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On a crystalline variational problem, part II: BV–regularity and structure of minimizers on facets

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Abstract

For a nonsmooth positively one homogeneous convex function $\phi : \mathbb{R}^n \to [0, +\infty[$, it is possible to introduce the class $\mathcal{R}_\phi(\mathbb{R}^n)$ of smooth boundaries with respect to ϕ, to define their ϕ-mean curvature κ_ϕ, and to prove that, for $E \in \mathcal{R}_\phi(\mathbb{R}^n)$, there holds $\kappa_\phi \in L^\infty(\partial E)$ [9]. Based on these results, we continue the analysis on the structure of ∂E and on the regularity properties of κ_ϕ. We prove that a facet F of ∂E is Lipschitz (up to negligible sets) and that κ_ϕ has bounded variation on F. Further properties of the jump set of κ_ϕ are inspected: in particular, in three space dimensions, we relate the sublevel sets of κ_ϕ on F with the geometry of the Wulff shape $\mathcal{W}_\phi := \{\phi \leq 1\}$.

1. Introduction

Let $\phi : \mathbb{R}^n \to [0, +\infty[$ be a nonsmooth one homogeneous convex function. In this paper we continue the analysis initiated in [9] on the properties of the class $\mathcal{R}_\phi(\mathbb{R}^n)$ of Lipschitz ϕ-regular sets (i.e. the "smooth" boundaries in the finite dimensional Banach space $(\mathbb{R}^n, \phi)$) and of their ϕ-mean curvature κ_ϕ. For $E \in \mathcal{R}_\phi(\mathbb{R}^n)$ and $g \in L^2(\partial E)$ we can consider a solution $N_{\min}$ of the following variational problem:

$$\min\{\mathcal{F}(N) : N \in H(\partial E; \mathbb{R}^n)\}, \qquad \mathcal{F}(N) := \int_{\partial E} (\mathrm{div}_{\phi,\tau} N - g)^2 \, dP_\phi, \quad (1)$$

where $H(\partial E; \mathbb{R}^n)$ is the class of ϕ-normal vector fields on ∂E with intrinsic tangential divergence $\mathrm{div}_{\phi,\tau}$ in $L^2(\partial E)$, and dP_ϕ denotes the (density of the) ϕ-perimeter, see [9] for all details. The function g has, in the evolution problem, the rôle of the forcing term. Setting $d_{\min} := \mathrm{div}_{\phi,\tau} N_{\min}$, the ϕ-mean curvature κ_ϕ of ∂E is defined as $\kappa_\phi := d_{\min}$ when $g = 0$. The basic result $\kappa_\phi \in L^\infty(\partial E)$ proved in [9] is the starting point of the present paper, which is focused on finer regularity properties of κ_ϕ (or, more generally, of $d_{\min}$) on suitable facets of ∂E. The importance of explicitely computing κ_ϕ (whenever this is possible) relies on the fact that κ_ϕ is exptected to be the initial velocity of ∂E in the evolution problem having ∂E as initial datum.

Denote by $\mathcal{W}_\phi := \{\phi \leq 1\}$ the Wulff shape. In Definition 32 we define what we mean by a facet F of ∂E corresponding to a facet of $\partial \mathcal{W}_\phi$ (we write $F \in \mathrm{Fct}_\phi(\partial E)$). If F is such a facet, it turns out that $d_{\min} - g$ has locally bounded variation on the interior of F (Theorem 33). To improve this regularity result, we need to investigate the regularity properties of the facets of $\mathrm{Fct}_\phi(\partial E)$. In general, it is clear that facets of a Lipschitz boundary may be very irregular. However Lipschitz ϕ-regular sets have a Lipschitz ϕ-normal vector field constrained to vary in a suitable family of convex compact cones. Using this information, in Theorem 44 we prove a first structure result on Lipschitz ϕ-regular sets which reads as follows. If $E \in \mathcal{R}_\phi(\mathbb{R}^n)$ and $F \in \mathrm{Fct}_\phi(\partial E)$, then F has finite perimeter in $\mathbb{R}^{n-1}$. Moreover, there exists a compact subset $Z_F \subset \partial F$ out of which ∂F can be written locally as the graph of a Lipschitz function and, if $n = 3$, Z_F is finite. In general, Z_F is non empty, see Figure 2, and we can exhibit an example in $n = 4$ dimensions where Z_F has two dimensional positive Hausdorff measure. In any case, the regularity properties of F coupled with the result that $d_{\min} \in L^\infty(\partial E)$ (for $g \in L^\infty(\partial E)$) are enough to prove that $d_{\min} - g$ has bounded variation on F (see Theorem 53). With this result at hand, we are allowed to consider the jump set of $d_{\min} - g$ on the interior of F. Section 6 is concerned with finer regularity properties of $d_{\min} - g$ and of its sublevel sets, denoted by Ω_t^F. In Theorem 64 we prove that each Ω_t^F solves a kind of anisotropic isoperimetric problem in the hyperplane containing the facet. This anisotropy, denoted with $\widetilde{\phi}$, has unit ball which is essentially the facet of $\mathcal{W}_\phi$ parallel to F. As a by-product of Theorem 64 and the results of [11], [12], [4] we obtain some interesting informations on the structure of Ω_t^F. We quote in particular the following result (Corollary 65) : in $n = 3$ space dimensions and if $g = 0$, every connected component of Ω_t^F is contained, up to a translation and a homotety, in the boundary of the corresponding facet of the Wulff shape.

In a forthcoming paper [8] we study necessary and sufficient conditions for a facet to subdivide in the subsequent evolution, and we make rigorous the second example discussed in [7].

2. Setting and notation

In this paper, we will follow the notation and the definitions of [9]. We recall that the duality mappings T and T^o are defined by

$$T(\xi) = \phi(\xi) D^- \phi(\xi), \qquad T^o(\xi^*) = \phi^o(\xi^*) D^- \phi^o(\xi^*), \qquad \xi, \xi^* \in \mathbb{R}^n,$$

where D^- is the subdifferential, and that

$$\mathcal{W}_\phi^o := \{\xi^* \in \mathbb{R}^n : \phi^o(\xi^*) \leq 1\}, \qquad \mathcal{W}_\phi := \{\xi \in \mathbb{R}^n : \phi(\xi) \leq 1\},$$

where ϕ^o is the dual of ϕ. We say that ϕ is crystalline if $\mathcal{W}_\phi$ is a convex polytope. By a facet of $\partial\mathcal{W}_\phi$ (or of $\partial\mathcal{W}_\phi^o$) we always mean a closed $(n-1)$-dimensional facet.

Given a nonempty Lipschitz set $E \subset \mathbb{R}^n$, we let d_ϕ^E be the oriented ϕ-distance function from ∂E negative inside E, and, on ∂E, we set $\nu_\phi^E := \frac{\nu^E}{\phi^o(\nu^E)} = \nabla d_\phi^E$, where ν^E is the outward normal to ∂E with euclidean unit length.

If E is Lipschitz we define

$$\mathrm{Nor}_\phi(\partial E;\mathbb{R}^n) = \{N : \partial E \to \mathbb{R}^n : N(x) \in T^o(\nu_\phi^E(x)) \text{ for } \mathcal{H}^{n-1}\text{– a.e. } x \in \partial E\},$$
$$\mathrm{Lip}_{\nu,\phi}(\partial E;\mathbb{R}^n) = \mathrm{Lip}(\partial E;\mathbb{R}^n) \cap \mathrm{Nor}_\phi(\partial E;\mathbb{R}^n).$$

Definition 21 *Let $E \subseteq \mathbb{R}^n$ be a Lipschitz set with compact boundary. We say that E is Lipschitz ϕ-regular if there exists a vector field $n_\phi \in \mathrm{Lip}_{\nu,\phi}(\partial E;\mathbb{R}^n)$. We denote by $\mathcal{R}_\phi(\mathbb{R}^n)$ the class of all Lipschitz ϕ-regular sets.*

We sometimes shall write $(E, n_\phi) \in \mathcal{R}_\phi(\mathbb{R}^n)$, and we shall say that (E, n_ϕ) is Lipschitz ϕ-regular.

We set

$$H(\partial E,\mathbb{R}^n) := \{N \in \mathrm{Nor}_\phi(\partial E;\mathbb{R}^n) : \mathrm{div}_{\phi,\tau} N \in L^2(\partial E)\},$$
$$\widehat{H}(\partial E,\mathbb{R}^n) := \{N \in \mathrm{Nor}_\phi(\partial E;\mathbb{R}^n) : \mathrm{div}_{\phi,\tau} N \in L^\infty(\partial E)\},$$

where the definition of $\mathrm{div}_{\phi,\tau}$ is given in [9], and will not be repeated here. We just make the following observation.

Remark 22 *Let $(E, n_\phi) \in \mathcal{R}_\phi(\mathbb{R}^n)$ and let F be a facet of ∂E. For any vector field $N \in \mathrm{Nor}_\phi(\partial E;\mathbb{R}^n)$ there holds*

$$\langle \mathrm{div}_{\phi,\tau} N, \psi\rangle = \int_{\mathrm{int}(F)} \psi \ \mathrm{div}_\tau N \ dP_\phi \qquad \forall \psi \in \mathrm{Lip}(\partial E),\ \mathrm{spt}(\psi) \subset \mathrm{int}(F). \tag{2}$$

Indeed, using Lemma 4.4 of [9], we have

$$\begin{aligned}&\langle \mathrm{div}_{\phi,\tau} N, \psi\rangle - \langle \mathrm{div}_{\phi,\tau} n_\phi, \psi\rangle = \langle \mathrm{div}_{\phi,\tau}(N - n_\phi), \psi\rangle \\ &= -\int_{\partial E} \nabla_\tau \psi \cdot (N - n_\phi)\, dP_\phi = -\int_{\partial E} \nabla_\tau\psi \cdot N\, dP_\phi - \int_{\partial E} \psi \ \mathrm{div}_\tau n_\phi \ dP_\phi.\end{aligned}$$

Then (2) follows using the fact that $\phi^o(\nu^E)$ is constant on $\mathrm{int}(F)$, that $\mathrm{spt}(\psi) \Subset \mathrm{int}(F)$ and performing a euclidean integration by parts.

Therefore, on $\mathrm{int}(F)$ (which is contained in a affine hyperplane of $\mathbb{R}^n$) the operator $\mathrm{div}_{\phi,\tau} N$ coincides with $\mathrm{div}_\tau N$, which denotes the usual weak divergence of $N \in \mathrm{Nor}_\phi(\partial E;\mathbb{R}^n)$ on $\mathrm{int}(F)$. We shall accordingly use the notation $\mathrm{div}_\tau N$ in place of $\mathrm{div}_{\phi,\tau} N$.

We let dP_ϕ be the measure supported on ∂E with density $\phi^o(\nu^E)$.

The following results have been proved in [9].

Theorem 23 *Let $E \in \mathcal{R}_\phi(\mathbb{R}^n)$ and assume that $g \in L^\infty(\partial E)$. Then*

$$d_{\min} \in L^\infty(\partial E). \tag{3}$$

Moreover

$$\int_{\partial E} (\mathrm{div}_{\phi,\tau} N_{\min} - g)\, \mathrm{div}_{\phi,\tau}(N_{\min} - N)\, dP_\phi \le 0 \qquad \forall N \in H(\partial E; \mathbb{R}^n). \tag{4}$$

Finally, if for any $t \in \mathbb{R}$ we define

$$A_t := \{d_{\min} - g > t\}, \qquad \Omega_t := \{d_{\min} - g < t\},$$

then

$$\int_{A_t} d_{\min}\, dP_\phi \le \int_{A_t} \mathrm{div}_{\phi,\tau}\, N\, dP_\phi \qquad \forall t \in \mathbb{R},\ \forall N \in H(\partial E; \mathbb{R}^n), \tag{5}$$

and

$$\int_{\Omega_t} d_{\min}\, dP_\phi \ge \int_{\Omega_t} \mathrm{div}_{\phi,\tau} N\, dP_\phi \qquad \forall t \in \mathbb{R},\ \forall N \in H(\partial E; \mathbb{R}^n). \tag{6}$$

Definition 24 *We say that F is a facet of ∂E if F is the closure of a connected component of the relative interior of $\partial E \cap T_x \partial E$ for some $x \in \partial E$ such that the tangent hyperplane $T_x \partial E$ to ∂E at x exists.*

If F is a facet of ∂E, we denote by ∂F (resp. $\mathrm{int}(F)$) the relative boundary (resp. the relative interior) of F. It is clear that, on $\mathrm{int}(F)$, the measure dP_ϕ coincides with $\mathcal{H}^{n-1}$, up to the positive constant $\phi^o(\nu^E)$.

We say that E is convex at F if ∂E, locally around F, meets the hyperplane H_F containing F only in F. We say that ∂E is concave at F if $\mathbb{R}^n \setminus E$ is convex at F. Whenever necessary, we identify H_F with the hyperplane parallel to H_F and passing through the origin, and F with its orthogonal projection on this latter hyperplane.

We often do not indicate the dependence on E of the unit normals ν^E and ν_ϕ^E, i.e. we set $\nu := \nu^E$, $\nu_\phi := \nu_\phi^E$.

Let $m \ge 1$ (throughout the paper, m has the rôle of $n-1$). Given a (scalar or vector valued) Radon measure μ on an open subset Ω of $\mathbb{R}^m$, we denote by $|\mu|$ the total variation of μ. If $p \in [1, +\infty]$, the symbol $L^p_\mu(\Omega)$ denotes the class of all functions f such that $|f|^p$ is integrable with respect to the measure μ if $p < +\infty$, and f is essentially bounded with respect to μ if $p = +\infty$. If ν is a positive Radon measure, and if μ is absolutely continuous with respect to ν, the density of μ with respect to ν will be indicated by $\frac{d\mu}{d\nu}$, and is usually called the Radon-Nikodym derivative of μ with respect to ν.

BV functions. The space $BV(\Omega)$ is defined as the set of all functions $u \in L^1(\Omega)$ whose distributional gradient Du is a Radon measure with bounded total variation on Ω, i.e. $|Du|(\Omega) = \int_\Omega |Du| < +\infty$. It is well known that $BV(\Omega)$ is contained in $L^{\frac{m}{m-1}}_{\mathrm{loc}}(\Omega)$ and that, if Ω is bounded and has Lipschitz boundary, then $BV(\Omega)$ is contained in $L^{\frac{m}{m-1}}(\Omega)$. Also, If Ω is bounded and Lipschitz and $u \in BV(\Omega)$, then u admits a trace (still denoted by u) on $\partial\Omega$, which belongs to $L^1(\partial\Omega)$.

We denote by $u^{\pm}(x)$ the essential upper and lower limits of u at $x \in \Omega$, and we let $J_u := \{u^- < u^+\}$ be the jump set of u (see [3]).

The space $BV_{\rm loc}(\Omega)$ is the class of all functions which are of bounded variation on each open set $A \Subset \Omega$.

We say that a set $B \subseteq \Omega$ is of finite perimeter in Ω, and we write $P(B,\Omega) < +\infty$, if $1_B \in BV(\Omega)$. We say that B is of locally finite perimeter in Ω if $1_B \in BV_{\rm loc}(B)$. Each set $B \subseteq \Omega$ of finite perimeter will be always identified with its representative consisting of points of density one. If B is of finite perimeter in Ω, $\partial^* B$ denotes the reduced boundary of B. $\partial^* B$ is rectifiable and can be endowed with a generalized exterior euclidean unit normal ν_B^* so that

$$D1_B(C) = -\int_{C\cap\partial^* B} \nu_B^* \, d\mathcal{H}^{m-1}$$

for any Borel set $C \subseteq \Omega$.

We recall the following result, proved in [6].

Theorem 25 *Let $\Omega \subseteq \mathbb{R}^m$ be a bounded open set. Let*

$$u \in BV(\Omega) \tag{7}$$

and

$$X \in L^\infty(\Omega;\mathbb{R}^m), \ {\rm div}X \in L^m(\Omega). \tag{8}$$

Then the linear functional

$$(X, Du) : \varphi \to -\int_\Omega u\varphi \, {\rm div}X \, dx - \int_\Omega uX\cdot\nabla\varphi \, dx, \qquad \varphi \in \mathcal{C}^1_c(\Omega)$$

defines a Radon measure (still denoted by (X,Du)) and satisfies

$$|(X,Du)|(B) \le \|X\|_{L^\infty(\Omega;\mathbb{R}^m)}|Du|(B)$$

for any Borel set $B \subseteq \Omega$.

We denote by $\theta(X,Du) \in L^\infty_{|Du|}(\Omega)$ the density of (X,Du) with respect to $|Du|$, that is

$$(X,Du)(B) = \int_B \theta(X,Du) \, d|Du| \qquad \text{for any Borel set } B \subseteq \Omega. \tag{9}$$

Note in particular that, if ${\rm spt}(X) \Subset \Omega$, then

$$(X,Du)(B) = -\int_B u \, {\rm div}X \, dx \qquad \text{for any Borel set } B \supseteq {\rm spt}(X). \tag{10}$$

Unless further regularity properties are assumed on u or on X, in general the function $\theta(X,Du)$ has not a pointwise expression almost everywhere with respect to the measure $|D^s u|$ (where $D^s u$ denotes the singular part of the measure Du with respect to the Lebesgue measure).

Remark 26 *Let u_1, u_2 satisfy (7) and let X satisfy (8). If $u_1 = u_2$ on an open set $A \Subset \Omega$, then*

$$(X, Du_1)(B) = (X, Du_2)(B) \qquad \forall \text{ Borel set } B \subseteq A.$$

Finally, we will often use the tilde to enphasize objects (such as normal vectors or positively one homogeneous convex functions) in a $(n-1)$-dimensional space.

3. BV_{loc}-regularity of minimizers on facets

We are interested in studying the behaviour of $d_{\min} - g$ on certain facets of ∂E. In order to do that, we need some preliminaries.

Let F be a facet of ∂E. Clearly, the vector $\nu_\phi^E(x)$ is independent of $x \in \mathrm{int}(F)$.

Definition 31 *Let $E \in \mathcal{R}_\phi(\mathbb{R}^n)$ and let F be a facet of ∂E. We define*

$$\begin{aligned} \nu_\phi(F) &:= \nu_\phi^E(x), \qquad x \in \mathrm{int}(F), \\ \widetilde{W}_\phi^F &:= T^o(\nu_\phi(F)). \end{aligned}$$

$\widetilde{W}_\phi^F$ is a closed convex set contained in $\partial \mathcal{W}_\phi$. Moreover, if F is parallel to a facet W of $\partial \mathcal{W}_\phi$ and has the same exterior unit normal, then $\widetilde{W}_\phi^F = W$. Indeed, $\nu_\phi(W) = \nu_\phi(F)$ implies $\widetilde{W}_\phi^F = T^o(\nu_\phi(W))$. Since $T^o(\nu_\phi(W)) = W$, it follows that $\widetilde{W}_\phi^F = W$.

Definition 32 *Let $E \in \mathcal{R}_\phi(\mathbb{R}^n)$. We define*

$$\mathrm{Fct}_\phi(\partial E) := \left\{ F : F \text{ is a facet of } \partial E \text{ and } \widetilde{W}_\phi^F \text{ is a facet of } \partial \mathcal{W}_\phi \right\}.$$

The class $\mathrm{Fct}_\phi(\partial E)$ is non empty only if $\partial \mathcal{W}_\phi$ has at least one facet: this assumption (obviously satisfied in the crystalline case) will be therefore tacitly assumed in the sequel.

The following result is a first regularity property of minimizers of $\mathcal{F}$ on facets corresponding to facets of the Wulff shape.

It is useful to recall that, by Remark 22, the ϕ-tangential divergence coincides with the euclidean tangential divergence on facets of ∂E, for vector fields in $H(\partial E; \mathbb{R}^n)$.

Theorem 33 *Let $E \in \mathcal{R}_\phi(\mathbb{R}^n)$ and let $F \in \mathrm{Fct}_\phi(\partial E)$. Then*

$$d_{\min} - g \in BV_{\mathrm{loc}}(\mathrm{int}(F)). \tag{11}$$

Proof. Let for simplicity of notation $V := d_{\min} - g$. Fix an open set A relatively compact in $\mathrm{int}(F)$. We have to prove that $V \in BV(A)$, i.e.

$$\sup \left\{ \int_A V \, \mathrm{div}_\tau \eta \, dP_\phi : \eta \in \mathcal{C}_c^1(A; \mathbb{R}^n), |\eta| \le 1, \eta \cdot \nu_\phi(F) = 0 \right\} < +\infty. \tag{12}$$

Choose $\rho > 0$ and a point $y \in \mathrm{int}\left(\widetilde{W}_\phi^F\right)$ such that

$$B_\rho(y) \cap \widetilde{W}_\phi^F \subseteq \mathrm{int}\left(\widetilde{W}_\phi^F\right). \tag{13}$$

Fix $\eta_0 \in \mathrm{Lip}_{\nu,\phi}(\partial E;\mathbb{R}^n)$ with the following properties:

$$\begin{cases} \eta_0 \equiv y & \text{in a neighbourhood of } A \text{ contained in } \mathrm{int}(F), \\ \eta_0 = n_\phi & \text{in } \partial E \setminus \mathrm{int}(F). \end{cases} \tag{14}$$

Let $\eta \in C_c^1(A;\mathbb{R}^n)$, $|\eta| \le 1$, $\eta \cdot \nu_\phi(F) = 0$, and set

$$\overline{\eta} := \eta_0 - \rho\eta.$$

Then $\overline{\eta} \in \mathrm{Lip}(\partial E;\mathbb{R}^n)$, and by (13), (14), $|\eta| \le 1$ and the fact that $\mathrm{spt}(\eta)$ is compact in A, it follows that $\overline{\eta} \in \mathrm{Lip}_{\nu,\phi}(\partial E;\mathbb{R}^n)$. In particular $\overline{\eta} \in H(\partial E;\mathbb{R}^n)$. ¿From (4) it follows that

$$\begin{aligned} \int_{\partial E} V \mathrm{div}_{\phi,\tau} N_{\min}\, dP_\phi &\le \int_{\partial E} V \mathrm{div}_\tau \overline{\eta}\, dP_\phi \\ &= \int_{\partial E} V \mathrm{div}_\tau \eta_0\, dP_\phi - \rho \int_A V \mathrm{div}_\tau \eta\, dP_\phi. \end{aligned}$$

Therefore

$$\begin{aligned} \int_A V \mathrm{div}_\tau \eta\, dP_\phi &\le \rho^{-1} \int_{\partial E} V \mathrm{div}_{\phi,\tau}(\eta_0 - N_{\min})\, dP_\phi \\ &\le C\rho^{-1} \|V\|_{L^2(\partial E)} \|\mathrm{div}_{\phi,\tau}(\eta_0 - N_{\min})\|_{L^2(\partial E)}, \end{aligned} \tag{15}$$

where $C := \max_{|\nu|=1} \phi^o(\nu)$. Passing to the supremum over η in (15) we deduce (12), and (11) follows.

We now want to give a pointwise version of inequality (6) on facets of ∂E corresponding to facets of $\partial\mathcal{W}_\phi$.

Definition 34 *Let $F \in \mathrm{Fct}_\phi(\partial E)$. For any $t \in \mathbb{R}$ we define*

$$\begin{aligned} A_t^F &:= \{x \in \mathrm{int}(F) : d_{\min}(x) - g(x) > t\}, \\ \Omega_t^F &:= \{x \in \mathrm{int}(F) : d_{\min}(x) - g(x) < t\}. \end{aligned} \tag{16}$$

Observe that from Theorem 33 and the coarea formula, Ω_t^F and A_t^F have locally finite perimeter in $\mathrm{int}(F)$ for almost every $t \in \mathbb{R}$. We denote by $\tilde{\nu}^*_{\Omega_t^F} = \tilde{\nu}^*_t := -\frac{D1_{\Omega_t^F}}{|D1_{\Omega_t^F}|}$ the exterior (generalized) unit normal to $\mathrm{int}(F) \cap \partial^* \Omega_t^F$.

The following proposition gives a pointwise version of the inequality (6) on facets $F \in \mathrm{Fct}_\phi(\partial E)$, and a pointwise expression of $\theta(N_{\min}, D1_{\Omega_t^F})$.

Proposition 35 *For almost every $t \in \mathbb{R}$ and for $\mathcal{H}^{n-2}$-almost every $x \in \mathrm{int}(F) \cap \partial^* A_t^F$ we have*

$$-\theta(N_{\min}, D1_{A_t^F})(x) = \min\left\{ z \cdot \tilde{\nu}_t^*(x) : z \in \widetilde{W}_\phi^F \right\}, \tag{17}$$

and for almost every $t \in \mathbb{R}$ and for $\mathcal{H}^{n-2}$-almost every $x \in \mathrm{int}(F) \cap \partial^* \Omega_t^F$ we have

$$-\theta(N_{\min}, D1_{\Omega_t^F})(x) = \max\left\{ z \cdot \tilde{\nu}_t^*(x) : z \in \widetilde{W}_\phi^F \right\}. \tag{18}$$

Proof. Set $\Omega := \mathrm{int}(F)$. Fix $t \in \mathbb{R}$ such that $1_{\Omega_t^F}$ has locally finite perimeter in Ω. Set $u := 1_{\Omega_t^F}$ and $J_u := \partial^* \Omega_t^F$. By (6) we have

$$\int_\Omega u \,\mathrm{div}_\tau(N_{\min} - N)\, d\mathcal{H}^{n-1} \geq 0 \quad \forall N \in \widehat{H}(\partial E; \mathbb{R}^n),\ \mathrm{spt}(N_{\min} - N) \Subset \mathrm{int}(F).$$

Applying (9) and (10) with $m = n-1$, $B = \Omega$ and $X = N_{\min} - N$, we get

$$\int_{J_u} \theta(N_{\min} - N, Du)\, d\mathcal{H}^{n-2} \leq 0. \tag{19}$$

Choose a subset $\mathcal{N}$ of $\mathrm{spt}(N_{\min} - N) \cap J_u$ such that $\mathcal{H}^{n-2}(\mathcal{N}) = 0$ and for any $x \in (\mathrm{spt}(N_{\min} - \eta) \cap J_u) \setminus \mathcal{N}$ there holds

$$\theta(N_{\min}, Du)(x) = \lim_{\rho \to 0^+} \frac{1}{\omega_{n-2}\rho^{n-2}} \int_{B_\rho(x) \cap J_u} \theta(N_{\min}, Du)\, d\mathcal{H}^{n-2} \tag{20}$$

and

$$\tilde{\nu}_t^*(x) = \lim_{\rho \to 0^+} \frac{1}{\omega_{n-2}\rho^{n-2}} \int_{B_\rho(x) \cap J_u} \tilde{\nu}_t^*\, d\mathcal{H}^{n-2}, \tag{21}$$

where ω_{n-2} denotes the Lebesgue measure of the unit ball in $\mathbb{R}^{n-2}$. Equality (20) follows from the fact that $\theta(N_{\min}, Du) \in L^\infty_{\mathcal{H}^{n-2}}(J_u)$ and Ω_t^F has locally finite perimeter, while equality (21) follows from the definition of $\tilde{\nu}_t^*$.

Fix $x \in (\mathrm{spt}(N_{\min} - N) \cap J_u) \setminus \mathcal{N}$ and $r > 0$ such that $B_r(x) \subset \Omega$. Choose $z \in \widetilde{W}_\phi^F$ such that

$$z \cdot \tilde{\nu}_t^*(x) = \max\left\{ w \cdot \tilde{\nu}_t^*(x) : w \in \widetilde{W}_\phi^F \right\}.$$

For any $\rho > 0$ with $\rho + \rho^2 < r$, we choose a function $\eta_\rho : \partial E \to \mathbb{R}^n$, $\eta_\rho \in \widehat{H}(\partial E; \mathbb{R}^n)$, such that

$$\eta_\rho(y) = \begin{cases} z & \forall y \in B_\rho(x), \\ N_{\min}(y) & \text{for } \mathcal{H}^{n-1} - \text{a.e. } y \notin B_{\rho+\rho^2}(x). \end{cases}$$

¿From (19) and the fact that $N_{\min} = \eta_\rho$ outside of $B_{\rho+\rho^2}(x)$, we have

$$\begin{aligned} 0 \geq &\lim_{\rho \to 0^+} \frac{1}{\omega_{n-2}\rho^{n-2}} \int_{B_\rho(x) \cap J_u} \theta(N_{\min} - \eta_\rho, Du)\, d\mathcal{H}^{n-2} \\ &+ \lim_{\rho \to 0^+} \frac{1}{\omega_{n-2}\rho^{n-2}} \int_{\left(B_{\rho+\rho^2}(x) \setminus B_\rho(x)\right) \cap J_u} \theta(N_{\min} - \eta_\rho, Du)\, d\mathcal{H}^{n-2}. \end{aligned}$$

Observing that the last limit at the right hand side vanishes, and that η_ρ is constant on $B_\rho(x)$, so that $\theta(\eta_\rho, Du)(y) = -z \cdot \tilde{\nu}_t^*(y)$ for $\mathcal{H}^{n-1}$-almost every $y \in B_\rho(x)$, using (20) and (19) we get

$$\theta(N_{\min}, Du)(x) \le -z \cdot \tilde{\nu}_t^*(x) = -\max\left\{w \cdot \tilde{\nu}_t^*(x) : \, w \in \widetilde{W}_\phi^F\right\}.$$

Let us prove the opposite inequality. Consider vector fields $N_\epsilon \in \widehat{H}(\partial E; \mathbb{R}^n)$ such that $N_{\epsilon|\Omega} \in \mathrm{Lip}(\Omega; \mathbb{R}^n)$, $N_\epsilon \rightharpoonup N_{\min}$ weakly-$*$ in $L^\infty(\Omega; \mathbb{R}^n)$ and $\mathrm{div}_\tau N_\epsilon \rightharpoonup \mathrm{div}_\tau N_{\min}$ weakly in $L^{n-1}(\Omega)$, as $\epsilon \to 0$. Then, one can check that

$$\theta(N_\epsilon, Du) \rightharpoonup \theta(N_{\min}, Du) \quad \text{in} \quad \text{weakly}-* \text{ in } L^\infty_{\mathcal{H}^{n-2} \llcorner J_u}(\Omega). \tag{22}$$

Since moreover $-\theta(N_\epsilon, Du) = N_\epsilon \cdot \tilde{\nu}_t^*$, we have that

$$-\theta(N_\epsilon, Du)(x) \le \max\{w \cdot \tilde{\nu}_t^*(x) : \, w \in \widetilde{W}_\phi^F\},$$

for $\mathcal{H}^{n-2}$–almost every $x \in J_u$. Passing to the limit as $\epsilon \to 0$ and using (22), we obtain

$$-\theta(N_{\min}, Du)(x) \le \max\{w \cdot \tilde{\nu}_t^*(x) : \, w \in \widetilde{W}_\phi^F\},$$

and the proposition is proved.

4. Regularity of facets of ∂E

The following three lemmas will be used to prove Theorem 44 which, in turn, is necessary to prove Theorem 53. Notice that $\xi^* \in T^o(\xi)$ if and only if $\xi \in T(\xi^*)$ for any $\xi, \xi^* \in \mathbb{R}^n$. We also recall that the map T is upper semicontinuous, in the sense that if $\xi_h, \xi \in \mathbb{R}^n$ and $\xi_h \to \xi$ as $h \to +\infty$, then

$$\bigcap_m \overline{\bigcup_{h \ge m} T(\xi_h)} \subseteq T(\xi). \tag{23}$$

Lemma 41 *Let $(E, n_\phi) \in \mathcal{R}_\phi(\mathbb{R}^n)$ and let $F \in \mathrm{Fct}_\phi(\partial E)$. Then*

$$x \in \partial F \Rightarrow n_\phi(x) \in \partial \widetilde{W}_\phi^F. \tag{24}$$

Proof. Let $x \in \partial F$. Since F is a facet of ∂E, ∂E is Lipschitz, and since a Lipschitz function with almost everywhere vanishing gradient is constant, it follows that, in a small neighbourhood of x, there are points where ν_ϕ^E exists and the set $T^o(\nu_\phi^E)$ does not intersect $\mathrm{int}(\widetilde{W}_\phi^F)$. Assume by contradiction that $n_\phi(x) \in \mathrm{int}(\widetilde{W}_\phi^F)$. As n_ϕ is continuous, there exists $\rho > 0$ such that $n_\phi(y) \in \mathrm{int}(\widetilde{W}_\phi^F)$ for any $y \in B_\rho(x) \cap \partial E$. Let $\overline{y} \in B_\rho(x) \cap \partial E$ be such that $T^o(\nu_\phi^E(\overline{y})) \cap \mathrm{int}(\widetilde{W}_\phi^F) = \emptyset$. Recalling that $n_\phi(\overline{y}) \in T^o(\nu_\phi^E(\overline{y}))$, we reach a contradiction.

Lemma 42 *Let $(E, n_\phi) \in \mathcal{R}_\phi(\mathbb{R}^n)$ and let $x \in \partial E$. Then*

$$\lim_{\rho \to 0^+} \sup_{y \in B_\rho(x) \cap \partial E,\ z \in T(n_\phi(y))} \mathrm{dist}\,(z, T(n_\phi(x))) = 0. \tag{25}$$

Moreover, if ϕ is crystalline, there exists $\rho_0 > 0$ such that

$$\nu_\phi^E(y) \in T(n_\phi(x)) \qquad \text{for } \mathcal{H}^{n-1} - \text{a.e. } y \in B_{\rho_0}(x) \cap \partial E. \tag{26}$$

Proof. Let $(y_h) \subseteq \partial E$ be a sequence of points converging to x. Since n_ϕ is continuous, we have $n_\phi(y_h) \to n_\phi(x)$; therefore, using (23), we have

$$\sup_{z \in T(n_\phi(y_h))} \mathrm{dist}\,(z, T(n_\phi(x))) \to 0 \quad \text{as } h \to +\infty, \tag{27}$$

and (25) follows.

Assume that ϕ is crystalline. Since n_ϕ is continuous, we can choose $\rho_0 > 0$ such that n_ϕ takes $B_{\rho_0}(x) \cap \partial E$ into the union of all the adiacent facets of $\partial \mathcal{W}_\phi$ at $n_\phi(x)$ (if $n_\phi(x)$ is interior to a facet of $\partial \mathcal{W}_\phi$, then this union reduces to that facet only). By the properties of the map T, we have $T(n_\phi(x)) \supseteq T(n_\phi(y))$ for any $y \in B_{\rho_0}(x) \cap \partial E$. Moreover, from the inclusion $n_\phi \in T^o(\nu_\phi^E)$ it follows $\nu_\phi^E \in T(n_\phi)$. Hence

$$T(n_\phi(x)) \supseteq T(n_\phi(y)) \ni \nu_\phi^E(y) \qquad \text{for } \mathcal{H}^{n-1} - \text{a.e. } y \in B_{\rho_0}(x) \cap \partial E.$$

Notice that (25) implies

$$\lim_{\rho \to 0^+} \sup_{y \in B_\rho(x) \cap \partial^* E} \mathrm{dist}\big(\nu_\phi^E(y), T(n_\phi(x))\big) = 0.$$

Given a set A, by $\overline{\mathrm{co}} A$ we mean the closed convex envelope of A.

Lemma 43 *Let W be a facet of $\partial \mathcal{W}_\phi$, let $\xi \in \partial W$, $\rho > 0$ and define the convex compact set K_ρ as*

$$K_\rho := \overline{\mathrm{co}} \bigcup_{\zeta \in B_\rho(\xi) \cap \partial \mathcal{W}_\phi} T(\zeta).$$

Then there exists $\rho_0 > 0$ such that the two following properties hold:

(i) $\nu_\phi(W)$ is an extreme point of K_ρ for any $\rho \in]0, \rho_0[$;
(ii) there exist a constant $c > 0$ and a vector $\tilde{n} \neq 0$ such that

$$\tilde{n} \cdot \nu_\phi(W) = 0, \qquad \tilde{n} \cdot \nu = \tilde{n} \cdot (\nu - \nu_\phi(W)) \geq c|\nu - \nu_\phi(W)|, \tag{28}$$

for any $\nu \in K_\rho$ and for any $\rho \in]0, \rho_0[$.

Proof. Assume first that ϕ is crystalline. In this case K_ρ is identically equal to $T(\xi)$ for $\rho > 0$ small enough (it suffices to apply (26) with $E := \mathcal{W}_\phi$ and $\xi := n_\phi(x)$) and is a convex polytope with dimension between 1 and $(n-1)$, having $\nu_\phi(W)$ as vertex. Therefore (i) is immediate. Let $\{W_1, \ldots, W_k\}$ be the facets of $\partial \mathcal{W}_\phi$ adiacent to W at ξ, and denote by ν_ϕ^i the exterior normal to W_i, with $\phi^o(\nu_\phi^i) = 1$, for $i = 1, \ldots, k$. Choose any euclidean unit vector $\tilde{n}$ with the following properties: $\tilde n$ lies in the hyperplane of W and points strictly inside the outward normal convex cone to ∂W at ξ, i.e.

$$\tilde n \cdot \nu_\phi(W) = 0, \quad \tilde n \cdot (\zeta - \xi) \le 0 \quad \forall \zeta \in W, \quad \tilde n \cdot (\gamma - \xi) \ge -c_1|\gamma - \xi|, \tag{29}$$

for any $\gamma \in B_r(\xi) \cap \partial W$ and for some constants $r > 0$ and $c_1 \in]0, 1[$ (independent of γ). Since $\nu_\phi^i - \frac{\nu_\phi^i \cdot \nu_\phi(W)}{|\nu_\phi(W)|^2}\nu_\phi(W)$ is the orthogonal projection of ν_ϕ^i in the hyperplane of W, a direct computation gives

$$\tilde n \cdot \nu_\phi^i \ge \sqrt{1 - c_1^2}\left|\nu_\phi^i - \frac{\nu_\phi^i \cdot \nu_\phi(W)}{|\nu_\phi(W)|^2}\nu_\phi(W)\right|, \tag{30}$$

and $\sqrt{1-c_1^2} > 0$. Moreover there exists $c > 0$ such that

$$\sqrt{1 - c_1^2}\left|\nu_\phi^i - \frac{\nu_\phi^i \cdot \nu_\phi(W)}{|\nu_\phi(W)|^2}\nu_\phi(W)\right| \ge c|\nu_\phi^i - \nu_\phi(W)|, \qquad i = 1, \ldots, k. \tag{31}$$

Using (30), (31) and the fact that $\overline{\mathrm{co}}\{\nu_\phi(W), \nu_\phi^1, \ldots, \nu_\phi^k\} = T(\xi) = K_\rho$ for $\rho > 0$ small enough, property (ii) follows.

Let now ϕ be generic. Choose a $(n-1)$-dimensional polytope $\tilde H$ such that $\tilde H \subseteq W$ and $\partial \tilde H \cap \partial W = \{\xi\}$ (in particular ξ is a vertex of $\tilde H$). Let us consider a n-dimensional polytope H such that $H \subseteq \mathcal{W}_\phi$ and $\tilde H$ is a facet of ∂H. Let $\{H_1, \ldots, H_j\}$ be the facets of ∂H adiacent to $\tilde H$ at ξ and let ν_ϕ^i be the exterior normals to H_i with $\phi^o(\nu_\phi^i) = 1$. Let also

$$L := \overline{\mathrm{co}}\left\{\frac{\nu}{\phi^o(\nu)} : \ \nu \in \overline{\mathrm{co}}\{\nu_\phi^1, \ldots, \nu_\phi^j\}\right\}.$$

Then $L \supseteq K_\rho$ for any $\rho > 0$ small enough. This follows from the following observation: if f_1 and f_2 are two convex functions with the property that $f_1 - f_2$ has a strict local minimum at some point z_0, then the outward normal convex cone to the graph of f_1 at z_0 contains the outward normal convex cone to the graph of f_2 at z_0. Moreover $\nu_\phi(W)$ is an extreme point for L. Reasoning as in the previous case, we can find a vector $\tilde n$ and a constant $c > 0$ such that (28) holds. Indeed, any non zero vector $\tilde n \in \nu_\phi(W)^\perp$, pointing strictly outside of $\tilde H$ (hence of W), satisfies (28) for some $c > 0$.

The following result is a regularity property of Lipschitz ϕ-regular sets, and is necessary to give a meaning to the normal traces of the vector fields in $\widehat H(\partial E; \mathbb{R}^n)$ on boundaries of facets corresponding to facets of $\mathcal{W}_\phi$.

Theorem 44 *Let $E \in \mathcal{R}_\phi(\mathbb{R}^n)$ and let $F \in \mathrm{Fct}_\phi(\partial E)$. Then F has finite perimeter in $\mathbb{R}^{n-1}$. Moreover there exists a compact set $Z_F \subset \partial F$ such that for any $x \in \partial F \setminus Z_F$, ∂F is a Lipschitz graph locally around x. Finally, if $n = 3$, then Z_F is finite.*

Figure 2 show an example of Lipschitz ϕ-regular set E, in $n = 3$ space dimensions, having a facet $F \in \mathrm{Fct}_\phi(\partial E)$ such that $Z_F \subset \partial F$ is not empty. We shall show also that, in $n \ge 4$ space dimensions, it may happen that $\mathcal{H}^{n-2}(Z_F) > 0$.

Proof. Let $x \in \partial F$. By Lemma 43 applied with $W := \widetilde{W}_\phi^F$ and $\xi := n_\phi(x)$, we can choose $\rho_0 > 0$, K_ρ with $\rho \in]0, \rho_0[$, $\tilde{n}$ and $c > 0$ satisfying (i)-(ii). Our aim is to write the set ∂F, locally around x, as a graph of a BV function with respect to $\tilde{n}^\perp \cap H_F$, and to use the inequality in (ii) of Lemma 43 to prove that, in three space dimensions, ∂F is locally Lipschitz, up to a finite number of points.

Denote by $\pi : \mathbb{R}^n \to \tilde{n}^\perp$ the orthogonal projection onto $\tilde{n}^\perp$. Notice that, since $\tilde{n} \cdot \nu_\phi(F) = 0$, it may happen that $\pi(B_\rho(x) \cap \partial E)$ is not an open neighbourhood of $\pi(x)$.

Choose a hyperplane $P \subset \mathbb{R}^n$ such that $B_\rho(x) \cap \partial E$ is the graph of a Lipschitz map $h : \Omega \subseteq P \to \mathbb{R}$, with Ω an open set. Note that $\nu_\phi^E \in K_\rho$, $\mathcal{H}^{n-1}$-almost everywhere on $B_\rho(x) \cap \partial E$. We split the proof into three intermediate steps.

Step 1. There exists a global Lipschitz graph Σ over P such that

(i) $B_\rho(x) \cap \Sigma = B_\rho(x) \cap \partial E$,
(ii) $\nu_\phi^\Sigma(y) \in K_\rho$ for any $y \in \Sigma$, where ν_ϕ^Σ is the normal vector field to Σ (normalized to have $\phi^o = 1$) which coincides with ν_ϕ^E on $B_\rho(x) \cap \partial E$;
(iii) the map $\pi_{|\Sigma}$ is surjective onto $\tilde{n}^\perp$.

Define

$$\mathcal{G} := \{u \in \mathrm{Lip}(P) : \nu_\phi^{\mathrm{graph}(u)} \in K_\rho, u \geq h \text{ on } \Omega\}.$$

It is immediate to check that $\mathcal{G}$ is non empty. Let

$$h^e := \inf \{u : u \in \mathcal{G}\}.$$

Then $h^e \in \mathcal{G}$. Moreover $h^e = h$ on Ω. This follows from the fact that, being h Lipschitz, for any $z \in \Omega$ there exists a piecewise linear function $g_z \in \mathcal{G}$ such that $g_z(z) = h(z)$.

To obtain property (iii) we need to further modify h^e outside of Ω as follows: define

$$h^*(z) := (w \cdot z - C) \vee h^e(z) \wedge (w \cdot z + C) \qquad \forall z \in P,$$

where $a \vee b := \max(a, b)$, $a \wedge b := \min(a, b)$ for $a, b \in \mathbb{R}$, and w is chosen in such a way that the normal vector to the graph of the map $z \to w \cdot z$ belongs to the relative interior of K_ρ, and $C > 0$ is such that $h^* = h$ on Ω.

Finally, we define $\Sigma := \mathrm{graph}(h^*)$. One can check that Σ satisfies properties (i)-(iii). This concludes the proof of step 1.

By (ii) of Lemma 43, it follows that Σ can be written as a graph over $\tilde{n}^\perp$, possibly with vertical parts. Since Σ has locally finite area, there exists a function $f : \tilde{n}^\perp \to \mathbb{R}\,\tilde{n}$, with $f \in BV_{\mathrm{loc}}(\tilde{n}^\perp)$, such that Σ is the boundary of the subgraph of f. Let $f^\pm(z)$ be the essential upper and lower limits of f at $z \in \tilde{n}^\perp$ and J_f be the jump set of f (see Figure 1). Notice that $F \cap B_\rho(x)$ is contained in the vertical part of the graph of f, i.e. $F \cap B_\rho(x) \subseteq \{z + t\tilde{n} : z \in \pi(F), t \in [f^-(x), f^+(x)]\}$.

Let $f_\epsilon := f * \rho_\epsilon$, where ρ_ϵ is the standard sequence of mollifiers in $\tilde{n}^\perp$.

Step 2. We have

$$|\nabla f_\epsilon(z) \cdot v| \leq \frac{|v|}{c} \qquad \forall z \in \tilde{n}^\perp,\ \forall v \in H_F \cap \tilde{n}^\perp,\ \forall \epsilon > 0. \tag{32}$$

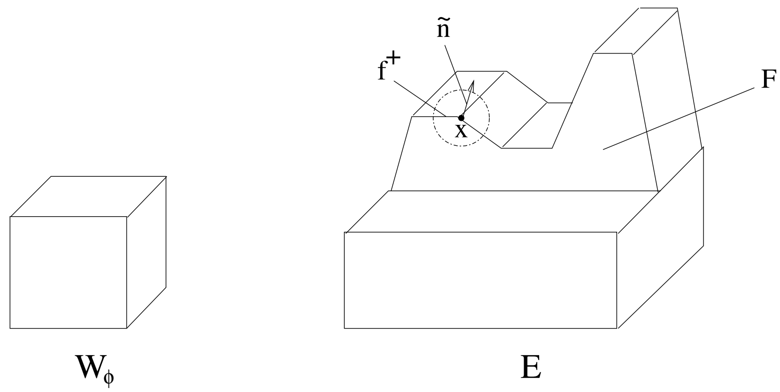

Figure 1. The function f^+ which defines (locally) a facet F

Define $\nu_\phi^\epsilon := \nu_\phi^{\mathrm{subgraph}(f_\epsilon)}$ at the points $(z, f_\epsilon(z))$ for $z \in B_\rho(\pi(x)) \cap \tilde{n}^\perp$. By a direct computation we have

$$\nabla f_\epsilon(z) = \frac{(\nu_\phi^\epsilon(z, f_\epsilon(z)) \cdot \tilde{n})\tilde{n} - \nu_\phi^\epsilon(z, f_\epsilon(z))}{\nu_\phi^\epsilon(z, f_\epsilon(z)) \cdot \tilde{n}} \qquad \forall z \in B_\rho(\pi(x)) \cap \tilde{n}^\perp. \quad (33)$$

Moreover, as $\nu_\phi^\Sigma \in K_\rho$ by step 1, we have $\nu_\phi^\epsilon \in K_\rho$. Using (ii) of Lemma 43 we then obtain

$$|\nu_\phi^\epsilon(z, f_\epsilon(z)) \cdot \tilde{n}|^{-1} \leq \frac{1}{c}|\nu_\phi^\epsilon(z, f_\epsilon(z)) - \nu_\phi(F)|^{-1}. \quad (34)$$

Let now $v \in H_F \cap \tilde{n}^\perp$; using (33), (34) and $v \cdot \tilde{n} = 0$, we get

$$|\nabla f_\epsilon(z, f_\epsilon(z)) \cdot v| \leq \frac{1}{c} \frac{|\nu_\phi^\epsilon(z, f_\epsilon(z)) \cdot v|}{|\nu_\phi^\epsilon(z, f_\epsilon(z)) - \nu_\phi(F)|}. \quad (35)$$

Since $v \in H_F$ we have $|\nu_\phi^\epsilon(z, f_\epsilon(z)) \cdot v| \leq |\nu_\phi^\epsilon(z, f_\epsilon(z)) - \nu_\phi(F)||v|$, which coupled with (35), concludes the proof of step 2.

Step 3. $f^\pm$ are Lipschitz continuous on $B_\rho(\pi(x)) \cap \pi(F)$. More precisely

$$|f^\pm(z_1) - f^\pm(z_2)| \leq \frac{1}{c}|z_1 - z_2| \qquad \forall z_1, z_2 \in B_\rho(\pi(x)) \cap \pi(F).$$

Let us consider the function f^+. Fix $z_1, z_2 \in B_\rho(\pi(x)) \cap \pi(F)$. In view of the properties of BV functions, we can pick two sequences $(z_m^{(i)})$, $i = 1, 2$, of points in $\tilde{n}^\perp \setminus J_f$, such that $z_m^{(i)} \to z_i$, $f(z_m^{(i)}) \to f^+(z_i)$, for $i = 1, 2$, and $z_m^{(1)} - z_m^{(2)} \in \nu_\phi(F)^\perp$ for any m; moreover, we can also assume that $f_\epsilon(z_m^{(i)}) \to f(z_m^{(i)})$, for $i = 1, 2$, as $\epsilon \to 0$.

Then, using the fact that $z_m^{(1)} - z_m^{(2)} \in H_F$ and step 2, we have

$$|f^+(z_1) - f^+(z_2)| = \lim_{m\to\infty} |f(z_m^{(1)}) - f(z_m^{(2)})| = \lim_{m\to\infty} \lim_{\epsilon\to 0} |f_\epsilon(z_m^{(1)}) - f_\epsilon(z_m^{(2)})|$$
$$\leq \lim_{m\to\infty} \lim_{\epsilon\to 0} \int_0^1 |\nabla f_\epsilon((1-t)z_m^{(1)} + t z_m^{(2)}) \cdot (z_m^{(1)} - z_m^{(2)})| \, dt \leq \frac{|z_1 - z_2|}{c}.$$

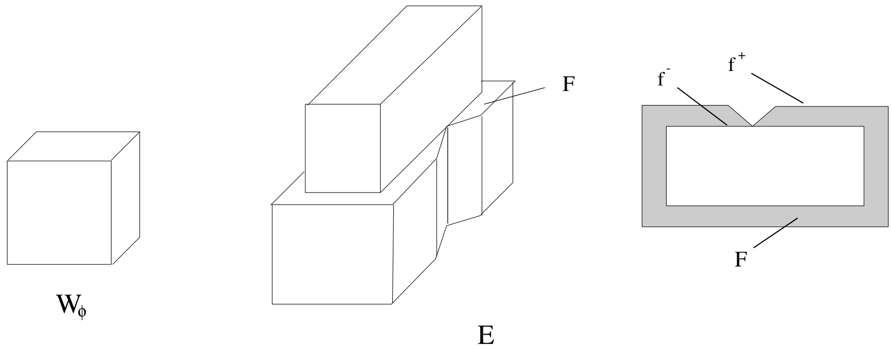

Figure 2. The Wulff shape $\mathcal{W}_\phi$ and a facet F of a Lipschitz ϕ-regular set having a singular point

Since $B_\rho(x) \cap \partial F \subseteq \mathrm{graph}(f^-) \cup \mathrm{graph}(f^+)$ and $f^\pm$ are Lipschitz, by compactness it follows that F is of finite perimeter in $\mathbb{R}^{n-1}$.

We define
$$Z_F := \{y \in \partial F : f^-(\pi(y)) = f^+(\pi(y))\}. \tag{36}$$
Notice that, for any $x \in \partial F \setminus Z_F$, ∂F is the graph of a Lipschitz function (namely f^+ or f^-) in a neighbourhood of x.

Let now $n = 3$. Since F is connected we have that $\pi(F \cap B_\rho(x))$ is an interval, and therefore f^- and f^+ can coincide in at most two points of $\overline{F \cap B_\rho(x)}$. Therefore $\mathcal{H}^0(Z_F \cap B_\rho(x)) \leq 2$, and so Z_F consists of isolated points of ∂F. By compactness, it follows that Z_F is finite.

Notice that, given a facet $F \in \mathrm{Fct}_\phi(\partial E)$ and defining the set Z_F as in (36), we have
$$Z_F \cap \partial^* F = \emptyset.$$
Indeed, Z_F is (locally) the intersection of the two Lipschitz graphs $\mathrm{graph}(f^+)$ and $\mathrm{graph}(f^+)$, and therefore F is contained, in a neighbourhood of any $x \in Z_F$, in a cone $C_x^- \cup C_x^+$ (with vertex at x), identified by the Lipschitz constants of f^-, f^+. This implies that the blow–up of ∂F at x cannot be a hyperplane of H_F, hence $x \notin \partial^* F$.

Definition 45 *Let $F \in \mathrm{Fct}_\phi(\partial E)$, Z_F, $f^\pm$, π be as in Theorem 44, and let $x \in \partial F \setminus Z_F$. If $x = \pi(x) + f^+(\pi(x))$ (resp. $x = \pi(x) + f^-(\pi(x))$) we say that ∂E is weakly convex (resp. weakly concave) at x.*

Notice that if $x \in \partial^* F = \partial^* F \setminus Z_F$, then ∂E is weakly convex at x (resp. weakly concave at x) if and only if $\tilde{\nu}_F^*(x)$ points outside $\overline{E}$ (resp. inside $\overline{E}$).

Corollary 46 *If $F \in \mathrm{Fct}_\phi(\partial E)$ and E is convex or concave at F, then F is Lipschitz.*

Proof. Assume that E is convex (resp. concave) at F. If $x \in \partial F$ then ∂E is weakly convex (resp. weakly concave) at x, which implies $f^+(x) \neq f^-(x)$. Therefore $Z_F = \emptyset$.

We now give an example of a set $E \in \mathcal{R}_\phi(\mathbb{R}^4)$ having a facet $F \in \mathrm{Fct}_\phi(\partial E)$ such that $\mathcal{H}^2(Z_F) > 0$.

Take $\phi(\xi) := \max\{|\xi_1| + |\xi_2| + |\xi_3|, |\xi_4|\}$, where $\xi = (\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4$. Then $\mathcal{W}_\phi = \Sigma \times [-1, 1]$, where $\Sigma \subset \mathbb{R}^3$ is given by $\Sigma = \{\zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{R}^3 : |\zeta_1| + |\zeta_2| + |\zeta_3| \le 1\}$. Take an open set $A \in [0,1] \times [0,1]$ with the following properties: A is connected, $\mathrm{int}(\overline{A}) = A$ and $\mathcal{H}^2(\overline{A}) > \mathcal{H}^2(A)$. For any $x \in \mathbb{R}^2$, let $d_A(x) := \mathrm{dist}(x, \mathbb{R}^2 \setminus A)$ be the euclidean distance function from $\mathbb{R}^2 \setminus A$. Then d_A is 1-Lipschitz, $d_A \ge 0$ and $d_A(x) > 0$ if and only if $x \in A$. Embed now $A \subset \mathbb{R}^2$ into $\mathbb{R}^4$ by identifying $\mathbb{R}^2$ with $\langle e_1, e_2 \rangle \subset \mathbb{R}^4$. Define

$$F := \big\{ x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \ x_4 = 0 \text{ and } 0 \le x_3 \le d_A((x_1, x_2)) \big\}.$$

One can check that ∂F is locally a Lipschitz graph out of a singular set $Z_F := \{x \in F : \ (x_1, x_2, 0, 0) \in \overline{A} \setminus A\}$, where the closure of A is taken in the subspace $\langle e_1, e_2 \rangle \subset \mathbb{R}^4$. Hence $\mathcal{H}^2(Z_F) \ge \mathcal{H}^2(\overline{A}) - \mathcal{H}^2(A) > 0$. Let us now construct a Lipschitz ϕ-regular set (E, n_ϕ) with the property that $F \in \mathrm{Fct}_\phi(\partial E)$. Let $E \subset \mathbb{R}^4$ be defined as

$$E := S \cap \rho \mathcal{W}_\phi,$$

where $\rho > 0$ is a real number sufficiently large and S is defined as

$$S := \big\{ x \in \mathbb{R}^4 : x_4 < 0 \text{ and } x_3 \le 0 \big\} \cup \big\{ x \in \mathbb{R}^4 : x_4 \ge 0 \text{ and } x_3 \le d_A((x_1, x_2)) \big\}.$$

Notice first that $F \subseteq \partial E$, F is connected, and F is a facet of ∂E, with $\nu_\phi(F) = -e_4$. Moreover, $F \in \mathrm{Fct}_\phi(\partial E)$, since $\widetilde{W}^F_\phi = T^o(-e_4) = (\Sigma, -1)$ is a facet of $\partial \mathcal{W}_\phi$. It remains to construct a vector field $n_\phi \in \mathrm{Lip}_{\nu,\phi}(\partial E; \mathbb{R}^4)$. First we choose n_ϕ constantly equal to $e_3 - e_4 \in (\Sigma, 1)$ in a neighbourhood of F in ∂E; using the fact that, out of F, E is a dilation of $\mathcal{W}_\phi$, we can extend n_ϕ on the whole of ∂E in such a way that $n_\phi \in \mathrm{Lip}_{\nu,\phi}(\partial E; \mathbb{R}^4)$. The example is complete.

If F is a facet not corresponding to any facet of $\partial \mathcal{W}_\phi$, less regularity than the one guaranteed by Theorem 44 is expected. In this respect, the worse situation is when F is such that $T^o(\nu_\phi(F))$ is a vertex of $\partial \mathcal{W}_\phi$ (if any): in this case no regularity property of ∂F is expected.

Thanks to Theorem 44, we can give the following definition.

Definition 47 *Let $F \in \mathrm{Fct}_\phi(\partial E)$. For any $x \in \partial^* F$ it is well–defined an exterior euclidean unit normal to $\partial^* F$, lying in H_F, which we will denote by $\tilde{\nu}^*_F(x)$. If $n = 3$, $\tilde{\nu}^*_F$ is defined $\mathcal{H}^1$-almost everywhere on ∂F and coincides with the usual normal vector $\tilde{\nu}^F$.*

We also define the function $c_F \in L^\infty(\partial^* F)$ as

$$c_F(x) := n_\phi(x) \cdot \tilde{\nu}^*_F(x) \qquad \forall x \in \partial^* F. \tag{37}$$

The next result shows that the funtion c_F is independent of the choice of $n_\phi \in \mathrm{Lip}_{\nu,\phi}(\partial E; \mathbb{R}^n)$, but depends only on F and on the geometry of $\mathcal{W}_\phi$.

Lemma 48 *Let* $(E, n_\phi) \in \mathcal{R}_\phi(\mathbb{R}^n)$*,* $F \in \mathrm{Fct}_\phi(\partial E)$ *and* $\eta \in \mathrm{Lip}_{\nu,\phi}(\partial E; \mathbb{R}^n)$*. Then, for any* $x \in \partial^* F$ *we have*

$$\eta(x) \cdot \tilde{\nu}^*_F(x) = c_F(x) = \begin{cases} \max\{p \cdot \tilde{\nu}^*_F(x) : \ p \in \widetilde{W}^F_\phi\} \text{ if } \partial E \text{ is w–convex at } x, \\ \min\{p \cdot \tilde{\nu}^*_F(x) : \ p \in \widetilde{W}^F_\phi\} \text{ if } \partial E \text{ is w–concave at } x. \end{cases} \tag{38}$$

In particular, c_F *is independent of* $n_\phi \in \mathrm{Lip}_{\nu,\phi}(\partial E; \mathbb{R}^n)$*.*

Proof. Fix $x \in \partial^* F = \partial^* F \setminus Z_F$. Assume first that ∂E is weakly convex at x. Notice that there exist a nonzero vector $\overline{\nu}$ in the orthogonal projection of $T(n_\phi(x))$ on F and $\lambda > 0$ such that

$$\tilde{\nu}^*_F(x) = \lambda\, \overline{\nu}. \tag{39}$$

Since $\eta(x) \in \widetilde{W}^F_\phi$, we have

$$\eta(x) \cdot q = \max_{p \in \widetilde{W}^F_\phi} p \cdot q \qquad \forall q \in T(\eta(x)). \tag{40}$$

Write $\overline{\nu} = \nu + \mu \nu_\phi(F)$ for some $\nu \in T(\eta(x))$ and $\mu \in \mathbb{R}$. ¿From (40) we get

$$\eta(x) \cdot \overline{\nu} = \max_{p \in \widetilde{W}^F_\phi} p \cdot \nu + \mu\, \eta(x) \cdot \nu_\phi(F) = \max_{p \in \widetilde{W}^F_\phi} p \cdot \nu + \mu = \max_{p \in \widetilde{W}^F_\phi} p \cdot \overline{\nu}, \tag{41}$$

where the last equality is a consequence of the equality $p \cdot \nu_\phi(F) = 1$, which holds for any $p \in \widetilde{W}^F_\phi$. Then (38) follows from (39).

Assume now that ∂E is weakly concave at x. The proof is the same as in the weakly convex case, by observing now that $\lambda < 0$, and therefore

$$\eta(x) \cdot \tilde{\nu}^*_F(x) = \lambda \max_{p \in \widetilde{W}^F_\phi} p \cdot \overline{\nu} = \min_{p \in \widetilde{W}^F_\phi} p \cdot \lambda \overline{\nu} = \min_{p \in \widetilde{W}^F_\phi} p \cdot \tilde{\nu}^*_F(x).$$

We conclude this section by observing that, under suitable assumptions, a facet F of ∂E is Lipschitz $\tilde{\phi}_y$-regular, where $\tilde{\phi}_y$ is the metric on H_F induced by $\widetilde{W}^F_\phi$. More precisely, let $F \in \mathrm{Fct}_\phi(\partial E)$, fix $y \in \mathrm{int}(\widetilde{W}^F_\phi)$ and let $\widetilde{W}^F_{\phi,y} := \widetilde{W}^F_\phi - y$. The $(n-1)$-dimensional convex body $\widetilde{W}^F_{\phi,y}$ contain the origin of H_F in its interior. Let $\tilde{\phi}_y$ be the convex positively one homogeneous function on H_F such that $\{\tilde{\phi}_y \le 1\} = \widetilde{W}^F_{\phi,y}$. Define also $\mathrm{sym}(\tilde{\phi}_y)$ as the convex positively one homogeneous function on H_F such that $\{\mathrm{sym}(\tilde{\phi}_y) \le 1\} = -\widetilde{W}^F_{\phi,y}$. Notice that the classes of Lipschitz $\tilde{\phi}_y$-regular sets and Lipschitz $\mathrm{sym}(\tilde{\phi}_y)$-regular sets do not depend on the choice of y.

Proposition 49 *Let* $(E, n_\phi) \in \mathcal{R}_\phi(\mathbb{R}^n)$ *and let* $F \in \mathrm{Fct}_\phi(\partial E)$*. If* E *is convex at* F *then* $(F, n_\phi - y)$ *is Lipschitz* $\tilde{\phi}_y$*-regular for any* $y \in \mathrm{int}(\widetilde{W}^F_\phi)$*. If* E *is concave at* F*, then* $(F, y - n_\phi)$ *is Lipschitz* $\mathrm{sym}(\tilde{\phi}_y)$*-regular for any* $y \in \mathrm{int}(\widetilde{W}^F_\phi)$*.*

Proof. Assume that E is convex at F and let $y \in \mathrm{int}(\widetilde{W}_\phi^F)$. ¿From Corollary 46 it follows that ∂F is Lipschitz; moreover $n_\phi - y \in \mathrm{Lip}(\partial F; H_F)$. Therefore we have only to prove that

$$n_\phi(x) - y \in \tilde{T}^o(\tilde{\nu}^F(x)) \qquad \text{for } \mathcal{H}^{n-2} - \text{a.e. } x \in \partial F, \tag{42}$$

where $\tilde{T}^o := \frac{1}{2}D^-((\tilde{\phi}_y^o)^2)$. For any $x \in \partial^* F$ (and therefore for $\mathcal{H}^{n-2}$-almost every $x \in \partial F$) ∂E is weakly convex at F, therefore by Lemma 48 there holds

$$(n_\phi(x) - y) \cdot \tilde{\nu}_F^*(x) = \max\left\{p \cdot \tilde{\nu}_F^*(x) : \; p \in \widetilde{W}_{\phi,y}^F\right\},$$

and relation (42) follows.

Assume now that E is concave at F. We have to prove that

$$y - n_\phi(x) \in \tilde{S}^o(\tilde{\nu}^F(x)) \qquad \text{for } \mathcal{H}^{n-2} - \text{a.e. } x \in \partial F, \tag{43}$$

where $\tilde{S}^o := \frac{1}{2}D^-(((\mathrm{Sym}(\tilde{\phi}_y))^o)^2)$. For any $x \in \partial^* F$ we have that ∂E is weakly convex at x, therefore

$$\begin{aligned}(y - n_\phi(x)) \cdot \tilde{\nu}_F^*(x) &= y - \min\left\{ -p \cdot \tilde{\nu}_F^*(x) : \; p \in -\widetilde{W}_\phi^F\right\} \\ &= \max\left\{p \cdot \tilde{\nu}_F^*(x) : p - y \in -\widetilde{W}_\phi^F\right\},\end{aligned}$$

and the assertion follows.

5. BV-regularity of minimizers on facets

The aim of this section is to prove Theorem 53. We begin with the following useful result proved in [6].

Theorem 51 *Let $\Omega \subset \mathbb{R}^m$ be a bounded open set with Lipschitz boundary. Let u, X satisfy (7) and (8) respectively. Then there is a function $[X \cdot \nu^\Omega] \in L^\infty(\partial\Omega)$ such that $\|[X \cdot \nu^\Omega]\|_{L^\infty(\partial\Omega)} \le \|X\|_{L^\infty(\Omega;\mathbb{R}^m)}$, and*

$$\int_\Omega u \,\mathrm{div} X \, dx + \int_\Omega \theta(X, Du)\, d|Du| = \int_{\partial\Omega} [X \cdot \nu^\Omega] u \, d\mathcal{H}^{m-1}. \tag{44}$$

Proposition 52 *Let $N_{\min}$ be a solution of (1), $F \in \mathrm{Fct}_\phi(\partial E)$, and let Ω_t^F be defined as in (16). Then there exists a constant $C > 0$ such that*

$$P\left(\Omega_t^F, \mathrm{int}(F)\right) \le C \qquad \text{for a.e. } t \in \mathbb{R}. \tag{45}$$

Proof. Fix $y \in \mathrm{int}\left(\widetilde{W}_\phi^F\right)$ and $\rho > 0$ such that $B_\rho(y) \subset \mathrm{int}(\widetilde{W}_\phi^F)$. Fix $t \in \mathbb{R}$ such that Proposition 35 holds; hence for $\mathcal{H}^{n-2}$-almost every $x \in \mathrm{int}(F) \cap J_u$ we have

$$\theta(N_{\min} - y, Du)(x) = -\max\left\{(z - y) \cdot \tilde{\nu}_t^*(x) : z \in \widetilde{W}_\phi^F\right\} \le -c_1\rho < 0,$$

for a suitable constant $c_1 > 0$ depending only on y and $\widetilde{W}_\phi^F$, where we have set $u := 1_{\Omega_t^F}$, $J_u := \partial^* \Omega_t^F$. Therefore

$$c_1\rho \leq \theta(y - N_{\min}, Du)(x) \qquad \text{for } \mathcal{H}^{n-2} - \text{a.e. } x \in \text{int}(F) \cap J_u.$$

Choose a Lipschitz open set $A \Subset \text{int}(F)$. We have

$$P(\Omega_t^F, A) \leq \frac{1}{c_1\rho} \int_{A\cap J_u} \theta(y - N_{\min}, Du)\, d\mathcal{H}^{n-2} =: \frac{1}{c_1\rho} I.$$

Applying Remark 26 on the open set A with the choice $m = n - 1$, $u_1 = u$, $u_2 = 1_{A\cap\Omega_t^F}$, $B = A$ and $X = y - N_{\min}$, we obtain

$$\begin{aligned} I &= (y - N_{\min}, Du)(A) = (y - N_{\min}, D1_{A\cap\Omega_t^F})(A) \\ &= \int_{A\cap\partial^*(A\cap\Omega_t^F)} \theta(y - N_{\min}, D1_{A\cap\Omega_t^F})\, d\mathcal{H}^{n-2}. \end{aligned}$$

Applying (44) with $m = n - 1$, $\Omega = A$, $u = 1_{A\cap\Omega_t^F}$, $X = y - N_{\min}$, we get

$$\begin{aligned} I &= \int_{\partial A} [(y - N_{\min}) \cdot \widetilde{\nu}^A]\, 1_{\Omega_t^F \cap A}\, d\mathcal{H}^{n-2} - \int_{A\cap\Omega_t^F} \text{div}_\tau (y - N_{\min})\, d\mathcal{H}^{n-1} \\ &\leq c_2 \mathcal{H}^{n-2}(\partial A) + \mathcal{H}^{n-1}(F) \left(\|\text{div}_\tau N_{\min}\|_{L^\infty(\text{int}(F))} + c_3 \right), \end{aligned}$$

where c_2 and c_3 depend only on $\mathcal{W}_\phi$.

Therefore

$$P(\Omega_t^F, A) \leq \frac{1}{c_1\rho} \Big(c_2 \mathcal{H}^{n-2}(\partial A) + \mathcal{H}^{n-1}(F) \left(\|\text{div}_\tau N_{\min}\|_{L^\infty(\text{int}(F))} + c_3 \right) \Big). \tag{46}$$

By Theorem 44 we have that F has finite perimeter in $\mathbb{R}^{n-1}$. Therefore, for any $\epsilon > 0$, we can find a Lipschitz open set $A_\epsilon \Subset \text{int}(F)$ such that $\mathcal{H}^{n-1}(\text{int}(F) \setminus A_\epsilon) < \epsilon$ and $|P(A_\epsilon, \text{int}(F)) - P(F, \mathbb{R}^{n-1})| < \epsilon$. Replacing A with A_ϵ in (46) and letting $\epsilon \to 0^+$, we obtain (45).

Theorem 53 *Let $E \in \mathcal{R}_\phi(\mathbb{R}^n)$ and let $F \in \text{Fct}_\phi(\partial E)$. Assume also $g \in L^\infty(\partial E)$. Then*

$$d_{\min} - g \in BV(\text{int}(F)). \tag{47}$$

Proof. Set $V := d_{\min} - g$. ¿From (3) we have $V \in L^\infty(\partial E)$. By the coarea formula and Proposition 52 we then have

$$\begin{aligned} \int_{\text{int}(F)} |DV| &= \int_{-\infty}^{+\infty} P\big(\Omega_t^F, \text{int}(F)\big)\, dt \\ &= \int_{-\|V\|_{L^\infty(\partial E)}}^{\|V\|_{L^\infty(\partial E)}} P\big(\Omega_t^F, \text{int}(F)\big)\, dt \leq 2C\|V\|_{L^\infty(\partial E)}. \end{aligned}$$

6. Further regularity properties of minimizers

Throughout this section $(E, n_\phi) \in \mathcal{R}_\phi(\mathbb{R}^n)$ and $F \in \mathrm{Fct}_\phi(\partial E)$. We always consider $t \in \mathbb{R}$ such that $\Omega_t^F \neq \emptyset$. We often identify Ω_t^F with its projection on the hyperplane parallel to H_F and passing through the origin of $\mathbb{R}^n$.

In order to obtain further regularity properties of the function $d_{\min} - g$ on F, we show that its sublevel sets solve a prescribed anisotropic mean curvature type problem (see Theorem 64).

Fix $y \in \mathrm{int}(\widetilde{W}_\phi^F)$. The following definition yields an $(n-1)$-dimensional notion of ϕ-perimeter for subsets of $\mathrm{int}(F)$.

Definition 61 *Let A be an open subset of H_F. For any Borel set $B \subseteq \mathrm{int}(F)$, we set*

$$\widetilde{P_\phi}(B, A) := \sup \left\{ \int_B \mathrm{div}_\tau \eta \, d\mathcal{H}^{n-1} : \eta \in \mathcal{C}_c^1(A; \mathbb{R}^n), \eta(x) \in \widetilde{W}_{\phi,y}^F \ \forall x \in A \right\}.$$

The above definition does not depend on the choice of $y \in \mathrm{int}(F)$. Notice that $\widetilde{P_\phi}(B, A) \geq 0$ for any B and A; moreover $\widetilde{P_\phi}(\mathrm{int}(F), H_F) < +\infty$ by Theorem 44. When $A = \mathrm{int}(F)$, we simply write $\widetilde{P_\phi}(B)$ instead of $\widetilde{P_\phi}(B, \mathrm{int}(F))$.

Remark 62 *One can check (see [2], Proposition 3.2) that we get an equivalent definition of $\widetilde{P_\phi}(B, A)$ if we let η vary in the set of all vector fields in $L^\infty(H_F; \mathbb{R}^n)$, with compact support in A, having bounded divergence in A and satisfying the constraint $\eta(x) \in \widetilde{W}_{\phi,y}^F$ for $\mathcal{H}^{n-1}$-almost every $x \in A$.*

Lemma 63 *Let $A \Subset \mathrm{int}(F)$ be a Lipschitz open set. Then, for almost every $t \in \mathbb{R}$ we have*

$$\widetilde{P_\phi}(\Omega_t^F, A) = \int_{A \cap \Omega_t^F} d_{\min} \, d\mathcal{H}^{n-1} - \int_{\partial A} [N_{\min} \cdot \widetilde{\nu}^A] 1_{\Omega_t^F} \, d\mathcal{H}^{n-2}. \tag{48}$$

Proof. Let $t \in \mathbb{R}$ be such that Proposition 35 holds. Using Theorem 51 and (18), we have

$$\begin{aligned} &\int_{A \cap \Omega_t^F} d_{\min} \, d\mathcal{H}^{n-1} \\ &= -\int_{A \cap \partial^* \Omega_t^F} \theta(N_{\min}, D1_{\Omega_t^F}) \, d\mathcal{H}^{n-2} + \int_{\partial A} [N_{\min} \cdot \widetilde{\nu}^A] 1_{\Omega_t^F} \, d\mathcal{H}^{n-2} \\ &= \int_{A \cap \partial^* \Omega_t^F} \max_{p \in \widetilde{W}_\phi^F} \{p \cdot \tilde{\nu}_t^*(x)\} \, d\mathcal{H}^{n-2} + \int_{\partial A} [N_{\min} \cdot \widetilde{\nu}^A] 1_{\Omega_t^F} \, d\mathcal{H}^{n-2}. \end{aligned}$$

Using Remark 62 and a commutation argument between supremum and integral (see for instance Lemma 4.3 in [10]) we have

$$\int_{A \cap \partial^* \Omega_t^F} \max_{p \in \widetilde{W}_\phi^F} \{p \cdot \tilde{\nu}_t^*(x)\} \, d\mathcal{H}^{n-2} = \widetilde{P_\phi}(\Omega_t^F, A),$$

and (48) follows.

By $\triangle$ we indicate the symmetric difference between sets.

Theorem 64 *For almost every $t \in \mathbb{R}$ the set Ω_t^F is a solution of the following variational problem:*

$$\inf \Big\{ \widetilde{P_\phi}(B) - \int_B (g+t)\, d\mathcal{H}^{n-1} : B \text{ Borel set } \subseteq \mathrm{int}(F),\ B\triangle\Omega_t^F \Subset \mathrm{int}(F) \Big\}. \tag{49}$$

Proof. Fix $t \in \mathbb{R}$ such that Proposition 35 holds. Let $B \subset \mathrm{int}(F)$ be a Borel set with $B\triangle\Omega_t^F \Subset \mathrm{int}(F)$. We have to show that

$$\widetilde{P_\phi}(\Omega_t^F) - \int_{\Omega_t^F} (g+t)\, d\mathcal{H}^{n-1} \le \widetilde{P_\phi}(B) - \int_B (g+t)\, d\mathcal{H}^{n-1}. \tag{50}$$

By inner approximation of $\mathrm{int}(F)$ with Lipschitz open sets as in Proposition 52, to show (50) it is enough to prove that for any Lipschitz set $A \Subset \mathrm{int}(F)$ such that $B\triangle\Omega_t^F \Subset A$, there holds

$$\widetilde{P_\phi}(\Omega_t^F, A) - \int_{A\cap\Omega_t^F} (g+t)\, d\mathcal{H}^{n-1} \le \widetilde{P_\phi}(B, A) - \int_{A\cap B} (g+t)\, d\mathcal{H}^{n-1}. \tag{51}$$

Fix such a set A. By Lemma 63 we have

$$\begin{aligned} &\widetilde{P_\phi}(\Omega_t^F, A) - \int_{A\cap\Omega_t^F} (g+t)\, d\mathcal{H}^{n-1} \\ &= \int_{A\cap\Omega_t^F} \big(d_{\min} - g - t\big)\, d\mathcal{H}^{n-1} - \int_{\partial A} [N_{\min}\cdot \widetilde{\nu}^A] 1_{\Omega_t^F}\, d\mathcal{H}^{n-2}. \end{aligned}$$

Since $d_{\min} - g - t < 0$ on Ω_t^F, we have

$$\int_{A\cap\Omega_t^F} \big(d_{\min} - g - t\big)\, d\mathcal{H}^{n-1} \le \int_{A\cap B} \big(d_{\min} - g - t\big)\, d\mathcal{H}^{n-1},$$

and since $B\triangle\Omega_t^F \Subset A$ we also have

$$\int_{\partial A} [N_{\min}\cdot \widetilde{\nu}^A] 1_{\Omega_t^F}\, d\mathcal{H}^{n-2} = \int_{\partial A} [N_{\min}\cdot \widetilde{\nu}^A] 1_B\, d\mathcal{H}^{n-2}.$$

Therefore

$$\begin{aligned} &\widetilde{P_\phi}(\Omega_t^F, A) - \int_{A\cap\Omega_t^F} (g+t)\, d\mathcal{H}^{n-1} \\ &\le \int_{A\cap B} \big(d_{\min} - g - t\big)\, d\mathcal{H}^{n-1} - \int_{\partial A} [N_{\min}\cdot \widetilde{\nu}^A] 1_B\, d\mathcal{H}^{n-2}. \end{aligned}$$

Using the definition of $\widetilde{P_\phi}(B, A)$, Remark 62 and an approximation argument we get

$$\widetilde{P_\phi}(B, A) \ge \int_{A\cap B} d_{\min}\, d\mathcal{H}^{n-1} - \int_{\partial A} [N_{\min}\cdot \widetilde{\nu}^A] 1_B\, d\mathcal{H}^{n-2}.$$

It follows that

$$\widetilde{P_\phi}(\Omega_t^F, A) - \int_{A\cap\Omega_t^F} (g+t)\, d\mathcal{H}^{n-1} \leq \widetilde{P_\phi}(B, A) - \int_{A\cap B} (g+t)\, d\mathcal{H}^{n-1},$$

and the proof of the theorem is concluded.

Note that, if we replace the weak inequality with the strict inequality in the definition of Ω_t^F, the assertion of Theorem 64 still holds.

We now list some regularity results on $d_{\min}$, which are consequences of Theorem 64 and the results in [11], [12], [13], [4]. Point (iv) of the next corollary, in the special case ϕ crystalline and E polyhedral, has been independently obtained by Yunger in [14].

Corollary 65 *The following properties hold.*

(i) For any $t \in \mathbb{R}$ the set Ω_t^F has finite perimeter in $\mathrm{int}(F)$, is a solution of the variational problem (49), and $\mathcal{H}^{n-2}\left(\partial\Omega_t^F \setminus \partial^*\Omega_t^F\right) = 0$.

(ii) Assume $g = 0$. Let $t \leq 0$. Then $\overline{\Omega}_t^F \cap \partial A \neq \emptyset$ for any open set $A \subseteq \mathrm{int}(F)$ such that $\Omega_t^F \cap A \neq \emptyset$.

(iii) Assume $n = 3$ and let $t \in \mathbb{R}$. Then $\partial\Omega_t^F$ is a Lipschitz graph in a neighbourhood of any $x \in (\mathrm{int}(F) \cap \partial\Omega_t^F) \setminus \Sigma$, where Σ is a closed subset of $\partial\Omega_t^F$ such that $\mathcal{H}^1(\Sigma) = 0$. Moreover, if $\widetilde{W}_\phi^F$ is neither a triangle nor a quadrilateral, or if there exists a constant $c > 0$ such that either $g \geq c$ or $g \leq -c$ $\mathcal{H}^2$-almost everywhere on $\mathrm{int}(F)$, then $\Sigma = \emptyset$.

(iv) Assume $n = 3$ and $g = 0$. Then, for any $t \neq 0$, every connected component of $\mathrm{int}(F) \cap \partial\Omega_t^F$ is contained, up to a translation, in $\frac{1}{t}\partial\widetilde{W}_\phi^F$.

(v) Assume $n = 3$ and $g = 0$. Assume also that $\widetilde{W}_\phi^F$ is strictly convex. Then κ_ϕ is continuous on $\mathrm{int}(F)$.

Proof. Let $t \in \mathbb{R}$. Write $\Omega_t^F = \bigcup_{\lambda\in I_t} \Omega_\lambda^F$, where I_t is the set of all real numbers $\lambda < t$ such that Ω_λ^F is a solution of (49). By a compactness property for minimizers for functionals of the form (49) (see [1, Section 3]) we obtain that Ω_t^F is also a solution of (49). Assertion (i) then follows using again the arguments in [1].

Let us prove (ii). Suppose by contradiction $\emptyset \neq \overline{\Omega_t^F \cap A} \Subset A$ for some open set $A \subseteq \mathrm{int}(F)$ and some $t \leq 0$. Let $\Omega' := \Omega_t^F \setminus A$ (note that Ω' could be empty). Since $\widetilde{P_\phi}(\Omega') < \widetilde{P_\phi}(\Omega_t^F)$ and $\mathcal{H}^{n-1}(\Omega') < \mathcal{H}^{n-1}(\Omega_t^F)$, as $t \leq 0$ we have $\widetilde{P_\phi}(\Omega') - t\mathcal{H}^{n-1}(\Omega') < \widetilde{P_\phi}(\Omega_t^F) - t\mathcal{H}^{n-1}(\Omega_t^F)$, which contradicts Theorem 64.

Assertions (iii) and (iv) follow from Theorem 64 and the results in [11], [12], [13]. It remains to prove (v). It is enough to check that if $t_1 < t_2$ are such that $\Omega_{t_1}^F \neq \emptyset$, $\Omega_{t_2}^F \neq \emptyset$, then $\partial\Omega_{t_1}^F \cap \partial\Omega_{t_2}^F \cap \mathrm{int}(F) = \emptyset$. Assume by contradiction that there exists $x \in \partial\Omega_{t_1}^F \cap \partial\Omega_{t_2}^F \cap \mathrm{int}(F)$. We can reduce to the case $t_1 \neq 0$ and $t_2 \neq 0$; indeed if for instance $t_2 = 0$, then any $t_3 \in]t_1, 0[$ is such that $\partial\Omega_{t_1}^F \cap \partial\Omega_{t_3}^F \cap \mathrm{int}(F) \ni x$. ¿From assertion (iv), we have that $\partial\Omega_{t_i}^F$, for $i = 1, 2$, are, around x, contained in $\frac{1}{t_i}\partial\widetilde{W}_\phi^F$. Therefore, around x, $\partial\Omega_{t_i}^F$ has $\tilde{\phi}_y$-curvature equal to t_i. This is a contradiction, since $t_1 < t_2$, $\Omega_{t_1}^F \subseteq \Omega_{t_2}^F$, and $\widetilde{W}_\phi^F$ is strictly convex.

The generalization of (iii)-(iv) of Corollary 65 to arbitrary dimensions remains an open problem, related to the general problem of regularity of area minimizers in crystalline geometry. We remark that assertion (v) does not hold in general in the crystalline case (see [7]).

We want now to prove assertion (ii) of Corollary 65 when $t > 0$ in $n = 3$ dimensions. In order to do that, we need a comparison-type result (Proposition 66). We begin with a technical observation.

¿From (v) of Theorem 3.8 in [6], if R and S are two sets of finite perimeter in Ω, and if C is a Borel set contained in $\partial^* R \cap \partial^* S$ and such that $\frac{D1_R}{|D1_R|} = \frac{D1_S}{|D1_S|}$ $\mathcal{H}^{m-1}$-almost everywhere on C, it follows that $\theta(N, D1_R) = \theta(N, D1_S)$ $\mathcal{H}^{m-1}$-almost everywhere on C, for any N satisfying (8). In particular, if $S \subseteq R$ and $C = \partial^* R \cap \partial^* S$

$$\theta(N, D1_R) = \theta(N, D1_S) \qquad \mathcal{H}^{m-1} - \text{a.e. on } \partial^* R \cap \partial^* S. \tag{52}$$

Recall also that, if B has finite perimeter, then $B \cap \partial^* B = \emptyset$.

Proposition 66 *Let $\Omega \subset \mathbb{R}^m$ be a bounded open set and let $K \subset \mathbb{R}^m$ be a compact set. Let*

$$N_1, N_2 \in L^\infty(\Omega; K), \qquad \mathrm{div} N_1, \mathrm{div} N_2 \in L^\infty(\Omega) \cap BV(\Omega).$$

Define

$$\Omega_t^i := \{x \in \Omega : \ \mathrm{div} N_i(x) < t\} \qquad \forall t \in \mathbb{R},$$

and for almost every $t \in \mathbb{R}$ let $\nu_t^i := -\frac{D1_{\Omega_t^i}}{|D1_{\Omega_t^i}|}$. Suppose that for almost every $t \in \mathbb{R}$ there holds

$$-\theta(N_i, D1_{\Omega_t^i})(x) = \max\{p \cdot \nu_t^i(x) : \ p \in K\} \quad \text{for } \mathcal{H}^{m-1}\text{– a.e. } x \in \Omega \cap \partial^* \Omega_t^i. \tag{53}$$

Let B be such that either B has finite perimeter and $B \Subset \Omega$, or $B \subseteq \Omega$ has Lipschitz boundary. In the latter case we set $\theta(N_i, D1_B) := -[N_i \cdot \nu^B]$, $i = 1, 2$. If

$$-\theta(N_1, D1_B) \geq -\theta(N_2, D1_B) \qquad \mathcal{H}^{m-1} - \text{a.e. on } \partial^* B, \tag{54}$$

then

$$\mathrm{div} N_1 \geq \mathrm{div} N_2 \qquad \mathcal{H}^m - \text{a.e. on } B.$$

Proof. Set $V_i := \mathrm{div} N_i$, for $i \in \{1, 2\}$. Assume by contradiction that there exists $\lambda \in \mathbb{R}$ such that $\mathcal{H}^m(\Omega_\lambda) > 0$, where $\Omega_\lambda := (\Omega_\lambda^1 \setminus \Omega_\lambda^2) \cap B$. Since $V_1, V_2 \in BV(\Omega)$, we can also assume that Ω_λ has finite perimeter. Clearly $1_{\Omega_\lambda} = 1_B \cdot 1_{\Omega_\lambda^1} \cdot 1_{\Omega \setminus \Omega_\lambda^2}$. Notice that if R and S are two sets of finite perimeter we have, up to sets of zero $\mathcal{H}^{m-1}$-measure,

$$\partial^*(R \cap S) = (\partial^* R \cap S) \cup (R \cap \partial^* S) \cup \big(\partial^*(R \cap S) \cap \partial^* R \cap \partial^* S\big), \tag{55}$$

where the three sets at the right hand side between parentheses are mutually disjoint.

We now split $\partial^*\Omega_\lambda = (B \cap \partial^*\Omega_\lambda) \cup (\partial^* B \cap \partial^*\Omega_\lambda)$, and using (55) with $R = \Omega^1_\lambda \setminus \Omega^2_\lambda$ and $S = B$ we write, always up to sets of zero $\mathcal{H}^{m-1}$-measure,

$$B\cap\partial^*\Omega_\lambda = (\partial^*\Omega^1_\lambda \cap B \cap (\Omega\setminus\Omega^2_\lambda)) \cup (\partial^*\Omega^2_\lambda \cap B \cap \Omega^1_\lambda) \cup (B\cap\partial^*\Omega^1_\lambda \cap \partial^*\Omega^2_\lambda \cap \partial^*\Omega_\lambda).$$

Hence we have, using (52),

$$\begin{aligned}\int_{\partial^*\Omega_\lambda} \theta(N_1, D1_{\Omega_\lambda})\, d\mathcal{H}^{m-1} &= \int_{\partial^*\Omega^1_\lambda\cap B\cap(\Omega\setminus\Omega^2_\lambda)} \theta(N_1, D1_{\Omega^1_\lambda})\, d\mathcal{H}^{m-1} \\ &- \int_{\partial^*\Omega^2_\lambda\cap B\cap\Omega^1_\lambda} \theta(N_1, D1_{\Omega^2_\lambda})\, d\mathcal{H}^{m-1} \\ &+ \int_{B\cap\partial^*\Omega^1_\lambda\cap\partial^*\Omega^2_\lambda\cap\partial^*\Omega_\lambda} \theta(N_1, D1_{\Omega^1_\lambda})\, d\mathcal{H}^{m-1} \\ &+ \int_{\partial^* B\cap\partial^*\Omega_\lambda} \theta(N_1, D1_B)\, d\mathcal{H}^{m-1}.\end{aligned}$$

By (53) and (54) we have

$$\int_{\partial^*\Omega^1_\lambda\cap B\cap(\Omega\setminus\Omega^2_\lambda)} \theta(N_1, D1_{\Omega^1_\lambda})\, d\mathcal{H}^{m-1} \le \int_{\partial^*\Omega^1_\lambda\cap B\cap(\Omega\setminus\Omega^2_\lambda)} \theta(N_2, D1_{\Omega^1_\lambda})\, d\mathcal{H}^{m-1},$$

$$-\int_{\partial^*\Omega^2_\lambda\cap B\cap\Omega^1_\lambda} \theta(N_1, D1_{\Omega^2_\lambda})\, d\mathcal{H}^{m-1} \le -\int_{\partial^*\Omega^2_\lambda\cap B\cap\Omega^1_\lambda} \theta(N_2, D1_{\Omega^2_\lambda})\, d\mathcal{H}^{m-1}$$

$$\int_{\partial^* B\cap\partial^*\Omega_\lambda} \theta(N_1, D1_B)\, d\mathcal{H}^{m-1} \le \int_{\partial^* B\cap\partial^*\Omega_\lambda} \theta(N_2, D1_B)\, d\mathcal{H}^{m-1}.$$

In addition, using (53) we also have

$$\int_{B\cap\partial^*\Omega^1_\lambda\cap\partial^*\Omega^2_\lambda\cap\partial^*\Omega_\lambda} \theta(N_1, D1_{\Omega^1_\lambda})\, d\mathcal{H}^{m-1} = \int_{B\cap\partial^*\Omega^1_\lambda\cap\partial^*\Omega^2_\lambda\cap\partial^*\Omega_\lambda} \theta(N_2, D1_{\Omega^2_\lambda})\, d\mathcal{H}^{m-1}.$$

Then

$$\begin{aligned}&-\lambda\mathcal{H}^m(\Omega_\lambda) < -\int_{\Omega_\lambda} V_1\, d\mathcal{H}^m = \int_{\partial^*\Omega_\lambda} \theta(N_1, D1_{\Omega_\lambda})\, d\mathcal{H}^{m-1} \\ &\le \int_{\partial^*\Omega^1_\lambda\cap B\cap(\Omega\setminus\Omega^2_\lambda)} \theta(N_2, D1_{\Omega^1_\lambda})\, d\mathcal{H}^{m-1} - \int_{\partial^*\Omega^2_\lambda\cap B\cap\Omega^1_\lambda} \theta(N_2, D1_{\Omega^2_\lambda})\, d\mathcal{H}^{m-1} \\ &+ \int_{B\cap\partial^*\Omega^1_\lambda\cap\partial^*\Omega^2_\lambda\cap\partial^*\Omega_\lambda} \theta(N_2, D1_{\Omega^2_\lambda})\, d\mathcal{H}^{m-1} + \int_{\partial^* B\cap\partial^*\Omega_\lambda} \theta(N_2, D1_B)\, d\mathcal{H}^{m-1} \\ &= \int_{\partial^*\Omega_\lambda} \theta(N_2, D1_{\Omega_\lambda})\, d\mathcal{H}^{m-1} = -\int_{\Omega_\lambda} V_2\, d\mathcal{H}^m \le -\lambda\mathcal{H}^m(\Omega_\lambda),\end{aligned}$$

which gives a contradiction.

The following result completes assertion (ii) of Corollary 65 in $n = 3$ dimensions.

Corollary 67 *Assume $g = 0$ and $n = 3$. Let $t > 0$. Then $\overline{\Omega}^F_t \cap \partial A \neq \emptyset$ for any open set $A \subseteq \mathrm{int}(F)$ such that $\Omega^F_t \cap A \neq \emptyset$.*

Proof. Suppose by contradiction $\emptyset \neq \overline{\Omega_t^F \cap A} \Subset A$ for some open set $A \subseteq \mathrm{int}(F)$ and some $t > 0$. Let $\Omega_A := \Omega_t^F \cap A$. Observe that the connected components [5] of Ω_A are simply connected (filling the holes decreases the functional in (49) when $t > 0$); moreover, using (iv) of Corollary 65, it follows that Ω_A consists of a finite number of connected components with pairwise disjoint closure, each of which coincides, up to a translation, with $\frac{1}{t}\widetilde{W}_\phi^F$. Let C be one of these connected components and let $y \in \mathrm{int}(C)$. Let $\sigma > 1$ be such that $\Omega' := y + \sigma(C - y) \Subset \mathrm{int}(F) \setminus (\overline{\Omega_t^F \setminus C})$. For any $z \in \Omega'$ define $N'(z) := N_{\min}(y + \frac{1}{\sigma}(z - y))$. Two cases are possible.

Case 1. There exists a Borel set $L \subseteq C$ with $\mathcal{H}^2(L) > 0$ and $\mathrm{div}_\tau N' > 0$ on L. As $\sigma > 1$ we have

$$\mathrm{div}_\tau N' < \kappa_\phi \qquad \mathcal{H}^2 - \text{a.e. on } L. \tag{56}$$

By (18) applied to N' we have $-\theta(N', D1_{\Omega'}) = \max\{z \cdot \widetilde{\nu}^*_{\Omega'} : z \in \widetilde{W}_\phi^F\} \geq -\theta(N_{\min}, D1_{\Omega'})$. Recalling also (18) and Theorem 53, we can apply Proposition 66 with $\Omega = \Omega' = B$, $K = \widetilde{W}_\phi^F$, $N_1 = N'$, $N_2 = N_{\min}$. It follows $\mathrm{div}_\tau N' \geq \kappa_\phi$ $\mathcal{H}^2$-almost everywhere on Ω', which contradicts (56), since $L \subseteq \Omega'$.

Case 2. $\Omega_A \subseteq \{x : \mathrm{div}_\tau N_{\min}(x) \leq 0\}$. Writing $\Omega_A = A \cap \left(\bigcap_{\mu > 0} \Omega_\mu^F\right)$ and reasoning as in Corollary 65 (i), we get that Ω_A minimize $\widetilde{P_\phi}$ among all compact subsets of A with finite perimeter. Therefore $\Omega_A = \emptyset$, which is a contradiction.

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