*) \leq 1\}, \qquad \mathcal{W}_\phi := \{\xi \in \mathbb{R}^n : \phi(\xi) \leq 1\}.$$

$\mathcal{W}^o_\phi$ and $\mathcal{W}_\phi$ are compact convex sets whose interior parts contain the origin. By a facet of $\partial\mathcal{W}_\phi$ (or of $\partial\mathcal{W}^o_\phi$) we always mean a $(n-1)$-dimensional facet.

We say that ϕ is crystalline if $\mathcal{W}_\phi$ is a convex polytope. If ϕ is crystalline, then also $\mathcal{W}^o_\phi$ is a convex polytope. $\mathcal{W}^o_\phi$ is sometimes called the Frank diagram and $\mathcal{W}_\phi$ the Wulff shape.

Duality mappings. By T and T^o we denote the possibly multivalued mappings defined by

$$\begin{aligned} T(\xi) &:= \{\xi^* \in \mathbb{R}^n : \xi^* \cdot \xi = \phi(\xi)^2 = (\phi^o(\xi^*))^2\}, \qquad \xi \in \mathbb{R}^n, \\ T^o(\xi^*) &:= \{\xi \in \mathbb{R}^n : \xi \cdot \xi^* = (\phi^o(\xi^*))^2 = \phi(\xi)^2\}, \qquad \xi^* \in \mathbb{R}^n, \end{aligned} \tag{3}$$

which are called the duality mappings. One can check that

$$T(\xi) = \frac{1}{2}D^-(\phi(\xi))^2 = \phi(\xi)D^-\phi(\xi), \qquad T(a\xi) = aT(\xi), \qquad \xi \in \mathbb{R}^n, a \geq 0,$$

and similarly for T^o and ϕ^o, where D^- denotes the subdifferential. Moreover T, T^o are maximal monotone operators, T (resp. T^o) takes $\partial\mathcal{W}_\phi$ (resp. $\partial\mathcal{W}^o_\phi$) onto $\partial\mathcal{W}^o_\phi$ (resp. onto $\partial\mathcal{W}_\phi$).

Notice that, if $\xi \in \partial\mathcal{W}_\phi$, $T(\xi)$ is the intersection of the closed outward normal cone to $\partial\mathcal{W}_\phi$ with $\partial\mathcal{W}^o_\phi$.

ϕ-distance function. Given a nonempty set $E \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, we set

$$\mathrm{dist}_\phi(x, E) := \inf_{y \in E} \phi(x - y), \qquad \mathrm{dist}_\phi(E, x) := \inf_{y \in E} \phi(y - x),$$

$$d^E_\phi(x) := \mathrm{dist}_\phi(x, E) - \mathrm{dist}_\phi(\mathbb{R}^n \setminus E, x).$$

The function d^E_ϕ is therefore the oriented intrinsic distance function negative inside E; note that, since in general ϕ is not symmetric, $-d^E_\phi$ does not necessarily coincide with $d^{\mathbb{R}^n \setminus E}_\phi$.

At each point x where d^E_ϕ is differentiable, there holds $\nabla d^E_\phi(x) \in \partial\mathcal{W}^o_\phi$, hence

$$\phi^o(\nabla d^E_\phi) = 1 \qquad \text{at } x; \tag{4}$$

we set $\nu_\phi^E(x) := \nabla d_\phi^E(x) = \frac{\nu^E(x)}{\phi^o(\nu^E(x))}$. As a consequence of (3), at $\mathcal{H}^{n-1}$-almost every $x \in \partial E$ we have

$$\nu_\phi^E(x) \cdot p = 1 \qquad \forall p \in T^o(\nu_\phi^E(x)). \tag{5}$$

If E is Lipschitz we define

$$\mathrm{Nor}_\phi(\partial E; \mathbb{R}^n) = \{N : \partial E \to \mathbb{R}^n : N(x) \in T^o(\nu_\phi^E(x)) \text{ for } \mathcal{H}^{n-1}-\text{ a.e.} x \in \partial E\},$$
$$\mathrm{Lip}_{\nu,\phi}(\partial E; \mathbb{R}^n) = \mathrm{Lip}(\partial E; \mathbb{R}^n) \cap \mathrm{Nor}_\phi(\partial E; \mathbb{R}^n).$$

Note that if $N \in \mathrm{Nor}_\phi(\partial E; \mathbb{R}^n)$, then $\phi(N(x)) = 1$ for $\mathcal{H}^{n-1}$-almost every $x \in \partial E$. Moreover, if $N_1, N_2 \in \mathrm{Nor}_\phi(\partial E; \mathbb{R}^n)$, then $N_1 - N_2$ is tangent to ∂E, since $N_1 \cdot \nu_\phi^E = 1 = N_2 \cdot \nu_\phi^E$.

We let dP_ϕ be the measure supported on ∂E with density $\phi^o(\nu^E)$ with respect to $\mathcal{H}^{n-1}$, i.e.

$$dP_\phi(B) := \int_B \phi^o(\nu^E)\, d\mathcal{H}^{n-1}, \quad B \text{ Borel set, } B \subseteq \partial E.$$

If E is Lipschitz and $\psi \in \mathrm{Lip}(\partial E)$ we denote by $\nabla_\tau \psi$ the euclidean tangential gradient of ψ on ∂E, which is defined at $\mathcal{H}^{n-1}$-almost every point of ∂E. If $v \in \mathrm{Lip}(\partial E; \mathbb{R}^n)$ we denote by $\mathrm{div}_\tau v$ the euclidean tangential divergence of v, which is defined (at $\mathcal{H}^{n-1}$-almost every point of ∂E).

In the following, whenever there is no risk of confusion, we often do not indicate the dependence on E of the unit normals ν^E and ν_ϕ^E, i.e. we set $\nu := \nu^E$, $\nu_\phi := \nu_\phi^E$.

3. Lipschitz ϕ-regular sets

Definition 31 *Let $E \subseteq \mathbb{R}^n$ be a Lipschitz set with compact boundary. We say that E is Lipschitz ϕ-regular if there exists a vector field $n_\phi \in \mathrm{Lip}_{\nu,\phi}(\partial E; \mathbb{R}^n)$. We denote by $\mathcal{R}_\phi(\mathbb{R}^n)$ the class of all Lipschitz ϕ-regular sets.*

With a little abuse of notation, we sometimes shall write $(E, n_\phi) \in \mathcal{R}_\phi(\mathbb{R}^n)$, and we shall say that (E, n_ϕ) is Lipschitz ϕ-regular.

Observe that if (E, n_ϕ) is Lipschitz ϕ-regular, then $\phi(n_\phi) = 1$ everywhere on ∂E. The canonical example of Lipschitz ϕ-regular set is given by $(\mathcal{W}_\phi, x)$.

Remark 32 *In [6] it is introduced the class of ϕ-regular sets, whose definition is different from Definition 31, since n_ϕ is required to belong to $L^\infty(\partial E; \mathbb{R}^n)$ and to admit an extension on an open set containing ∂E which is bounded and has bounded divergence. In view of Lemmas 34 and 45 below, it follows that $\mathcal{R}_\phi(\mathbb{R}^n)$ is contained in the class of ϕ-regular sets. One can also prove that, if ϕ is crystalline and E is a ϕ-regular polyhedron, then E is Lipschitz ϕ-regular.*

The Lipschitz regularity requirement on n_ϕ in Definition 31 allows to define the ϕ-tangential divergence $\mathrm{div}_{\phi,n_\phi,\tau}$ on the elements of $\mathrm{Nor}_\phi(\partial E;\mathbb{R}^n)$, see Definition 41. Assuming less regularity on n_ϕ should require a non trivial modification of Definition 41, see Remark 42.

The following lemma is well known in the smooth case. We need its version in our nonsmooth case in order to prove Corollary 47 and to compute the first variation of P_ϕ (see Section 5).

Lemma 33 *Let $E \subseteq \mathbb{R}^n$ be a Lipschitz set with compact boundary. Let $\eta \in \mathrm{Lip}(\partial E;\mathbb{R}^n)$ be a vector field with the property that there is a constant $c > 0$ verifying*

$$|\eta(x)\cdot\nu^E(x)| \geq c \qquad \text{for } \mathcal{H}^{n-1}-\text{a.e. } x \in \partial E. \tag{6}$$

Then there exist $\epsilon > 0$ and an open set $U(\partial E)$ containing ∂E such that the following properties hold.

(i) The map $F_\eta : \partial E\times\,]-\epsilon,\epsilon[\, \to U(\partial E)$ defined by $F_\eta(x,t) := x + t\eta(x)$ is bilipschitz.

(ii) Set $G_\eta := F_\eta^{-1}$, $G_\eta(\cdot) := (\pi_\eta(\cdot), t_\eta(\cdot)) \in \partial E\times\,]-\epsilon,\epsilon[$ on $U(\partial E)$. Let $z \in U(\partial E)$. If η is differentiable at $\pi_\eta(z)$ and there exists $T_{\pi_\eta(z)}\partial E$, then G_η is differentiable at z.

Proof. Let $x_0 \in \partial E$. Up to a rotation of coordinates, a neighbourhood of x_0 in ∂E can be written as $(s, f(s))$, for $s \in \Omega \subseteq \mathbb{R}^{n-1}$ and $f : \Omega \to \mathbb{R}$ a Lipschitz function. Write $\eta(s, f(s)) = (\eta_1(s, f(s)), \eta_2(s, f(s)) \in \mathbb{R}^{n-1}\times\mathbb{R}$, and set $\widehat{F}(s,t) := F_\eta(x,t) = (s + t\eta_1(s,f(s)), f(s) + t\eta_2(s,f(s)))$. We want to apply the Implicit Function Theorem in the Lipschitz case to the function $\widehat{F}$, see [14]. In order to fulfill all the assumptions, we need to prove that, if $\{(s_n,t_n)\}$ is a sequence of points in the domain of $\widehat{F}$ converging to (s,t) as $n \to \infty$ such that $\widehat{F}$ is differentiable at each (s_n,t_n) and there is the limit M of $\nabla\widehat{F}(s_n,t_n)$, then M is nonsingular. Observe that

$$\nabla\widehat{F}(s_n,t_n) = \begin{pmatrix} Id + O(t_n) & \eta_1(s_n, f(s_n)) \\ \nabla f(s_n) + O(t_n) & \eta_2(s_n, f(s_n)) \end{pmatrix}.$$

Using (6) one can check that $\left|\det\left(\nabla\widehat{F}(s_n,t_n)\right)\right| \geq c/2$, for any $n \in \mathbb{N}$ and $\epsilon > 0$ small enough, and therefore M is nonsingular. By [14] it follows that $\widehat{F}$ is locally invertible with a Lipschitz inverse $\widehat{G}$. Let us verify that F_η (hence $\widehat{F}$) is globally injective for $\epsilon > 0$ small enough. Assume by contradiction that $F_\eta(x,t) = x + t\eta(x) = y + r\eta(y) = F_\eta(y,r)$ for some $x,y \in \partial E$ and $t,r \in\,]-\epsilon,\epsilon[$, $(x,t) \neq (y,r)$. Then $|x-y| \leq 2\epsilon\|\eta\|_{L^\infty(\partial E;\mathbb{R}^n)}$, which contradicts the local invertibility of F_η. Using the compactness of ∂E, property (i) follows.

Let us prove (ii). Notice that if η is differentiable at $\pi_\eta(z)$ and there exists the tangent hyperplane to ∂E at $\pi_\eta(z)$ (i.e. f is differentiable at the point of Ω corresponding to $\pi_\eta(z)$), then $\widehat{F}$ is differentiable at $\widehat{G}(z)$. Differentiating the identity $\widehat{F}(\widehat{G}) = \mathrm{Id}$ we get that $\widehat{G}$ is differentiable at the point corresponding to z, which implies that G_η is differentiable at z.

Lemma 34 *Let $E \in \mathcal{R}_\phi(\mathbb{R}^n)$ and let $\eta \in \mathrm{Lip}_{\nu,\phi}(\partial E;\mathbb{R}^n)$. Let $\epsilon > 0$ be given by Lemma 33. Then for any $t \in]-\epsilon,\epsilon[$ we have*

$$d_\phi^E(x + t\eta(x)) = t, \qquad x \in \partial E. \tag{7}$$

In particular

$$\{z \in \mathbb{R}^n : d_\phi^E(z) = t\} = \{x + t\eta(x) : x \in \partial E\}, \qquad t \in]-\epsilon,\epsilon[. \tag{8}$$

Moreover,

$$\nabla d_\phi^E(x + t\eta(x)) = \nu_\phi(x), \qquad \nabla d_\phi^E(x + t\eta(x)) \cdot \eta(x) = 1, \tag{9}$$

for any $t \in]-\epsilon,\epsilon[$ and for $\mathcal{H}^{n-1}$-almost every $x \in \partial E$. Finally, if π_η is as in (ii) of Lemma 33, and if we define $\eta^e(z) := \eta(\pi_\eta(z))$ for $z \in U(\partial E)$, then $(\{z \in U(\partial E) : d_\phi^E(z) = t\}, \eta^e)$ is Lipschitz ϕ-regular, for any $t \in]-\epsilon,\epsilon[$.

Proof. Let us consider the case $t > 0$, since the case $t < 0$ is similar. Since $d_\phi^E(x+t\eta(x)) \le \phi(x+t\eta(x)-x) = t$, in order to prove (7) we need to show that $d_\phi^E(x+t\eta(x)) \ge t$. Set for simplicity $\pi := \pi_\eta$, and let $U(\partial E)$ be as in Lemma 33. Fix $z \in U(\partial E) \cap (\mathbb{R}^n \setminus E)$ such that there exists $T_{\pi(z)}\partial E$ and η is differentiable at $\pi(z)$ (hence the function F_η is differentiable at $(\pi(z), 0)$). By (ii) of Lemma 33, it follows that $t(\cdot) := t_\eta(\cdot)$ is differentiable at z. We need two intermediate steps.

Step 1. Let us prove that

$$\nabla t(z) \perp T_{\pi(z)}\partial E.$$

Fix $\tau \in T_{\pi(z)}\partial E$ and set $z_\epsilon := z + \epsilon\tau$, for $\epsilon \in \mathbb{R}$ small enough. ¿From the relation $\pi(z) + t(z)\eta(\pi(z)) = z$ we get

$$\begin{aligned}\big(t(z_\epsilon) - t(z)\big)\eta(\pi(z)) &= t(z_\epsilon)\eta(\pi(z_\epsilon)) - t(z)\eta(\pi(z)) \\ &\quad + t(z_\epsilon)\big[\eta(\pi(z)) - \eta(\pi(z_\epsilon))\big] \\ &= z_\epsilon - z + \pi(z) - \pi(z_\epsilon) + t(z_\epsilon)\big[\eta(\pi(z)) - \eta(\pi(z_\epsilon))\big].\end{aligned} \tag{10}$$

Taking the scalar product of both sides of (10) with $\nu_\phi(\pi(z))$, using (5) and recalling that $z_\epsilon - z$ is a tangent vector to ∂E at $\pi(z)$, we obtain

$$\begin{aligned}t(z_\epsilon) - t(z) &= (\pi(z) - \pi(z_\epsilon)) \cdot \nu_\phi(\pi(z)) \\ &\quad + t(z_\epsilon)\big[\eta(\pi(z)) - \eta(\pi(z_\epsilon))\big] \cdot \nu_\phi(\pi(z)).\end{aligned} \tag{11}$$

The existence of $T_{\pi(z)}\partial E$ and the fact that $z_\epsilon - z$ is a tangent vector imply

$$\big(\pi(z) - \pi(z_\epsilon)\big) \cdot \nu_\phi(\pi(z)) = o(\epsilon), \tag{12}$$

so that (11) becomes

$$t(z_\epsilon) - t(z) = t(z_\epsilon)\big[\eta(\pi(z)) - \eta(\pi(z_\epsilon))\big] \cdot \nu_\phi(\pi(z)) + o(\epsilon). \tag{13}$$

Let us now prove that

$$\big[\eta(\pi(z)) - \eta(\pi(z_\epsilon))\big] \cdot \nu_\phi(\pi(z)) = o(\epsilon). \tag{14}$$

If $\eta = (\eta^1, \dots, \eta^n)$, $\nu_\phi = (\nu_{\phi_1}, \dots, \nu_{\phi_n})$ and $\pi = (\pi_1, \dots, \pi_n)$, summing over repeated indices and using (12) we get

$$\left[\eta(\pi(z)) - \eta(\pi(z_\epsilon))\right] \cdot \nu_\phi(\pi(z)) = \nabla_j \eta^i(\pi(z)) \nu_{\phi_i}(\pi(z))(\pi_j(z) - \pi_j(z_\epsilon)) + o(\epsilon). \tag{15}$$

Recalling that $\phi(\eta(\pi(z)))^2 = 1$, one can check that

$$0 = D^-(\phi(\eta(\pi(z)))^2) = \nu_\phi(\pi(z)) \nabla\eta(\pi(z)) \tag{16}$$

and therefore (14) follows from (15).

We conclude, using (13) and (14) that $t(z_\epsilon) - t(z) = o(\epsilon)$. Recalling that $\epsilon\nabla t(z) \cdot \tau = t(z_\epsilon) - t(z) + o(\epsilon)$, we get $\epsilon\nabla t(z) \cdot \tau = o(\epsilon)$, which proves step 1.

Step 2. Let us prove that

$$\nabla t(z) = \nu_\phi(\pi(z)).$$

¿From step 1 it follows that $\nabla t(z) = \lambda\nu_\phi(\pi(z))$ for some $\lambda \in \mathbb{R}$. Precisely, $\lambda = \nabla t(z) \cdot \eta(\pi(z))$. Differentiating with respect to r the relation $t(\pi(z) + r\eta(\pi(z))) = r$, we get $\lambda = 1$ and step 2 is proved.

By step 2 we deduce $\phi^o(\nabla t(z)) = 1$, which implies

$$t(z_2) - t(z_1) \leq \phi(z_2 - z_1) \qquad \text{for any } z_1, z_2 \in U(\partial E) \cap (\mathbb{R}^n \setminus E). \tag{17}$$

Taking $z_2 := z$ and $z_1 \in \partial E$ such that $d_\phi^E(z) = \phi(z - z_1)$, inequality (17) becomes

$$d_\phi^E(z) \geq t(z) - t(z_1) = t(z),$$

and (7) follows.

Equality (8) is a direct consequence of (7). Finally, the set $\{z \in U(\partial E) : d_\phi^E(z) = t\}$ is the image of ∂E through a bilipschitz map, η^e is a Lipschitz vector field on $U(\partial E)$ and $\eta^e(z) = \eta(\pi(z)) \in T^o(\nu_\phi(\pi(z))) = T^o(\nu_\phi(z))$.

We conclude this section with a result which will be useful in the computation of the first variation of the functional P_ϕ. Let $\eta \in \mathrm{Lip}_{\nu,\phi}(\partial E; \mathbb{R}^n)$, let $U(\partial E)$ and π_η be as in Lemma 33. Let $\psi \in \mathrm{Lip}(\partial E)$ and define $\psi^e \in \mathrm{Lip}(U(\partial E))$, $\eta^e \in \mathrm{Lip}(U(\partial E); \mathbb{R}^n)$ as $\psi^e(z) := \psi(\pi_\eta(z))$, $\eta^e(z) := \eta(\pi_\eta(z))$. For $t \in \mathbb{R}$ with $|t| < \epsilon$, $\epsilon > 0$ small enough, define

$$\widetilde{F} \in \mathrm{Lip}(U(\partial E) \times \,]-\epsilon, \epsilon[; \mathbb{R}^n), \qquad \widetilde{F}(z,t) := z + t\psi^e(z)\eta^e(z), \tag{18}$$

and set $\widetilde{F}^t(\cdot) := \widetilde{F}(\cdot, t)$. Finally, let $E_t := \widetilde{F}^t(E)$.

Lemma 35 *There exists $\epsilon > 0$ such that for $|t| < \epsilon$ the following properties hold.*

(1) The set E_t is Lipschitz continuous, and $\partial E_t = \{z : z = x + t\psi(x)\eta(x), x \in \partial E\}$.
(2) For $\mathcal{H}^{n-1}$-almost every $x \in \partial E$, there exists $\nabla d_\phi^{E_t}(x + t\psi(x)\eta(x))$. We define

$$\nu_t(x) := \frac{\nabla d_\phi^{E_t}(x + t\psi(x)\eta(x))}{|\nabla d_\phi^{E_t}(x + t\psi(x)\eta(x))|}. \tag{19}$$

(3) For $\mathcal{H}^{n-1}$-almost every $x \in \partial E$, there exists $\frac{d}{dt}\nu_t(x)_{|t=0}$, and

$$\frac{d}{dt}\nu_{t\,|t=0} = -\nu^E \nabla(\psi^e \eta^e) + (\nu^E \cdot \nu^E \nabla(\psi^e \eta^e))\nu^E \qquad \mathcal{H}^{n-1} - \text{a.e. on } \partial E. \tag{20}$$

(4) For $\mathcal{H}^{n-1}$-almost every $x \in \partial E$, there exists the right derivative $\frac{d^+}{dt}\phi^o(\nu_t(x))$ at time zero, and there holds

$$\frac{d^+}{dt}\phi^o(\nu_t(x))_{|t=0} = \max_{p \in T^o(\nu_\phi(x))} p \cdot \frac{d}{dt}\nu_t(x)_{|t=0}. \tag{21}$$

(5) $\nu_\phi(x)\nabla\eta^e(x) = 0$ for $\mathcal{H}^{n-1}$-almost every $x \in \partial E$.

Proof. The field $\psi\eta$ does not verify property (6), since ψ may vanish somewhere on ∂E. However, write $t\psi\eta = -n_\phi + (n_\phi + t\psi\eta)$, where $n_\phi \in \mathrm{Lip}_{\nu,\phi}(\partial E; \mathbb{R}^n)$; for $|t|$ small enough, both $-n_\phi$ and $n_\phi + t\psi\eta$ satisfy property (6), and therefore ∂E_t is a bilipschitz image of ∂E from Lemma 33. In addition, it is clear that $\partial E_t := \{x + t\psi(x)\eta(x) : \, x \in \partial E\}$.

Notice that $d_\phi^{E_t} = d_\phi^E - t$, and therefore, for $\mathcal{H}^{n-1}$-almost every $x \in \partial E$, there exists $\nabla d_\phi^{E_t}(x) = \nabla d_\phi^E(x) = \nu_\phi^E(x)$, and $\nu_\phi^E(x) = \nabla d_\phi^E(x + t\psi(x)\eta(x))$ by (9). This proves (2).

Let us prove (3). The equality $d_\phi^E(z) = d_\phi^{E_t}(\widetilde{F}^t(z))$ for $z \in U(\partial E)$ yields, by differentiation,

$$\nabla d_\phi^{E_t}(\widetilde{F}^t(z)) = \nabla d_\phi^E(z)\Big(\mathrm{Id} + t\nabla(\psi^e\eta^e)(z) + o(t)\Big)^{-1}$$

for almost every $z \in U(\partial E)$ (precisely for any $z \in U(\partial E)$ such that $\nu^E(\pi_\eta(z))$ exists and such that η is differentiable at $\pi_\eta(z)$). Therefore

$$\nabla d_\phi^{E_t}(\widetilde{F}^t(z)) = \nabla d_\phi^E(z) - t\nabla d_\phi^E(z)\nabla(\psi^e\eta^e)(z) + o(t).$$

Then (3) follows by a direct computation and using the definition of ν_t.

Assertion (4) follows using (3) and the properties of subdifferential of convex functions, and (5) follows by differentiating the equality $\nu_\phi \cdot \eta^e = 1$, see (16).

4. ϕ-tangential divergence

The definition of weak ϕ-tangential divergence on a Lipschitz ϕ-regular set E with respect to a nonsmooth Finsler metric ϕ is quite involved. To justify our definition we start from the smooth case, i.e. for strictly convex smooth ϕ^2 and ϕ^{o2} and smooth sets E. We recall from [8] and [3] that in the smooth case there holds

$$\int_{\partial E} \mathrm{tr}\big[(\mathrm{Id} - n_\phi \otimes \nu_\phi)\nabla v\big]\, dP_\phi = \int_{\partial E} v \cdot \nu_\phi \,\mathrm{div}\, n_\phi \, dP_\phi, \tag{22}$$

for any $v \in C^1(U(\partial E); \mathbb{R}^n)$, where $U(\partial E)$ is a suitable open neighbourhood of ∂E and $n_\phi := T^o(\nu_\phi^E)$ on $U(\partial E)$. One can check that $\mathrm{tr}\big[(\mathrm{Id} - n_\phi \otimes \nu_\phi)\nabla v\big]$ depends only on the values of v on ∂E. If $\psi \in C^1(U(\partial E); \mathbb{R}^n)$, then $\mathrm{tr}\big[(\mathrm{Id} -$

$n_\phi \otimes \nu_\phi)\nabla(\psi v)\big] = \psi \mathrm{tr}\big[(\mathrm{Id} - n_\phi \otimes \nu_\phi)\nabla v\big] + \big[(\mathrm{Id} - \nu_\phi \otimes n_\phi)\nabla \psi\big] \cdot v$ (notice the switch of the rôle of ν_ϕ and n_ϕ). Therefore, from (22) we get

$$\int_{\partial E} \psi \; \mathrm{tr}\big[(\mathrm{Id} - n_\phi \otimes \nu_\phi)\nabla v\big] \; dP_\phi \tag{23}$$

$$= \int_{\partial E} \psi \; v \cdot \nu_\phi \; \mathrm{div} \; n_\phi \; dP_\phi - \int_{\partial E} \big[(\mathrm{Id} - \nu_\phi \otimes n_\phi)\nabla \psi\big] \cdot v \; dP_\phi. \tag{24}$$

Formula (23) will be the starting point for our definition of ϕ-tangential divergence in the nonsmooth case. Let us introduce the ϕ-tangential divergence for vector fields $v \in L^2(\partial E; \mathbb{R}^n)$ as bounded linear operator on $\mathrm{Lip}(\partial E)$.

Definition 41 *Let $(E, n_\phi) \in \mathcal{R}_\phi(\mathbb{R}^n)$ and let $v \in L^2(\partial E; \mathbb{R}^n)$. We define the function $\mathrm{div}_{\phi, n_\phi, \tau} \; v : \mathrm{Lip}(\partial E) \to \mathbb{R}$ as follows: for any $\psi \in \mathrm{Lip}(\partial E)$ we set*

$$\langle \mathrm{div}_{\phi, n_\phi, \tau} \; v, \psi \rangle := \int_{\partial E} \psi \; v \cdot \nu_\phi \; \mathrm{div}_\tau n_\phi \; dP_\phi - \int_{\partial E} \big[(\mathrm{Id} - \nu_\phi \otimes n_\phi)\nabla_\tau \psi\big] \cdot v \; dP_\phi. \tag{25}$$

Remark 42 *Definition 41 cannot be easily adapted when (E, n_ϕ) is a ϕ-regular set in the sense of [6] (see Remark 32). For instance, if we assume that n_ϕ is only bounded, the definition of $\mathrm{div}_\tau n_\phi$ seems to require an integration by parts formula involving the euclidean mean curvature of ∂E, which cannot be in principle computed, since ∂E is only Lipschitz continuous.*

The following observations are immediate.

(i) If ψ is extended out of ∂E by a differentiable function ψ^e, then $(\mathrm{Id} - \nu_\phi \otimes n_\phi)\nabla_\tau \psi = (\mathrm{Id} - \nu_\phi \otimes n_\phi)\nabla \psi^e$ on ∂E, i.e. any euclidean normal component of the gradient of ψ is killed in formula (25) by the operator $\mathrm{Id} - \nu_\phi \otimes n_\phi$. This is a consequence of (5).

(ii) $\mathrm{div}_{\phi, n_\phi, \tau} v$ is a linear continuous map on $\mathrm{Lip}(\partial E)$.

(iii) In the smooth case, i.e. for strictly convex smooth ϕ^2 and $(\phi^o)^2$ and smooth sets E, $\mathrm{div}_{\phi, n_\phi, \tau} v = \mathrm{tr}\big[(\mathrm{Id} - n_\phi \otimes \nu_\phi)\nabla v\big]$, for any $v \in C^1(U(\partial E); \mathbb{R}^n)$.

The operator $\mathrm{div}_{\phi, n_\phi, \tau}$ depends on ϕ and could depend, in general, on n_ϕ.

Definition 43 *Let $(E, n_\phi) \in \mathcal{R}_\phi(\mathbb{R}^n)$ and let $v \in L^2(\partial E; \mathbb{R}^n)$. We say that $\mathrm{div}_{\phi, n_\phi, \tau} v$ is independent of the choice of n_ϕ if, given $n^*_\phi \in \mathrm{Lip}_{\nu, \phi}(\partial E; \mathbb{R}^n)$, we have*

$$\langle \mathrm{div}_{\phi, n_\phi, \tau} v, \psi \rangle = \langle \mathrm{div}_{\phi, n^*_\phi, \tau} v, \psi \rangle \qquad \forall \psi \in \mathrm{Lip}(\partial E).$$

When $\mathrm{div}_{\phi, n_\phi, \tau} v$ is independent of the choice of n_ϕ, we simply set $\mathrm{div}_{\phi, \tau} v := \mathrm{div}_{\phi, n_\phi, \tau} v$.

If $X \in L^2(\partial E; \mathbb{R}^n)$ is tangent to ∂E and $N \in \mathrm{Nor}_\phi(\partial E; \mathbb{R}^n)$ it turns out that $\mathrm{div}_{\phi, n_\phi, \tau} X$ and $\mathrm{div}_{\phi, n_\phi, \tau} N$ are independent of the choice of n_ϕ, see (2) of Lemma 44 and Corollary 47.

Lemma 44 *Let $(E, n_\phi) \in \mathcal{R}_\phi(\mathbb{R}^n)$. The following assertions hold.*

(1) $\langle \mathrm{div}_{\phi,n_\phi,\tau} n_\phi, \psi\rangle = \int_{\partial E} \psi \, \mathrm{div}_\tau n_\phi \, dP_\phi$ *for any* $\psi \in \mathrm{Lip}(\partial E)$.
(2) *Let* $X \in L^2(\partial E; \mathbb{R}^n)$ *be tangent to* ∂E. *Then*

$$\langle \mathrm{div}_{\phi,n_\phi,\tau} X, \psi\rangle = -\int_{\partial E} \nabla_\tau \psi \cdot X \, dP_\phi \qquad \forall \psi \in \mathrm{Lip}(\partial E).$$

In particular, $\mathrm{div}_{\phi,n_\phi,\tau} X$ *is independent of the choice of* n_ϕ.
(3) *Let* $X \in L^2(\partial E; \mathbb{R}^n)$ *be tangent to* ∂E. *Then* $\langle \mathrm{div}_{\phi,\tau} X, 1\rangle = 0$.
(4) *Let* $N_1, N_2 \in \mathrm{Nor}_\phi(\partial E; \mathbb{R}^n)$. *Then* $\langle \mathrm{div}_{\phi,n_\phi,\tau}(N_1 - N_2), 1\rangle = 0$.

Proof. If v is such that $v \cdot n_\phi = 1$, (25) becomes

$$\langle \mathrm{div}_{\phi,n_\phi,\tau} \, v, \psi\rangle = \int_{\partial E} \psi \, \mathrm{div}_\tau n_\phi \, dP_\phi - \int_{\partial E} \nabla_\tau \psi \cdot (v - n_\phi) \, dP_\phi.$$

Letting $v = n_\phi$, assertion (1) follows. Assertion (2) is immediate, and (3) follows from (2). Assertion (4) follows from (3), since $N_1 - N_2 \in L^2(\partial E; \mathbb{R}^n)$ is tangent to ∂E.

A refinement of property (1) of Lemma 44 is given in Proposition 46 below.

The following lemma, which follows from (5) of Lemma 35, shows that for any $\eta \in \mathrm{Lip}_{\nu,\phi}(\partial E; \mathbb{R}^n)$ the euclidean tangential divergence of η coincides with the divergence of the extension η^e of η.

Lemma 45 *Let* $(E, n_\phi) \in \mathcal{R}_\phi(\mathbb{R}^n)$ *and let* $\eta \in \mathrm{Lip}_{\nu,\phi}(\partial E; \mathbb{R}^n)$. *Let* $U(\partial E)$ *and* π_η *be as in Lemma 33, and set* $\eta^e(z) := \eta(\pi_\eta(z))$ *for any* $z \in U(\partial E)$. *Then for* $\mathcal{H}^{n-1}$*-almost every* $x \in \partial E$ *there holds*

$$\mathrm{div}_\tau \eta(x) = \mathrm{div} \, \eta^e(x) = \mathrm{tr}\left(\Big(\mathrm{Id} - n_\phi(x) \otimes \nu_\phi(x)\Big) \nabla \eta^e(x)\right). \tag{26}$$

The following proposition shows that the operator $\mathrm{div}_{\phi,n_\phi,\tau}$ coincides with the operator div_τ on vector fields of $\mathrm{Lip}_{\nu,\phi}(\partial E; \mathbb{R}^n)$.

Proposition 46 *Let* $(E, n_\phi) \in \mathcal{R}_\phi(\mathbb{R}^n)$ *and let* $\eta \in \mathrm{Lip}_{\nu,\phi}(\partial E; \mathbb{R}^n)$. *Then*

$$\langle \mathrm{div}_{\phi,n_\phi,\tau} \eta, \psi\rangle = \int_{\partial E} \psi \, \mathrm{div}_\tau \, \eta \, dP_\phi \qquad \forall \psi \in \mathrm{Lip}(\partial E). \tag{27}$$

In particular $\mathrm{div}_{\phi,n_\phi,\tau} \eta$ *is independent of the choice of* n_ϕ.

Proof. Fix $\psi \in \mathrm{Lip}(\partial E)$, and write

$$\int_{\partial E} \psi \, \mathrm{div}_\tau \, \eta \, dP_\phi = \int_{\partial E} \psi \, \mathrm{div}_\tau \, n_\phi \, dP_\phi + \int_{\partial E} \psi \, \mathrm{div}_\tau \, (\eta - n_\phi) \, dP_\phi. \tag{28}$$

Set $\zeta := \eta - n_\phi$. Let $U(\partial E)$ and $\pi := \pi_\eta$ be as in Lemma 34, and set $\psi^e(z) := \psi(\pi(z))$, $\eta^e(z) := \eta(\pi(z))$, $\zeta^e(z) := \zeta(\pi(z))$ for $z \in U(\partial E)$. Let $\epsilon > 0$ be small enough. Recalling (4), the coarea formula for Lipschitz maps [15], the one-homogeneity of ϕ^o and (9), we have

$$\begin{aligned}\int_{\{-\epsilon<d_\phi^E<\epsilon\}} \psi^e \mathrm{div}\zeta^e \, dz &= \int_{\{-\epsilon<d_\phi^E<\epsilon\}} \psi^e \mathrm{div}\zeta^e \, \phi^o(\nabla d_\phi^E)\, dz \\ &= \int_{-\epsilon}^{\epsilon}\int_{\{d_\phi^E=t\}} \psi^e \mathrm{div}\zeta^e \, \phi^o\left(\frac{\nabla d_\phi^E}{|\nabla d_\phi^E|}\right)\, d\mathcal{H}^{n-1}(z)\, dt \\ &= \int_{-\epsilon}^{\epsilon}\int_{\{d_\phi^E=t\}} \psi^e \mathrm{div}\zeta^e \, dP_\phi \, dt.\end{aligned} \tag{29}$$

Recalling (8) we have

$$\begin{aligned}\int_{-\epsilon}^{\epsilon}\int_{\{d_\phi^E=t\}} \psi^e \mathrm{div}\zeta^e \, dP_\phi \, dt &= \int_{-\epsilon}^{\epsilon}\int_{\{x+t\eta(x):x\in\partial E\}} \psi^e \mathrm{div}\zeta^e \, dP_\phi \, dt \qquad (30)\\ = \int_{-\epsilon}^{\epsilon}\int_{\partial E} &\psi^e(x+t\eta(x))\mathrm{div}\zeta^e(x+t\eta(x)) \\ &\phi^o(\nu(x+t\eta(x)))\, d\mathcal{H}^{n-1}(x+t\eta(x))\, dt.\end{aligned}$$

¿From (9) we have $\phi^o(\nu(x+t\eta(x))) = \phi^o(\nu(x))$ for $\mathcal{H}^{n-1}$-almost every $x \in \partial E$. Moreover $\psi^e(x+t\eta(x)) = \psi(x)$ by definition, and $d\mathcal{H}^{n-1}(x+t\eta(x)) = d\mathcal{H}^{n-1}(x) + O(t)$ by the area formula [15]. Finally

$$\mathrm{div}\zeta^e(x+t\eta(x)) = \mathrm{div}_\tau\zeta(x) + O(t). \tag{31}$$

Indeed, letting $z := x+t\eta(x)$, since $\zeta^e(z) = \zeta^e(z - d_\phi^E(z)\eta^e(z))$, it follows

$$\nabla\zeta^e(z) = \nabla\zeta^e(z - d_\phi^E(z)\eta^e(z))(\mathrm{Id} - \nabla d_\phi^E(z)\otimes\eta^e(z) - d_\phi^E(z)\nabla\eta^e(z))$$

$$= \nabla\zeta^e(x)(\mathrm{Id} - \nabla d_\phi^E(x)\otimes\eta(x) - d_\phi^E(z)\nabla\eta^e(z)),$$

where the last equality follows from (9). Then (31) is a consequence of (26).

¿From (29), (30) and the above considerations, we then get

$$\int_{\partial E} \psi \, \mathrm{div}_\tau\zeta \, dP_\phi = \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\int_{\{-\epsilon<d_\phi^E<\epsilon\}} \psi^e \mathrm{div}\zeta^e \, dz. \tag{32}$$

Applying the Gauss-Green Theorem for fixed positive ϵ, we obtain

$$\begin{aligned}\int_{\{-\epsilon<d_\phi^E<\epsilon\}} \psi^e \mathrm{div}\zeta^e \, dz = &-\int_{\{-\epsilon<d_\phi^E<\epsilon\}} \zeta^e\cdot\nabla\psi^e \, dz \\ &+\int_{\partial\{-\epsilon<d_\phi^E<\epsilon\}} \psi^e\zeta^e\cdot\nu_\epsilon \, d\mathcal{H}^{n-1},\end{aligned}$$

where ν_ϵ is the euclidean outward unit normal to $\partial\{-\epsilon < d_\phi^E < \epsilon\}$.

Observe now that for $\mathcal{H}^{n-1}$-almost every $z \in \partial\{-\epsilon < d_\phi^E < \epsilon\}$, if $z = x + t\eta(x)$ for $x \in \partial E$, by (9) we have $\zeta^e(z)\cdot\nu_\epsilon(z) = \eta(x)\cdot\nu^E(x) - n_\phi(x)\cdot\nu^E(x) = 0$. Hence from (32)

$$\int_{\partial E} \psi \, \mathrm{div}_\tau \zeta \, dP_\phi = -\lim_{\epsilon\to 0} \frac{1}{2\epsilon} \int_{\{-\epsilon<d_\phi^E<\epsilon\}} \nabla\psi^e \cdot \zeta^e \, dz = -\int_{\partial E} \nabla_\tau\psi \cdot \zeta \, dP_\phi, \tag{33}$$

which gives

$$\int_{\partial E} \psi \, \mathrm{div}_\tau \, \eta \, dP_\phi = \int_{\partial E} \psi \, \mathrm{div}_\tau \, n_\phi \, dP_\phi - \int_{\partial E} \nabla_\tau\psi \cdot (\eta - n_\phi) \, dP_\phi,$$

which is (27).

We can now prove that the operator $\mathrm{div}_{\phi,n_\phi,\tau}$ is independent of the choice of n_ϕ on the whole of $\mathrm{Nor}_\phi(\partial E;\mathbb{R}^n)$.

Corollary 47 *Let* $(E, n_\phi) \in \mathcal{R}_\phi(\mathbb{R}^n)$. *If* $N \in \mathrm{Nor}_\phi(\partial E;\mathbb{R}^n)$, *it follows that* $\mathrm{div}_{\phi,n_\phi,\tau}N$ *does not depend on the choice of* n_ϕ.

Proof. By (5) we have $N \cdot \nu_\phi = 1$, hence for $\psi \in \mathrm{Lip}(\partial E)$ formula (25) reduces to

$$\langle \mathrm{div}_{\phi,n_\phi,\tau} \, N, \psi\rangle = \int_{\partial E} \psi \, \mathrm{div}_\tau \, n_\phi \, dP_\phi - \int_{\partial E} \nabla_\tau\psi \cdot (N - n_\phi) \, dP_\phi. \tag{34}$$

Let now $n_\phi^* \in \mathrm{Lip}_{\nu,\phi}(\partial E;\mathbb{R}^n)$ and set $\zeta := n_\phi - n_\phi^*$. From (34) we deduce

$$\langle \mathrm{div}_{\phi,n_\phi,\tau}N - \mathrm{div}_{\phi,n_\phi^*,\tau} \, N, \psi\rangle = \int_{\partial E} \psi \, \mathrm{div}_\tau \, \zeta \, dP_\phi + \int_{\partial E} \nabla_\tau\psi \cdot \zeta \, dP_\phi. \tag{35}$$

Since $n_\phi, n_\phi^* \in \mathrm{Lip}_{\nu,\phi}(\partial E;\mathbb{R}^n)$ we have that $\zeta \in \mathrm{Lip}(\partial E;\mathbb{R}^n)$ is tangent to ∂E, hence from (2) of Lemma 44 we get

$$\langle \mathrm{div}_{\phi,n_\phi,\tau}N - \mathrm{div}_{\phi,n_\phi^*,\tau} \, N, \psi\rangle = \int_{\partial E} \psi \, \mathrm{div}_\tau \, \zeta \, dP_\phi - \langle \mathrm{div}_{\phi,n_\phi,\tau}\zeta, \psi\rangle.$$

Recalling the definition of ζ and applying Proposition 46, it follows that the right hand side of the above equality vanishes, and the assertion of the corollary follows.

5. The first variation of the functional P_ϕ

In this section we compute the first variation of the functional P_ϕ. This computation is quite delicate, because both the integrand $\phi^o(\nu)$ and the manifold ∂E are not smooth. A partial result in this direction can be found in [6].

Let $(E, n_\phi) \in \mathcal{R}_\phi(\mathbb{R}^n)$. The set $\{v \in L^2(\partial E;\mathbb{R}^n) \, : \, \mathrm{div}_{\phi,n_\phi,\tau} \, v \in L^2(\partial E)\}$ is nonempty and is a Hilbert space endowed with the norm

$$\left(\|v\|^2_{L^2(\partial E;\mathbb{R}^n)} + \|\mathrm{div}_{\phi,n_\phi,\tau} \, v\|^2_{L^2(\partial E)}\right)^{1/2}.$$

We define

$$H(\partial E;\mathbb{R}^n) := \{N \in \mathrm{Nor}_\phi(\partial E;\mathbb{R}^n) : \mathrm{div}_{\phi,\tau} N \in L^2(\partial E)\}.$$

Let $v \in \mathrm{Lip}(\partial E;\mathbb{R}^n)$. We first compute the first variation of the functional $E \to P_\phi(E)$ along the field v. As in the smooth case, P_ϕ does not change under (infinitesimal) tangential variations, and therefore it is enough to consider ϕ-normal fields, i.e. we can assume that v can be written as $v = \psi\eta$, where $\psi \in \mathrm{Lip}(\partial E)$ and $\eta \in \mathrm{Lip}_{\nu,\phi}(\partial E;\mathbb{R}^n)$.

Let $U(\partial E)$ and π_η be as in Lemma 33, and define $\psi^e \in \mathrm{Lip}(U(\partial E))$, $\eta^e \in \mathrm{Lip}(U(\partial E);\mathbb{R}^n)$ as $\psi^e(z) := \psi(\pi_\eta(z))$, $\eta^e(z) := \eta(\pi_\eta(z))$, and set $v^e := \psi^e\eta^e$. For $t \in \mathbb{R}$ with $|t| < \epsilon$, $\epsilon > 0$ small enough, define $\widetilde{F}$ as in (18), and set $\widetilde{F}^t(\cdot) := \widetilde{F}(\cdot, t)$, $E_t := \widetilde{F}^t(E)$. Define also

$$\mathrm{Var}(P_\phi, E)(\psi\eta) := \liminf_{t\to 0^+} \frac{P_\phi(E_t) - P_\phi(E)}{t}.$$

Theorem 51 *We have*

$$\mathrm{Var}(P_\phi, E)(\psi\eta) = \lim_{t\to 0^+} \frac{P_\phi(E_t) - P_\phi(E)}{t} = \sup_{N\in \mathrm{Nor}_\phi(\partial E;\mathbb{R}^n)} \langle \mathrm{div}_{\phi,\eta,\tau}\, N, \psi\rangle. \tag{36}$$

Proof. By (1) of Lemma 35 E_t is Lipschitz; therefore, using the area formula we have

$$P_\phi(E_t) = \int_{\partial E} \phi^o(\nu_t)\, d\mathcal{H}^{n-1} + \int_{\partial E} \phi^o(\nu)\, \mathrm{div}_\tau v\, d\mathcal{H}^{n-1} + o(t),$$

where ν_t is defined in (19). Hence, by (21) and (20), $\mathrm{Var}(P_\phi, E)(\psi\eta)$ is equal to

$$\begin{aligned}
&\int_{\partial E} \lim_{t\to 0^+} \frac{\phi^o(\nu_t) - \phi^o(\nu)}{t}\, d\mathcal{H}^{n-1} + \int_{\partial E} \phi^o(\nu)\, \mathrm{div}_\tau v\, d\mathcal{H}^{n-1} \qquad (37)\\
&= \int_{\partial E} \Big\{ \max_{p\in T^o(\nu_\phi(x))} p \cdot \Big(- \nu_\phi(x)\nabla v^e(x) \\
&\qquad + \big[\nu(x)\cdot \nu(x)\nabla v^e(x)\big]\nu_\phi(x)\Big) + \mathrm{div}_\tau v \Big\}\, dP_\phi \\
&= \int_{\partial E} \Big\{ \max_{p\in T^o(\nu_\phi(x))} -p\cdot \nu_\phi(x)\nabla v^e(x) + \nu(x)\cdot\nu(x)\nabla v^e(x) + \mathrm{div}_\tau v \Big\}\, dP_\phi,
\end{aligned}$$

where the last equality follows from $p \cdot \nu_\phi(x) = 1$. It is not difficult to prove now that the map $x \to T^o(\nu_\phi(x))$, defined for $\mathcal{H}^{n-1}$-almost every $x \in \partial E$, is the smallest closed-valued $\mathcal{H}^{n-1}$-measurable multifunction with the property that, for any $N \in \mathrm{Nor}_\phi(\partial E;\mathbb{R}^n)$, $N(x) \in T^o(\nu_\phi(x))$ for $\mathcal{H}^{n-1}$-almost every $x \in \partial E$. Using this observation, a commutation argument between supremum and integral (see [9], Lemma 4.3) allows to prove that the last member of (37) equals

$$\begin{aligned}
&\sup_{N\in \mathrm{Nor}_\phi(\partial E;\mathbb{R}^n)} \int_{\partial E} \Big\{ - N\cdot \nu_\phi \nabla v^e + \nu\cdot\nu\nabla v^e + \mathrm{div}_\tau v \Big\}\, dP_\phi \\
&=: \sup_{N\in \mathrm{Nor}_\phi(\partial E;\mathbb{R}^n)} \int_{\partial E} I_N\, dP_\phi.
\end{aligned}$$

Fix $N \in \mathrm{Nor}_\phi(\partial E; \mathbb{R}^n)$: recalling the expression of the euclidean tangential divergence, I_N can be written as $-N \cdot \nu_\phi \nabla v^e + \mathrm{div}\, v^e = -N \cdot \nu_\phi \nabla v^e + \psi^e \mathrm{div}\, \eta^e + \nabla \psi^e \cdot \eta^e$. Therefore, since Lemma 45 implies $\mathrm{div}\, \eta^e = \mathrm{div}_\tau\, \eta$ on ∂E, we obtain

$$\int_{\partial E} I_N \, dP_\phi = \int_{\partial E} \Big\{ - N \cdot \nu_\phi \nabla v^e + \psi \, \mathrm{div}_\tau \eta + \nabla \psi^e \cdot \eta \Big\} \, dP_\phi .$$

On the other hand, substituting $v^e = \psi^e \eta^e$ into $-N \cdot \nu_\phi \nabla v^e$ and using (5) of Lemma 35, we get $-N \cdot \nu_\phi \nabla v^e = -\nabla \psi^e \cdot N$ on ∂E. Since $N - \eta$ is a tangent vector field, we have $\nabla \psi^e \cdot (N - \eta) = \nabla_\tau \psi \cdot (N - \eta)$ on ∂E. Therefore we conclude that

$$\int_{\partial E} I_N \, dP_\phi = \int_{\partial E} \Big\{ \psi \, \mathrm{div}_\tau \eta - \nabla_\tau \psi \cdot (N - \eta) \Big\} \, dP_\phi = \langle \mathrm{div}_{\phi,\eta,\tau} N, \psi \rangle,$$

and the theorem follows.

Recall that $\mathrm{div}_{\phi,\eta,\tau}\, N$ in (36) actually does not depend on η (see Corollary 47), so that $\mathrm{Var}(P_\phi, E)(\psi\eta)$ is independent of η. We define (with a little abuse of notation) the functional $\mathrm{Var}(P_\phi, E) : L^2(\partial E) \to]-\infty, +\infty]$ as

$$\mathrm{Var}(P_\phi, E)(\psi) := \begin{cases} \displaystyle\sup_{N \in \mathrm{Nor}_\phi(\partial E; \mathbb{R}^n)} \langle \mathrm{div}_{\phi,\tau}\, N, \psi \rangle & \text{if } \psi \in \mathrm{Lip}(\partial E), \\ \\ +\infty & \text{if } \psi \in L^2(\partial E) \setminus \mathrm{Lip}(\partial E). \end{cases}$$

Define also

$$\begin{aligned} B_\phi &:= \Big\{ \psi \in \mathrm{Lip}(\partial E) : \int_{\partial E} \psi^2 \, dP_\phi \le 1 \Big\}, \\ B_\phi^n &:= \Big\{ \psi\eta : \ \psi \in B_\phi, \ \eta \in \mathrm{Lip}_{\nu,\phi}(\partial E; \mathbb{R}^n) \Big\}. \end{aligned} \tag{38}$$

The next result gives, roughly speaking, the expression of minus the norm of the gradient of the functional P_ϕ.

Proposition 52 *We have*

$$\inf_{\psi\eta \in B_\phi^n} \mathrm{Var}(P_\phi, E)(\psi\eta) = - \min_{N \in H(\partial E; \mathbb{R}^n)} \left(\int_{\partial E} (\mathrm{div}_{\phi,\tau} N)^2 \, dP_\phi \right)^{\frac{1}{2}} .$$

Proof. Using Theorem 51 we get

$$\begin{aligned} \inf_{\psi\eta \in B_\phi^n} \mathrm{Var}(P_\phi, E)(\psi\eta) &= \inf_{\psi \in B_\phi} \mathrm{Var}(P_\phi, E)(\psi) \\ &= \inf_{\psi \in B_\phi} \sup_{N \in \mathrm{Nor}_\phi(\partial E; \mathbb{R}^n)} \langle \mathrm{div}_{\phi,\tau}\, N, \psi \rangle =: I. \end{aligned} \tag{39}$$

Notice that the functional $(\psi, N) \in B_\phi \times \mathrm{Nor}_\phi(\partial E; \mathbb{R}^n) \to \langle \mathrm{div}_{\phi,\tau}\, N, \psi \rangle$ is bilinear. Moreover both B_ϕ and $\mathrm{Nor}_\phi(\partial E; \mathbb{R}^n)$ are convex sets, and B_ϕ (resp.

$\mathrm{Nor}_\phi(\partial E;\mathbb{R}^n)$) is closed in the Lip-topology (resp. strongly and weakly closed and weakly compact in the $L^2(\partial E;\mathbb{R}^n)$-topology). Therefore, by a commutation result between sup and inf (see [10, Proposition 1.1]), we get

$$I = \sup_{N\in \mathrm{Nor}_\phi(\partial E;\mathbb{R}^n)} \inf_{\psi\in B_\phi} \langle \mathrm{div}_{\phi,\tau}\, N, \psi\rangle.$$

Since $N \notin H(\partial E;\mathbb{R}^n)$ implies $\inf_{\psi\in B_\phi}\langle \mathrm{div}_{\phi,\tau}\, N,\psi\rangle = -\infty$, we deduce

$$I = \sup_{N\in H(\partial E;\mathbb{R}^n)} \inf_{\psi\in B_\phi} \langle \mathrm{div}_{\phi,\tau}\, N, \psi\rangle.$$

Using the fact that $\inf_{\psi\in B_\phi}\langle \mathrm{div}_{\phi,\tau}\, N,\psi\rangle = \inf_{\psi\in \overline{B_\phi}^{L^2}}\langle \mathrm{div}_{\phi,\tau}\, N,\psi\rangle$, it follows

$$\begin{aligned} I &= \sup_{N\in H(\partial E;\mathbb{R}^n)} -\frac{\int_{\partial E}(\mathrm{div}_{\phi,\tau} N)^2\, dP_\phi}{\left(\int_{\partial E}(\mathrm{div}_{\phi,\tau} N)^2\, dP_\phi\right)^{\frac{1}{2}}} \\ &= -\inf_{N\in H(\partial E;\mathbb{R}^n)} \left(\int_{\partial E}(\mathrm{div}_{\phi,\tau} N)^2\, dP_\phi\right)^{\frac{1}{2}}. \end{aligned}$$

Thanks to Proposition 61 below, the infimum in the above inequality is a minimum, and the proposition is proved.

Given $v \in L^2(\partial E;\mathbb{R}^n)$ such that $v = \psi\widetilde{N}$ for some $\psi \in L^2(\partial E)$ and $\widetilde{N} \in H(\partial E;\mathbb{R}^n)$, we define

$$V^E_\phi(v) := \sup_{N\in H(\partial E;\mathbb{R}^n)} \int_{\partial E} \psi\, \mathrm{div}_{\phi,\tau} N\, dP_\phi =: V^E_\phi(\psi),$$

where $V^E_\phi : L^2(\partial E) \to]-\infty,+\infty]$.

The following observation shows that V^E_ϕ is the lower semicontinuous envelope of $\mathrm{Var}(P_\phi, E)$, with respect to the $L^2(\partial E)$-topology.

Proposition 53 *The functional V^E_ϕ is the greatest $L^2(\partial E)$–lower semicontinuous functional less than or equal to $\mathrm{Var}(P_\phi,E)$. In particular*

$$\inf_{\psi\in B_\phi} \mathrm{Var}(P_\phi,E)(\psi) = \inf_{\psi\in B_\phi} V^E_\phi(\psi).$$

Proof. Given $\psi\in L^2(\partial E)$ and $\epsilon > 0$, we set

$$B_\epsilon(\psi) := \left\{\tilde\psi \in \mathrm{Lip}(\partial E) :\ ||\tilde\psi - \psi||_{L^2(\partial E)} < \epsilon\right\}.$$

The $L^2(\partial E)$–lower semicontinuous envelope of $\mathrm{Var}(P_\phi,E)$ is, by definition

$$\sup_{\epsilon>0} \inf_{\tilde\psi\in B_\epsilon(\psi)} \mathrm{Var}(P_\phi,E)(\tilde\psi).$$

By Theorem 51, this is equal to

$$\sup_{\epsilon>0} \inf_{\tilde\psi\in B_\epsilon(\psi)} \sup_{N\in \mathrm{Nor}_\phi(\partial E;\mathbb{R}^n)} \langle \mathrm{div}_{\phi,\tau}\, N, \tilde\psi\rangle.$$

Commuting the sup and the inf [10], we then have

$$
\begin{aligned}
&\sup_{\epsilon>0} \inf_{\tilde{\psi}\in B_\epsilon(\psi)} \sup_{N\in \mathrm{Nor}_\phi(\partial E;\mathbb{R}^n)} \langle \mathrm{div}_{\phi,\tau}\, N, \tilde{\psi}\rangle \\
&= \sup_{\epsilon>0} \sup_{N\in \mathrm{Nor}_\phi(\partial E;\mathbb{R}^n)} \inf_{\tilde{\psi}\in B_\epsilon(\psi)} \langle \mathrm{div}_{\phi,\tau}\, N, \tilde{\psi}\rangle \\
&= \sup_{\epsilon>0} \sup_{N\in H(\partial E;\mathbb{R}^n)} \inf_{\tilde{\psi}\in B_\epsilon(\psi)} \int_{\partial E} \tilde{\psi}\, \mathrm{div}_{\phi,\tau} N \, dP_\phi \\
&= \sup_{N\in H(\partial E;\mathbb{R}^n)} \sup_{\epsilon>0} \inf_{\tilde{\psi}\in B_\epsilon(\psi)} \int_{\partial E} \tilde{\psi}\, \mathrm{div}_{\phi,\tau} N \, dP_\phi \\
&= \sup_{N\in H(\partial E;\mathbb{R}^n)} \int_{\partial E} \psi\, \mathrm{div}_{\phi,\tau} N \, dP_\phi = V_\phi^E(\psi),
\end{aligned}
$$

which proves the assertion.

The following result, which follows from Proposition 53, shows that the direction of minimal slope for the functional P_ϕ is given by $N_{\min}$, and is one of the motivation for introducing and studying the functional in (1) in connection with motion by crystalline mean curvature.

Corollary 54 *We have*

$$
\inf_{\psi\eta\in B_\phi^n} \mathrm{Var}(P_\phi, E)(\psi\eta) = V_\phi^E(\overline{\psi} N_{\min}),
$$

where $N_{\min}$ is a minimizer of the functional $N \to \int_{\partial E} (\mathrm{div}_{\phi,\tau} N)^2 \, dP_\phi$ (see Proposition 61 below, with $g = 0$) and

$$
\overline{\psi} = -\frac{\mathrm{div}_{\phi,\tau} N_{\min}}{\left(\int_{\partial E} (\mathrm{div}_{\phi,\tau} N_{\min})^2 \, dP_\phi\right)^{\frac{1}{2}}} \,.
$$

6. The minimum problem on ∂E: L^∞-regularity

Let $g \in L^2(\partial E)$ and let $\mathcal{F} : H(\partial E;\mathbb{R}^n) \to [0,+\infty[$ be the functional defined as

$$
\mathcal{F}(N) := \int_{\partial E} \left(\mathrm{div}_{\phi,\tau}\, N - g\right)^2 dP_\phi. \tag{40}
$$

We are interested in studying the following minimum problem on ∂E:

$$
\inf \left\{ \mathcal{F}(N) : N \in H(\partial E;\mathbb{R}^n) \right\}. \tag{41}
$$

It is clear that problem (41) is equivalent, up to constants, to the problem

$$
\inf \left\{ \int_{\partial E} (\mathrm{div}_{\phi,\tau}\, N)^2 - 2g \mathrm{div}_{\phi,\tau} N \, dP_\phi : N \in H(\partial E;\mathbb{R}^n) \right\}. \tag{42}
$$

Proposition 61 *Problem (41) admits a solution. Moreover, if N_1 and N_2 are two minimizers of (41), then $\mathrm{div}_{\phi,\tau}\ N_1(x) = \mathrm{div}_{\phi,\tau}\ N_2(x)$ for $\mathcal{H}^{n-1}$-almost every $x \in \partial E$.*

Proof. Define

$$C := \{\mathrm{div}_{\phi,\tau} N : N \in H(\partial E; \mathbb{R}^n)\}.$$

Then C is a convex subset of $L^2(\partial E)$. Let us prove that C is closed in $L^2(\partial E)$. Let $f_k := \mathrm{div}_{\phi,\tau} N_k \in C$ be such that $f_k \to f$ in $L^2(\partial E)$ as $k \to \infty$. We have to prove that $f \in C$. Since $\sup_k \|N_k\|_{L^2(\partial E;\mathbb{R}^n)} < +\infty$, possibly passing to a subsequence (still denoted by (N_k)) we can assume that (N_k) converges weakly in $L^2(\partial E; \mathbb{R}^n)$ to a vector field $N \in L^2(\partial E; \mathbb{R}^n)$. Since $N_k \in T^o(\nu_\phi^E)$, by Mazur's Theorem we obtain that $N \in \mathrm{Nor}_\phi(\partial E; \mathbb{R}^n)$. Moreover, for any $\psi \in \mathrm{Lip}(\partial E)$ we have, using (25),

$$\begin{aligned}\int_{\partial E} \psi f\, dP_\phi &= \lim_{k\to+\infty} \int_{\partial E} \psi\, \mathrm{div}_{\phi,\tau}\ N_k\, dP_\phi \\ &= \int_{\partial E} \psi\, \mathrm{div}_\tau n_\phi\, dP_\phi - \lim_{k\to+\infty} \int_{\partial E} \nabla_\tau \psi \cdot (N_k - n_\phi)\, dP_\phi \\ &= \int_{\partial E} \psi\, \mathrm{div}_\tau n_\phi\, dP_\phi - \int_{\partial E} \nabla_\tau \psi \cdot (N - n_\phi)\, dP_\phi.\end{aligned}$$

It follows that $f = \mathrm{div}_{\phi,\tau} N$, hence C is closed in $L^2(\partial E; \mathbb{R}^n)$. The proof of the proposition is then is a standard consequence of minimization on convex sets of strictly convex functionals on Hilbert spaces.

Let $N_{\min} \in H(\partial E; \mathbb{R}^n)$ be a minimizer of $\mathcal{F}$. A direct computation yields that the Euler-Lagrange inequality of $\mathcal{F}$ reads as follows:

$$\int_{\partial E} (\mathrm{div}_{\phi,\tau} N_{\min} - g)\ \mathrm{div}_{\phi,\tau} (N_{\min} - N)\, dP_\phi \le 0 \qquad \forall N \in H(\partial E; \mathbb{R}^n). \tag{43}$$

We now give the definition of mean curvature of a Lipschitz ϕ-regular set with respect to the metric ϕ.

Definition 62 *Let $E \in \mathcal{R}_\phi(\mathbb{R}^n)$. Let $N_{\min}$ be a solution of (41). We set*

$$d_{\min} := \mathrm{div}_{\phi,\tau} N_{\min} \in L^2(\partial E).$$

When $g = 0$, we define the ϕ-mean curvature κ_ϕ of ∂E as

$$\kappa_\phi := \mathrm{div}_{\phi,\tau} N_{\min} \in L^2(\partial E).$$

We shall see in Theorem 67 that Lipschitz ϕ-regular sets have actually bounded ϕ-mean curvature.

Remark 63 *Since $\mathcal{F}$ is strictly convex if considered as a function of the divergence, if f is of the form $f = \mathrm{div}_{\phi,\tau} \overline{N}$ for some $\overline{N} \in H(\partial E; \mathbb{R}^n)$ and if*

$$\int_{\partial E} (f - g) \mathrm{div}_{\phi,\tau} (\overline{N} - N)\, dP_\phi \le 0 \qquad \forall N \in H(\partial E; \mathbb{R}^n),$$

then $\overline{N}$ is a solution of (41).

The next step is to prove that $d_{\min} \in L^\infty(\partial E)$. To this aim we begin with the following auxiliary lemma, whose standard proof is omitted.

Lemma 64 *Let $a, b \in \mathbb{R}$, $a < b$ and let $f, \beta \in L^2(\partial E)$. Let μ be a measure on ∂E absolutely continuous with respect to the restriction of $\mathcal{H}^{n-1}$ to ∂E. Assume that $\int_{\partial E} \beta \, d\mu = 0$. Then*

$$\int_{\partial E} f\beta \, d\mu = \int_{-\infty}^{+\infty} \int_{\{f>t\}} \beta \, d\mu \, dt, \tag{44}$$

$$\int_{\{a<f<b\}} f\beta \, d\mu = \int_a^b \int_{\{f>t\}} \beta \, d\mu \, dt + a \int_{\{f>a\}} \beta \, d\mu - b \int_{\{f>b\}} \beta \, d\mu.$$

The crucial result of this section is the following.

Theorem 65 *Let $E \in \mathcal{R}_\phi(\mathbb{R}^n)$. For any $t \in \mathbb{R}$, we define*

$$A_t := \{d_{\min} - g > t\}, \qquad \Omega_t := \{d_{\min} - g < t\}.$$

Then

$$\int_{A_t} d_{\min} \, dP_\phi \le \int_{A_t} \mathrm{div}_{\phi,\tau} \, N \, dP_\phi \qquad \forall t \in \mathbb{R}, \ \forall N \in H(\partial E; \mathbb{R}^n), \tag{45}$$

and

$$\int_{\Omega_t} d_{\min} \, dP_\phi \ge \int_{\Omega_t} \mathrm{div}_{\phi,\tau} N \, dP_\phi \qquad \forall t \in \mathbb{R}, \ \forall N \in H(\partial E; \mathbb{R}^n). \tag{46}$$

Proof. We shall prove only (45), since the proof of (46) is similar. For simplicity of notation, set

$$V := d_{\min} - g.$$

Moreover, if $\chi \in H(\partial E; \mathbb{R}^n)$ and $B \subseteq \partial E$ is a Borel set, we let

$$D(B, \chi) := \int_B \mathrm{div}_{\phi,\tau}(\chi - N_{\min}) \, dP_\phi.$$

Assume by contradiction that there exist $\lambda \in \mathbb{R}$, $N \in H(\partial E; \mathbb{R}^n)$, and $c > 0$ such that

$$D(A_\lambda, N) = -4c < 0.$$

Since $A_\lambda = \bigcup_{t>\lambda} A_t$, we have $1_{A_t} \to 1_{A_\lambda}$ in $L^1(\partial E)$ (hence, being characteristic functions, also in $L^2(\partial E)$) as $t \downarrow \lambda$. Therefore there exists $\epsilon > 0$ such that

$$D(A_t, N) \le -2c, \qquad \forall t \in [\lambda, \lambda + \epsilon]. \tag{47}$$

Fix $\lambda' \in \mathbb{R}$ and $\epsilon' > 0$ with the following properties:

$$[\lambda', \lambda' + \epsilon'] \subseteq [\lambda, \lambda + \epsilon], \tag{48}$$

$$\mathcal{H}^{n-1}\Big(\{V=\lambda'\}\cup\{V=\lambda'+\epsilon'\}\Big)=0. \tag{49}$$

Clearly from (47) and (48) we get

$$\int_{\lambda'}^{\lambda'+\epsilon'} D(A_t,N)\,dt\leq -2c\epsilon'. \tag{50}$$

Let $\{f_i\}$ be a sequence of functions in $\mathrm{Lip}(\partial E)$ converging to V in $L^2(\partial E)$ and almost everywhere. For any $i\in\mathbb{N}$ and $t\in\mathbb{R}$ we define

$$A^i_t:=\{f_i>t\}.$$

We split the proof into three intermediate steps.

Step 1. Let us prove that there exists $i_0\in\mathbb{N}$ such that

$$\int_{\lambda'}^{\lambda'+\epsilon'} D(A^i_t,N)\,dt\leq -c\epsilon' \qquad \forall i\geq i_0. \tag{51}$$

We claim that

$$\lim_{i\to+\infty}\int_{\lambda'}^{\lambda'+\epsilon'} D(A^i_t,N)\,dt=\int_{\lambda'}^{\lambda'+\epsilon'} D(A_t,N)\,dt. \tag{52}$$

By (4) of Lemma 44 we have $D(\partial E,N)=0$, hence applying Lemma 64 with f, β, $d\mu$, a, b replaced by f_i, $\mathrm{div}_{\phi,\tau}(N-N_{\min})$, dP_ϕ, λ', $\lambda'+\epsilon'$ in the order, from (44) we find

$$\int_{\lambda'}^{\lambda'+\epsilon'} D(A^i_t,N)\,dt=\int_{\{\lambda'<f_i<\lambda'+\epsilon'\}} f_i\mathrm{div}_{\phi,\tau}(N-N_{\min})\,dP_\phi-\lambda' D(A^i_{\lambda'},N)$$

$$+(\lambda'+\epsilon')D(A^i_{\lambda'+\epsilon'},N).$$

In view of this equality and of the corresponding one with f_i and A^i_t replaced by V and A_t, to prove (52) it is enough to show that $1_{A^i_{\lambda'}}$ (resp. $1_{\{\lambda'<f_i<\lambda'+\epsilon'\}}$, $1_{A^i_{\lambda'+\epsilon'}}$) converges to $1_{A_{\lambda'}}$ (resp. $1_{\{\lambda'<V<\lambda'+\epsilon'\}}$, $1_{A_{\lambda'+\epsilon'}}$) in $L^1(\partial E)$ as $i\to+\infty$. We show this property for $A^i_{\lambda'}$, the other cases being similar. Define $A_i:=(A^i_{\lambda'}\setminus A_{\lambda'})\cup(A_{\lambda'}\setminus A^i_{\lambda'})$. It is enough to check that $\mathcal{H}^{n-1}(\cap_i\cup_{m\geq i}A_m)=0$. Let $x\in\cap_i\cup_{m\geq i}A_m$. Then there exists a subsequence (i_k) such that $f_{i_k}(x)>\lambda'$ and $V(x)\leq\lambda'$, or $f_{i_k}(x)\leq\lambda'$ and $V(x)>\lambda'$ for any $k\in\mathbb{N}$.

Since $f_i\to V$ almost everywhere, we get $\mathcal{H}^{n-1}(\cap_i\cup_{m\geq i}A_m)\leq\mathcal{H}^{n-1}(\{V=\lambda'\})=0$, by (49). The claim is proved. Then (51) follows from (52) and (50), and step 1 is proved.

Step 2. Let us prove that

$$\liminf_{i\to+\infty}\int_{\partial E} f_i\mathrm{div}_{\phi,\tau}\,(\chi-N_{\min})\,dP_\phi\geq 0 \qquad \forall\chi\in H(\partial E;\mathbb{R}^n). \tag{53}$$

Let $\chi \in H(\partial E;\mathbb{R}^n)$. Since $f_i \to V$ in $L^2(\partial E)$, we have

$$\liminf_{i\to+\infty} \int_{\partial E} f_i \mathrm{div}_{\phi,\tau}\,(\chi - N_{\min})\, dP_\phi = \int_{\partial E} V \mathrm{div}_{\phi,\tau}(\chi - N_{\min})\, dP_\phi \geq 0, \quad (54)$$

where the last inequality follows from the Euler inequality (43). Step 2 is proved.

Fix now $\delta > 0$ and define $\tilde\eta = \tilde\eta(\eta, \lambda', \epsilon', i, \delta) : \partial E \to \mathbb{R}^n$ as follows:

$$\tilde\eta := \begin{cases} N & \text{on } \{\lambda' < f_i < \lambda' + \epsilon'\} \\ \\ N_{\min} & \text{on } \{f_i < \lambda' - \delta\} \cup A^i_{\lambda'+\epsilon'+\delta}, \end{cases}$$

and

$$\tilde\eta(x) := \begin{cases} \psi\left(\frac{f_i(x)-\lambda'+\delta}{\delta}\right) N(x) + \left(1 - \psi\left(\frac{f_i(x)-\lambda'+\delta}{\delta}\right)\right) N_{\min}(x) \\ \text{for } x \in \{\lambda' - \delta \leq f_i \leq \lambda'\}, \\ \\ \psi\left(\frac{f_i(x)-\lambda'-\epsilon'}{\delta}\right) N_{\min}(x) + \left(1 - \psi\left(\frac{f_i(x)-\lambda'-\epsilon'}{\delta}\right)\right) N(x) \\ \text{for } x \in \{\lambda' + \epsilon' \leq f_i \leq \lambda' + \epsilon' + \delta\}, \end{cases}$$

where ψ has the following properties: $\psi \in C^\infty([0,1];[0,1])$, there is $\sigma \in]0, \frac{1}{2}[$ such that $\psi(s) = 0$ for $s \in [0,\sigma]$, $\psi(s) = 1$ for $s \in [1-\sigma, 1]$, and $\psi_{|[\sigma,1-\sigma]}$ is strictly increasing.

Note that $\tilde\eta \in H(\partial E;\mathbb{R}^n)$ and that $(\tilde\eta - N)$ has compact support in A^i_t, for any $t \in [\lambda', \lambda' + \epsilon']$. It follows from (4) of Lemma 44 that

$$\int_{A^i_t} \mathrm{div}_{\phi,\tau}(\tilde\eta - N)\, dP_\phi = 0 \qquad \forall t \in [\lambda', \lambda' + \epsilon]. \quad (55)$$

Therefore from (55) and (51)

$$\int_{\lambda'}^{\lambda'+\epsilon'} D(A^i_t, \tilde\eta)\, dt = \int_{\lambda'}^{\lambda'+\epsilon'} D(A^i_t, N)\, dt \leq -c\epsilon' \qquad \forall i \geq i_0. \quad (56)$$

Step 3. Let us prove that

$$0 = \lim_{\delta\downarrow 0} \int_{\lambda'-\delta}^{\lambda'} D(A^i_t, \tilde\eta)\, dt = \lim_{\delta\downarrow 0} \int_{\lambda'+\epsilon'}^{\lambda'+\epsilon'+\delta} D(A^i_t, \tilde\eta)\, dt \qquad \forall i \geq i_0. \quad (57)$$

Since $(\tilde\eta - N_{\min})$ has compact support in $A^i_{\lambda'-\delta}$, using (4) of Lemma 44 we have $D(A^i_{\lambda'-\delta}, \tilde\eta) = 0$. Therefore, by Lemma 64 applied with f, β, $d\mu$, a, b replaced by f_i, $\mathrm{div}_{\phi,\tau}(\tilde\eta - N_{\min})$, dP_ϕ, $\lambda' - \delta$, λ' in the order, we have

$$\int_{\lambda'-\delta}^{\lambda'} D(A^i_t, \tilde\eta)\, dt = \int_{\{\lambda'-\delta<f_i<\lambda'\}} f_i\, \mathrm{div}_{\phi,\tau}(\tilde\eta - N_{\min})\, dP_\phi + \lambda' D(A^i_{\lambda'}, \tilde\eta). \quad (58)$$

Define now

$$h_i(x) := \begin{cases} f_i(x) & \text{if } x \in \{f_i < \lambda'\}, \\ \lambda' & \text{if } x \in \{f_i \geq \lambda'\}. \end{cases}$$

Then $h_i \in \mathrm{Lip}(\partial E)$, and recalling that $\tilde{\eta} = N_{\min}$ in a neighbourhood of $\{f_i < \lambda' - \delta\}$, we get

$$\int_{\{\lambda'-\delta<f_i<\lambda'\}} f_i \,\mathrm{div}_{\phi,\tau}(\tilde{\eta} - N_{\min})\, dP_\phi + \lambda' D(A^i_{\lambda'}, \tilde{\eta}) = \int_{\partial E} h_i \mathrm{div}_{\phi,\tau}(\tilde{\eta} - N_{\min})\, dP_\phi. \tag{59}$$

Hence, by (58), (59), and using (25) we deduce

$$\int_{\lambda'-\delta}^{\lambda'} D(A^i_t, \tilde{\eta})\, dt = -\int_{\partial E} \nabla_\tau h_i \cdot (\tilde{\eta} - N_{\min})\, dP_\phi$$

$$= -\int_{\{\lambda'-\delta<f_i<\lambda'\}} \nabla_\tau h_i \cdot (\tilde{\eta} - N_{\min})\, dP_\phi$$

$$\leq \|\tilde{\eta} - N_{\min}\|_{L^\infty(\partial E)} \|\nabla_\tau h_i\|_{L^\infty(\partial E)} \mathcal{H}^{n-1}(\{\lambda' - \delta < f_i < \lambda'\}).$$

Now the first equality of (57) follows by observing that $\mathcal{H}^{n-1}(\{\lambda' - \delta < f_i < \lambda'\}) \downarrow 0$ as $\delta \to 0$. The other equality is similar. Step 3 is proved.

We can now conclude the proof of the theorem. Recalling step 2, we can fix a natural number $i_1 \geq i_0$, such that

$$\int_{\partial E} f_{i_1} \,\mathrm{div}_{\phi,\tau}(\tilde{\eta} - N_{\min})\, dP_\phi > -c\frac{\epsilon'}{4}.$$

We have

$$-c\frac{\epsilon'}{4} < \int_{\partial E} f_{i_1} \mathrm{div}_{\phi,\tau}(\tilde{\eta} - N_{\min})\, dP_\phi = \int_{-\infty}^{\lambda'-\delta} D(A^{i_1}_t, \tilde{\eta})\, dt$$

$$+ \int_{\lambda'-\delta}^{\lambda'} D(A^{i_1}_t, \tilde{\eta})\, dt + \int_{\lambda'}^{\lambda'+\epsilon'} D(A^{i_1}_t, \tilde{\eta})\, dt + \int_{\lambda'+\epsilon'}^{\lambda'+\epsilon'+\delta} D(A^{i_1}_t, \tilde{\eta})\, dt$$

$$+ \int_{\lambda'+\epsilon'+\delta}^{+\infty} D(A^{i_1}_t, \tilde{\eta})\, dt := I + II + III + IV + V.$$

By the definition of $\tilde{\eta}$ and Lemma 44 we have $I = V = 0$. In addition $III \leq -c\epsilon'$ by (56), and II and IV tends to zero as $\delta \downarrow 0$ by step 3. Therefore, taking $\delta > 0$ small enough, we get $II + IV \leq c\frac{\epsilon'}{2}$. Then

$$-\epsilon\frac{c'}{4} < \int_{\partial E} f_{i_1} \mathrm{div}_{\phi,\tau}(\tilde{\eta} - N_{\min})\, dP_\phi \leq -\epsilon' c + \epsilon' \frac{c}{2} = -\epsilon' \frac{c}{2},$$

which is a contradiction.

Since $\int_{\partial E} \mathrm{div}_{\phi,\tau}(N - N_{\min})\, dP_\phi = 0$, from (45) and (46) it follows that the statement of Theorem 65 holds also if, in the definitions of A_t and Ω_t, we place the weak inequalities in place of the strict inequalities.

The following observation, which follows from Lemma 64 and Remark 63, is a kind of converse of Theorem 65.

Proposition 66 *Let $N \in H(\partial E; \mathbb{R}^n)$ and define $B_t := \{\mathrm{div}_{\phi,\tau}\, N - g > t\}$, for any $t \in \mathbb{R}$. If*

$$\int_{B_t} \mathrm{div}_{\phi,\tau} N\, dP_\phi \le \int_{B_t} \mathrm{div}_{\phi,\tau}\, \chi\, dP_\phi \qquad \forall t \in \mathbb{R},\ \forall \chi \in H(\partial E; \mathbb{R}^n), \tag{60}$$

then $\mathrm{div}_{\phi,\tau}\, N = d_{\min}$.

Proof. Recalling (4) of Lemma 44, we have

$$\int_{\partial E} \mathrm{div}_\tau n_\phi\, dP_\phi = \int_{\partial E} \mathrm{div}_{\phi,\tau} N\, dP_\phi =: c.$$

Set $f := \mathrm{div}_{\phi,\tau} N - g$ and $\beta := \mathrm{div}_{\phi,\tau} N - c$. Assumption (60) can be rewritten as

$$\int_{\{f>t\}} \beta\, dP_\phi \le \int_{\{f>t\}} (\mathrm{div}_{\phi,\tau}\chi - c)\, dP_\phi \qquad \forall t \in \mathbb{R},\ \forall \chi \in H(\partial E; \mathbb{R}^n).$$

Clearly $\int_{\partial E} \beta\, dP_\phi = 0$. Applying (44) we get

$$\begin{aligned} \int_{\partial E} f\beta\, dP_\phi &= \int_{-\infty}^{\infty} \int_{\{f>t\}} \beta\, dP_\phi\, dt \\ &\le \int_{-\infty}^{\infty} \int_{\{f>t\}} (\mathrm{div}_{\phi,\tau}\chi - c)\, dP_\phi\, dt = \int_{\partial E} f(\mathrm{div}_{\phi,\tau}\chi - c)\, dP_\phi. \end{aligned}$$

It follows

$$\int_{\partial E} f(\beta - \mathrm{div}_{\phi,\tau}\chi + c)\, dP_\phi \le 0 \qquad \forall \chi \in H(\partial E; \mathbb{R}^n),$$

that is

$$\int_{\partial E} (\mathrm{div}_{\phi,\tau} N - g)\, \mathrm{div}_{\phi,\tau}(N - \chi)\, dP_\phi \le 0 \qquad \forall \chi \in H(\partial E; \mathbb{R}^n).$$

Using Remark 63 the assertion follows.

Note that Proposition 66 still holds if, in the definition of B_t, we replace the weak inequality with the strict inequality.

We are now in a position to prove the L^∞-regularity of the divergence of solutions of (41).

Theorem 67 *Let $E \in \mathcal{R}_\phi(\mathbb{R}^n)$ and assume that $g \in L^\infty(\partial E)$. Then*

$$d_{\min} \in L^\infty(\partial E). \tag{61}$$

More precisely,

$$\|d_{\min} - g\|_{L^\infty(\partial E)} \le \|\mathrm{div}_\tau n_\phi - g\|_{L^\infty(\partial E)}. \tag{62}$$

Proof. Set $V := d_{\min} - g$. By (45) we have

$$\int_{A_t} V\, dP_\phi \le \int_{A_t} \Big(\mathrm{div}_\tau n_\phi - g\Big)\, dP_\phi \le \|\mathrm{div}_\tau n_\phi - g\|_{L^\infty(\partial E)} P_\phi(A_t) \qquad \forall t \in \mathbb{R},$$

so that

$$t \le \frac{1}{P_\phi(A_t)} \int_{A_t} V\, dP_\phi \le \|\mathrm{div}_\tau n_\phi - g\|_{L^\infty(\partial E)} \qquad \forall t \in \mathbb{R} \text{ with } A_t \neq \emptyset,$$

which implies $\mathcal{H}^{n-1} - \mathrm{esssup}_{\partial E} V \le \|\mathrm{div}_\tau n_\phi - g\|_{L^\infty(\partial E)}$.

Since $\int_{\partial E} \mathrm{div}_{\phi,\tau}(N_{\min} - n_\phi)\, dP_\phi = 0$, using (46) we also get

$$\int_{\Omega_t} V\, dP_\phi \ge \int_{\Omega_t} \Big(\mathrm{div}_\tau n_\phi - g\Big)\, dP_\phi, \qquad \forall t \in \mathbb{R},$$

which implies

$$t \ge \frac{1}{P_\phi(\Omega_t)} \int_{\Omega_t} V\, dP_\phi \ge -\|\mathrm{div}_\tau n_\phi - g\|_{L^\infty(\partial E)} \qquad \forall t \in \mathbb{R} \text{ with } \Omega_t \neq \emptyset.$$

It follows $\mathcal{H}^{n-1} - \mathrm{essinf}_{\partial E} V \ge -\|\mathrm{div}_\tau n_\phi - g\|_{L^\infty(\partial E)}$. This concludes the proof of (62).

Remark 68 *Thanks to Theorem 67, if $g \in L^\infty(\partial E)$ the functional $\mathcal{F}$ can be equivalently minimized on the space*

$$\widehat{H}(\partial E; \mathbb{R}^n) := \big\{N \in \mathrm{Nor}_\phi(\partial E; \mathbb{R}^n) : \mathrm{div}_{\phi,\tau} N \in L^\infty(\partial E)\big\}.$$

Moreover,

$$\|d_{\min} - g\|_{L^\infty(\partial E)} = \min\big\{\|\mathrm{div}_{\phi,\tau} N - g\|_{L^\infty(\partial E)} :\ N \in \widehat{H}(\partial E; \mathbb{R}^n)\big\}.$$

References

1. F.J. Almgren and J. Taylor. Flat flow is motion by crystalline curvature for curves with crystalline energies. *J. Diff. Geom.*, 42:1–22, 1995.
2. F.J. Almgren, J.E. Taylor, and L. Wang. Curvature-driven flows: a variational approach. *SIAM J. Control Optim.*, 31:387–437, 1993.
3. G. Bellettini and I. Fragala. Elliptic approximations of prescribed mean curvature surfaces in Finsler geometry. *Asympt. Anal.*, 22:87–111, 2000.
4. G. Bellettini, R. Goglione, and M. Novaga. Approximation to driven motion by crystalline curvature in two dimensions. *Adv. Math. Sc. Appl.*, 10:467–493, 2000.
5. G. Bellettini and M. Novaga. Approximation and comparison for non-smooth anisotropic motion by mean curvature in $\mathbb{R}^n$. *Math. Mod. Methods Appl. Sc.*, 10:1–10, 2000.
6. G. Bellettini, M. Novaga, and M. Paolini. Facet–breaking for three–dimensional crystals evolving by mean curvature. *Interfaces and Free Boundaries*, pages 39–55, 1999.
7. G. Bellettini, M. Novaga, and M. Paolini. On a crystalline variational problem, part II: BV–regularity and structure of minimizers on facets. *Arch. Rational Mech. Anal.*, to appear.
8. G. Bellettini and M. Paolini. Anisotropic motion by mean curvature in the context of Finsler geometry. *Hokkaido Math. J*, 25:537–566, 1996.
9. G. Bouchitté and G. Dal Maso. Integral representation and relaxation of convex local functionals on $BV(\Omega)$. *Annali Sc. Norm. Sup. di Pisa Cl. Sc.(4)*, 20:483–533, 1993.
10. H. Brezis. *Operateurs Maximaux Monotones*. North-Holland, Amsterdam, 1973.
11. J.W. Cahn, C.A. Handwerker, and J.E. Taylor. Geometric models of crystal growth. *Acta Metall. Mater.*, 40:1443–1474, 1992.
12. J.W. Cahn and D.W. Hoffman. A vector thermodynamics for anisotropic interfaces. 1. Fundamentals and applications to plane surface junctions. *Surface Sci.*, 31:368–388, 1972.
13. J.W. Cahn and D.W. Hoffman. A vector thermodynamics for anisotropic interfaces. 2. Curved and faceted surfaces. *Acta Metall. Mater.*, 22:1205–1214, 1974.
14. F. H. Clarke. *Optimization and Non Smooth Analysis*. J. Wiley & Sons, Inc., New York, 1983.
15. L.C. Evans and R. F. Gariepy. *Measure Theory and Fine Properties of Functions*. Studies in Advanced Math., CRC Press, Ann Harbor, 1992.
16. T. Fukui and Y. Giga. Motion of a graph by nonsmooth weighted curvature. In V. Lakshmikantham, editor, *World Congress of Nonlinear Analysis '92*, pages 47–56. Walter de Gruyter, Berlin, 1996.
17. M.-H. Giga and Y. Giga. Evolving graphs by singular weighted curvature. *Arch. Rational Mech. Anal.*, 141:117–198, 1998.
18. M.-H. Giga and Y. Giga. A subdifferential interpretation of crystalline motion under nonuniform driving force. *Dynamical Systems and Differential Equations*, 1:276–287, 1998.
19. Y. Giga and M.E. Gurtin. A comparison theorem for crystalline evolutions in the plane. *Quarterly of Applied Mathematics*, LIV:727–737, 1996.
20. Y. Giga, M.E. Gurtin, and J. Matias. On the dynamics of crystalline motion. *Japan Journal of Industrial and Applied Mathematics*, 15:7–50, 1998.
21. P.M. Girao and R.V. Kohn. Convergence of a crystalline algorithm for the heat equation in one dimension and for the motion of a graph. *Numer. Math.*, 67:41–70, 1994.
22. P.M. Girao and R.V. Kohn. The crystalline algorithm for computing motion by curvature. *In Variational Methods for Discontinuous Structures (eds. R. Serapioni and F. Tomarelli), Birkäuser*, 1996.
23. M. Gurtin. *Thermomechanics of Evolving Phase Boundaries in the Plane*. Clarendon Press, Oxford, 1993.
24. M. Novaga and E. Paolini. A computational approach to fractures in crystal growth. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei*, 10(9):47–56, 1999.

25. A.R. Roosen and J. Taylor. Modeling crystal growth in a diffusion field using fully faceted interfaces. *J. Comput. Phys.*, 114:444–451, 1994.
26. P. Rybka. A crystalline motion: uniqueness and geometric properties. *SIAM J. Appl. Math.*, 57:53–72, 1997.
27. J. Taylor. Complete catalog of minimizing embedded crystalline cones. *Proc. Symposia Math.*, 44:379–403, 1986.
28. J.E. Taylor. Crystalline variational problems. *Bull. Amer. Math. Soc. (N.S.)*, 84:568–588, 1978.
29. J.E. Taylor. Geometric crystal growth in 3D via facetted interfaces. *In: Computational Crystal Growers Workshop, Selected Lectures in Mathematics, Amer. Math. Soc.*, pages 111–113, 1992.
30. J.E. Taylor. II-mean curvature and weighted mean curvature. *Acta Metall. Mater.*, 40:1475–1485, 1992.
31. J.E. Taylor. Motion of curves by crystalline curvature, including triple junctions and boundary points. *Proc. of Symposia in Pure Math.*, 54:417–438, 1993.
32. W.A. Tiller. *The Science of Crystallization*. Cambridge University Press, 1991.
33. J. Yunger. *Facet stepping and motion by crystalline curvature*. PhD thesis, Rutgers University, 1998.

Dipartimento di Matematica, Università di Roma “Tor Vergata”,
via della Ricerca Scientifica, 00133 Roma, Italy,
e-mail: belletti@axp.mat.uniroma2.it

and

Dipartimento di Matematica,Università di Pisa,
via Buonarroti 2, 56127 Pisa, Italy,
e-mail: novaga@dm.unipi.it

and

Dipartimento di Matematica,Università Cattolica “Sacro Cuore”,
via Musei 41, 25121 Brescia, Italy,
e-mail: paolini@dmf.bs.unicatt.it