The level set method for systems of PDEs

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Abstract

We propose a level set method for systems of PDEs which is consistent with the previous research pursued by Evans in [12] for the heat equation and by Giga and Sato in [21] for Hamilton-Jacobi equations. Our approach follows a geometric construction related to the notion of barriers introduced by De Giorgi. The main idea is to force a comparison principle between manifolds of different codimension and require each sublevel of a solution of the level set equation to be a barrier for the graph of a solution of the corresponding system. We apply the method to a class of systems of first order quasilinear equations. We compute the level set equation associated with suitable first order systems of conservation laws, with the mean curvature flow of a manifold of arbitrary codimension and with systems of reaction-diffusion equations. Finally, we provide a level set equation associated with the parametric curvature flow of planar curves.

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1 Introduction

Let $m, n \geq 1$ and consider a smooth function $\mathbf{u} : [0,T] \times \mathbb{R}^n \to \mathbb{R}^m$ which solves the system of partial differential equations

$$\mathbf{u}_t + \mathbf{F}(t, x, \mathbf{u}, \nabla \mathbf{u}, \nabla^2 \mathbf{u}) = 0. \tag{1.1}$$

The graph $\Gamma_{\mathbf{u}(t,\cdot)} = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m : y = \mathbf{u}(t,x)\}$ of $\mathbf{u}(t,\cdot)$ is a smooth m-codimensional manifold embedded in $\mathbb{R}^{n+m}$ which smoothly evolves in time. System (1.1) can be expressed as a geometric evolution law for the graph of $\mathbf{u}$ in the form

$$V^{\perp}(x, \mathbf{u}(t,x)) = \Psi \tag{1.2}$$

where $V^{\perp}$ denotes the normal velocity, and Ψ is an appropriate function of t, x, the tangent space to $\Gamma_{\mathbf{u}(t,\cdot)}$ and the second fundamental form of $\Gamma_{\mathbf{u}(t,\cdot)}$ at $(x, \mathbf{u}(t,x))$. A first question is to ask whether it is possible to look at $\Gamma_{\mathbf{u}(t,\cdot)}$ as the zero-level set $\{(x,y) \in \mathbb{R}^{n+m} : w(t,x,y) = 0\} =: \Sigma_{w(t,\cdot)}$ of a

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scalar function $w(t,\cdot) : \mathbb{R}^{n+m} \to \mathbb{R}$ solving an appropriate partial differential equation (called level set equation) of the form

$$w_t(t,x,y) + \widehat{F}(t,x,y,\nabla w(t,x,y),\nabla^2 w(t,x,y)) = 0. \tag{1.3}$$

It is then natural to try to relate $\Sigma_{w(t,\cdot)}$ with the graph of an $\mathbb{R}^m$-valued function of n variables, which could be interpreted as a solution of the original system (1.1). Note that if $\widehat{F}$ is geometric (Proposition 3.6), then each level set of w evolves in time with a law depending only on the geometry of that set, so that its evolution is unaffected by the neighboring level sets of w. There are several problems that arise in implementing such a program, some of which have been already considered in the literature. Even if $m = 1$, the level set equation can be strongly degenerate so that the available theories of weak solutions (viscosity solutions, for instance) cannot be directly applied. Furthermore, the zero-level set of w may develop *overturning* in finite time and cannot be written as the graph of a real-valued function u on $\mathbb{R}^n$. This is the case of Burgers' equation ($n = 1$)

$$u_t + u\nabla u = 0 \qquad\qquad u : [0,T] \times \mathbb{R} \to \mathbb{R}. \tag{1.4}$$

The corresponding level set equation is

$$w_t + y w_x = 0 \qquad\qquad w : [0,T] \times \mathbb{R}^2 \to \mathbb{R}, \tag{1.5}$$

and it is well known that $\Sigma_{w(t,\cdot)}$ is not, in general, a graph [15]. Indeed, the most that one can hope is that the graph of any reasonable solution to (1.1) is contained in $\Sigma_{w(t,\cdot)}$. A further difficulty (and this is our main motivation here) is that if $m > 1$, $\Gamma_{\mathbf{u}(t,\cdot)}$ has codimension *higher* than one. In spite of these problems, there are reasons for which this program is of some interest: for instance, it may help to understand for which degenerate equations a certain theory of weak solutions applies. In addition, finding a solution of graph-type inside $\Sigma_{w(t,\cdot)}$ could be related to selection principles for problems of the form (1.1) which do not have uniqueness. In these directions two main examples have already been considered. In [12] Evans studied the case $m = 1$ and $F(X) = -\mathrm{tr}(X)$, X an $(n \times n)$ matrix, for which (1.1) is the classical heat equation. In this case the author computed explicitly equation (1.3), and the corresponding geometric evolution of $\Sigma_{w(t,\cdot)}$ which, in the case $n = 1$, reads as

$$V^\perp = \frac{\kappa}{(\nu_2)^2}, \qquad \nu := (\nu_1,\nu_2) = \frac{\nabla w}{|\nabla w|},$$

κ the curvature of $\Sigma_{w(t,\cdot)}$. The function $\widehat{F}$ has a singularity of the type $(w_y)^{-2}$ near $\{w_y = 0\}$ where w_y stands for the vertical part of the gradient of w. Such a degeneration of $\widehat{F}$ prevents, as far as we know, a direct application of the theory of viscosity solutions, and a special regularization of the equation is needed. In the case $n = 1$ the author proved that the level set of solutions of the approximating equations suitably unfold multivalued initial data to become graphs and any limit of approximating solutions can be viewed as solution of the heat equation.

The other example in which the program has been carried out is for $m = 1$ and for first order equations. In particular, Giga and Sato in [21] considered Hamilton-Jacobi equations, for which (1.3) takes the form

$$w_t(t,x,y) - w_y F\left(t,x,y,-\frac{w_x(t,x,y)}{w_y(t,x,y)}\right) = 0,$$

hence $\widehat{F}$ is singular at $w_y = 0$. Under a suitable monotonicity assumption on F and on the initial datum, the authors proved that it is possible to remove the singularity of $\widehat{F}$ and, using viscosity theory, to obtain that $\Sigma_{w(t,\cdot)}$ is the graph of a unique function u (called L-solution) which is consistent with other notions of solution considered in the literature.

In [17] Giga introduced the notion of proper viscosity solution for a class of equations (including scalar conservation laws) the solutions of which may develop discontinuities in finite time, and proved several comparison principles. In general, the graph of a proper viscosity solution does not represent the zero-level set of a solution of the associated level set equation since the overturning phenomenon may occur. In [18] Giga showed that ($m = 1$) the graph of a proper viscosity solution can be obtained through the level set approach developed in [21] by adding the vertical singular diffusion term

$$w_t - w_y F\left(t, x, y, -\frac{w_x}{w_y}\right) = C|\nabla w| \left(\frac{w_y}{|w_y|}\right)_y . \tag{1.6}$$

The author showed that there is a threshold of the value of $C > 0$ that prevents overturning of the zero-level set. In the case of Burgers' equation the solution of (1.6) does not depend on C for C large enough, and its zero-level set is the graph of the proper viscosity solution, which is also the unique entropy solution. Giga also extended the notion of proper viscosity solutions to a class of second order problems and modified the level set method to have coincidence between the level set of solutions and the graph of proper solutions.
Independently, Ambrosio and Soner [1], following a suggestion of De Giorgi [11], described the mean curvature flow of a manifold of arbitrary codimension as the zero-level set of the viscosity solution of a scalar equation whose level set $\Sigma_{w(t,\cdot)}$ evolves with normal velocity equal to the sum of the n smallest principal curvatures.

What we describe here is a temptative extension to systems of the viscosity theory for solutions of partial differential equations. The level set method we propose follows a geometric approach based on the idea of barriers introduced by De Giorgi in [11]. More precisely, to find the expression of the level set equation for w, we force a comparison principle between evolving manifolds of different codimension and require each sub-level set of w to be a *barrier* for the solutions of the corresponding system. The velocity w_t shall be equal to the minimum of the projections on ∇w of the velocities $(0, \mathbf{u}_t)$ of all smooth graphs $(x, \mathbf{u}(t, x))$ evolving according to (1.1), which are locally tangent *from one side* to the level set. We remark that, when $m = 1$, the level set equation of w is such that the geometric evolution law (1.2) of $\Gamma_{u(t,\cdot)}$ coincides with the one of $\Sigma_{w(t,\cdot)}$, provided that (1.2) verifies the comparison principle. We also recall that the idea of a comparison principle between evolving manifolds of different codimension appeared, for instance, in [4].
One of the advantages of this approach relies on the fact that, in general, equations may be easier to solve and study than systems, both theoretically and numerically. On the other hand, there are serious limitations which do not arise in the usual viscosity theory. For instance, even if system (1.1) is well-posed and the function $\mathbf{F}$ is smooth, the corresponding level set equation can be strongly undefined, i.e. $\widehat{F}$ can assume the value $+\infty$ in large regions. However, there are few cases of first order systems where (1.3) is well defined and the usual theory can be applied (see Section 4).
The paper is organized as follows. In Section 2 we fix the notation and recall some preliminaries of differential geometry. In Section 3 we present the general procedure to devise the level set equation (3.3) associated with system (1.1). We illustrate the geometric idea underlying this approach and we list the main properties of $\widehat{F}$. In Example 3.4 we compare the method with some results known in the literature, when (1.1) is a scalar equation, in particular for the heat equation [12] and the Hamilton-Jacobi equation [21]. In Section 4 we compute the level set equation associated with a system of first order PDEs and we prove the consistency of the method under suitable assumption on $\widehat{F}$ (Proposition 4.2). In Section 4.1 we show that in the case of quasi-linear first order systems (1.1) where $\mathbf{F}$ is of the form

$$\mathbf{F}(t, x, y, P) = P\,\mathbf{f}(t, x, y) - \mathbf{g}(t, x, y) \in \mathbb{R}^m, \qquad (t, x, y, P) \in [0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^m \times \mathrm{M}^{m\times n},$$

with $\mathbf{f} : [0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $\mathbf{g} : [0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$, the level set equation (3.3) is a transport equation, i.e. $w_t + \mathbf{f}(t, x, y) \cdot w_x + \mathbf{g}(t, x, y) \cdot w_y = 0$ and admits a unique viscosity solution

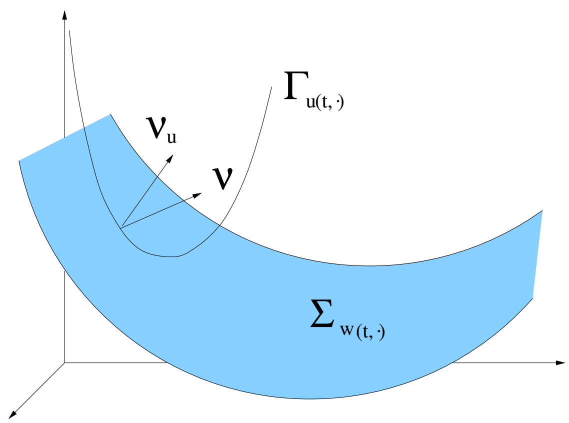

Figure 1: $n = 1$, $m = 2$.

provided that the initial data are sufficiently smooth. We also investigate systems of quasilinear equations of the form

$$\mathbf{F}(t,x,y,P) = \sum_{j=1}^{n} B_j(t,x,y)Pe_j - \mathbf{g}(t,x,y) \in \mathbb{R}^m, \qquad (t,x,y,P) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathrm{M}^{m\times n},$$

with $B_j : [0,+\infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathrm{M}^{m\times m}$, $\mathbf{g} : [0,+\infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ and $\{e_1, ..., e_n\}$ the canonical basis of $\mathbb{R}^n$. The associated level set equation (3.3) assumes the value $+\infty$ in large regions; for instance in Example 4.8 concerning the wave equation, $\widehat{F}(q)$, $q \in \mathbb{R} \times (\mathbb{R}^2 \setminus \{0\})$, is finite only in two hyperplanes of $\mathbb{R} \times (\mathbb{R}^2 \setminus \{0\})$.
In Section 5.1 we show that the method is consistent with the mean curvature flow of a graph in arbitrary codimension. Systems of reaction-diffusion equations

$$\mathbf{u}_t - \Delta\mathbf{u} + \mathbf{f}(u) = 0, \qquad (t,x) \in [0,+\infty) \times \mathbb{R}^n$$

are considered in Section 5.3. In particular, when $\mathbf{f}$ is Lipschitz, classical sub- and supersolutions for the associated level set equation are detected. Finally, we deal with 1-dimensional ($n = 1$) systems of quasi-linear equations in Section 5.2 and we produce a level set equation associated with the *parametric* curvature flow of a planar curve.

2 Notation

In the following m and n denote positive integers, $\cdot$ is the Euclidean canonical inner product in $\mathbb{R}^d$, and $\mathcal{H}^k$ the k-dimensional Hausdorff measure. Vectors will be always denoted by small letters, matrices by capital letters, and higher order tensor sometimes by greek capital letters. We set $\mathbb{R}^d_0 := \mathbb{R}^d \setminus \{0\}$ and we denote by $\{e_1, ..., e_n\}$ the canonical basis of $\mathbb{R}^n$. Given two vectors $a \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ we denote by $a \otimes b$ the $(n \times m)$ matrix $(a \otimes b)_{ij} = a_i b_j$; in particular, when $n = m$, $a \otimes b$ is the tensor product of a and b.
We denote by $\mathrm{M}^{m\times n}$ (resp. Sym^d) the vector space of real $(m \times n)$ matrices (resp. $(d \times d)$ symmetric matrices) equipped with the norm $|M| := \sqrt{\sum_{ij}(M_{ij})^2}$. By $A \succeq B$ (resp. $A \succ B$), $A, B \in \mathrm{Sym}^d$, we mean that the quadratic form corresponding to $A - B$ is nonnegative definite (resp. positive definite). Given $M \in \mathrm{M}^{m\times n}$ (resp. $q \in \mathbb{R}^d$) we denote by M^* (resp. q^*) the transposed of M (resp. of q).

We denote by $\mathcal{T}^{n\times n\times m}$ the space of tensors $\mathcal{X} = (X^1,\dots,X^m)$, $X^i \in \mathrm{Sym}^n$, $i = 1,\dots,m$. Given $\mathcal{X} \in \mathcal{T}^{n\times n\times m}$, by $\mathrm{tr}(\mathcal{X})$ we mean $(\mathrm{tr}(X^1),\dots,\mathrm{tr}(X^m)) \in \mathbb{R}^m$; in particular, when $n = 1$, $\mathrm{tr}(\mathcal{X}) = \mathcal{X}$. Given $q \in \mathbb{R}^m$, by $q\mathcal{X}$ we mean $\sum_{i=1}^m q_i X^i \in \mathrm{Sym}^n$. Finally, we sometimes denote by $\mathrm{O}_{m\times n}$ (resp. O_d) the $(m \times n)$ (resp. $(d \times d)$) zero matrix, and by Id_d the $(d \times d)$ identity matrix. For $q \in \mathbb{R}^d_0$ we let $\Pi_{q^\perp} := \mathrm{Id}_d - q \otimes q/|q|^2$ be the orthogonal projection on the hyperplane $q^\perp$ orthogonal to q. If $d = 1$ then $\Pi_{q^\perp} = 0$.

We identify $\mathrm{M}^{1\times n}$ with $\mathbb{R}^n$; consequently $P \in \mathrm{M}^{1\times n}$ will be also denoted by p. Similarly we identify $\mathcal{T}^{n\times n\times 1}$ (resp. $\mathcal{T}^{1\times 1\times m}$) with Sym^n (resp. $\mathbb{R}^m$). When we need to emphasize the splitting $\mathbb{R}^{n+m} = \mathbb{R}^n \oplus \mathbb{R}^m$, a vector $q \in \mathbb{R}^{n+m}$ can be indicated as $q = (q_\mathrm{h}, q_\mathrm{v})$, q_h the "horizontal" part and q_v the "vertical" part of q. Similarly, a matrix $M \in \mathrm{M}^{n\times m}$ can be denoted by $M = \begin{pmatrix} M_{\mathrm{hh}} & M_{\mathrm{hv}} \\ M_{\mathrm{vh}} & M_{\mathrm{vv}} \end{pmatrix}$. To emphasize the vector character of a map, boldface symbols will be sometimes used.

Given a $\mathcal{C}^2$ function $\mathbf{f} = (f^1,\dots,f^d) : (0,\infty) \times \mathbb{R}^h \to \mathbb{R}^d$, we denote by $\mathbf{f}_t \in \mathrm{M}^{1\times d}$ the time derivative of $\mathbf{f}$, and by $\nabla\mathbf{f} \in \mathrm{M}^{d\times h}$ (resp. $\nabla^2 f^\alpha \in \mathrm{Sym}^h$, $\alpha = 1,\dots,d$) the gradient of $\mathbf{f}$ (resp. the hessian of f^α, $\alpha = 1,\dots,d$) with respect to $z \in \mathbb{R}^h$. If $z = (x,y) \in \mathbb{R}^{k\times l}$ then by $\mathbf{f}_x \in \mathrm{M}^{d\times k}$ (resp. $f^\alpha_{xx} \in \mathrm{Sym}^k$) we denote the gradient of $\mathbf{f}$ (resp. the hessian of f^α) with respect to $x = (x_1,\dots,x_k)$. Coherently, $\nabla\mathbf{f} = \mathbf{f}_x$, $\nabla^2 f^\alpha = f^\alpha_{xx}$ and $\nabla^2\mathbf{f} = \mathbf{f}_{xx} \in \mathcal{T}^{k\times k\times d}$. Furthermore, by f_{xy} and f_{yx} we mean respectively $(f_x)_y \in \mathrm{M}^{k\times l}$ and $(f_y)_x \in \mathrm{M}^{l\times k}$.

Given $\mathcal{F} : \mathrm{dom}(\mathcal{F}) \subseteq [0,+\infty) \times \mathbb{R}^d_x \times \mathbb{R}_r \times \mathbb{R}^d_q \times \mathrm{Sym}^d \to \mathbb{R}$, we recall that $\mathcal{F}$ is said to be degenerate elliptic if

$$\mathcal{F}(t,x,r,q,M) \le \mathcal{F}(t,x,r,q,N)$$

for all (t,x,r,q,M), $(t,x,r,q,N) \in \mathrm{dom}(\mathcal{F})$, $M \succeq N$.

Lemma 2.1. *Let A, B, G be $(n \times n)$ symmetric matrices with $G \succeq 0$ and $A \succeq B$. Then $\mathrm{tr}(GA) \ge \mathrm{tr}(GB)$.*

Proof. Let $\sqrt{G} \succeq 0$ be the unique $(n \times n)$ symmetric matrix such that $G = \sqrt{G}\sqrt{G}$. Then

$$\sqrt{G}(A-B)\sqrt{G}\xi \cdot \xi = (A-B)\sqrt{G}\xi \cdot \sqrt{G}\xi \ge 0.$$

Since $\mathrm{tr}(GC) = \mathrm{tr}(\sqrt{G}C\sqrt{G})$ for $C \in \mathrm{Sym}^n$, the assertion follows. □

2.1 Preliminaries of differential geometry

Let $U \subseteq \mathbb{R}^n$ be open, $\phi : U \to \mathbb{R}^{n+m}$ be a smooth embedding, and $\Gamma := \phi(U)$ be the smooth manifold in $\mathbb{R}^{n+m}$ with codimension m parametrized by ϕ. Given a vector $v \in \mathbb{R}^{n+m}$ and $z \in \Gamma$, we denote with $\Pi_{\mathrm{N}_z\Gamma} v$ the orthogonal projection of v on the normal space $\mathrm{N}_z\Gamma$ to Γ at z.

Given $x \in U$, let $g_{ij}(x) := \phi_{x_i}(x) \cdot \phi_{x_j}(x)$, and denote by $(g^{ij}(x))$ the inverse matrix of $(g_{ij}(x))$. The second fundamental form $\mathrm{B}^\Gamma_{\phi(x)}$ of Γ at $\phi(x)$ is defined by

$$\mathrm{B}^\Gamma_{\phi(x)}[\phi_{x_i}(x), \phi_{x_j}(x)] := \Pi_{\mathrm{N}_{\phi(x)}\Gamma}(\phi_{x_i x_j}(x)), \quad i,j = 1,...,n. \tag{2.1}$$

The mean curvature vector $\kappa^\Gamma_{\phi(x)}$ of Γ at $\phi(x)$ is defined by

$$\begin{aligned} \kappa^\Gamma_{\phi(x)} &:= \sum_{i=1}^n \mathrm{B}^\Gamma_{\phi(x)}[\tau_i,\tau_i] = \sum_{i,j=1}^n g^{ij}(x)\mathrm{B}^\Gamma_{\phi(x)}[\phi_{x_i}(x), \phi_{x_j}(x)] \\ &= \Pi_{\mathrm{N}_{\phi(x)}\Gamma}\left(\sum_{i,j=1}^n g^{ij}(x)\phi_{x_i x_j}(x)\right), \end{aligned} \tag{2.2}$$

where $\{\tau_1, ..., \tau_n\}$ is an orthonormal basis of the tangent space $\mathrm{T}_{\phi(x)}\Gamma$ to Γ at $\phi(x)$.
If $\{\phi(t,\cdot)\}_t$ is a smooth family of n-dimensional embeddings of U in $\mathbb{R}^{n+m}$, parametrized by time,

$$\mathrm{V}^\perp := \Pi_{\mathrm{N}_{\phi(x)}\phi(t,U)}(\phi_t) \tag{2.3}$$

is the normal velocity of $\{\phi(t,\cdot)\}_t$.

Let $\mathbf{u} : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^m$ be a $\mathcal{C}^2$ function. Then the graph of $\mathbf{u}(t,\cdot)$,

$$\Gamma_{\mathbf{u}(t,\cdot)} := \{(x, \mathbf{u}(t,x)) \,:\, x \in \mathbb{R}^n\}$$

is a $\mathcal{C}^2$ manifold in $\mathbb{R}^{n+m}$ with codimension m. Letting $\phi(t,x) := (x, \mathbf{u}(t,x))$ in (2.1) and (2.2), the second fundamental form of $\Gamma_{\mathbf{u}(t,\cdot)}$ at $(x, \mathbf{u}(t,x))$ is given by

$$\mathrm{B}^{\Gamma_{\mathbf{u}(t,\cdot)}}_{(x,\mathbf{u}(t,x))}[\phi_{x_i}(t,x), \phi_{x_j}(t,x)] = \Pi_{\mathrm{N}_{(x,\mathbf{u}(t,x))}\Gamma_{\mathbf{u}(t,\cdot)}}(0, \mathbf{u}_{x_i x_j}(t,x)), \qquad i,j = 1, ..., n, \tag{2.4}$$

and the mean curvature vector of $\Gamma_{\mathbf{u}(t,\cdot)}$ at $(x, \mathbf{u}(t,x))$ by

$$\kappa^{\Gamma_{\mathbf{u}(t,\cdot)}}_{(t,x)} = \Pi_{\mathrm{N}_{(x,\mathbf{u}(t,x))}\Gamma_{\mathbf{u}(t,\cdot)}}\left(\sum_{i,j=1}^n g^{ij}(t,x)(0, \mathbf{u}_{x_i x_j}(t,x))\right). \tag{2.5}$$

The mean curvature flow equation of $\Gamma_{\mathbf{u}(t,\cdot)}$ reads as

$$V^\perp = \Pi_{\mathrm{N}_{(x,\mathbf{u}(t,x))}\Gamma_{\mathbf{u}(t,\cdot)}}(0, \mathbf{u}_t) = \Pi_{\mathrm{N}_{(x,\mathbf{u}(t,x))}\Gamma_{\mathbf{u}(t,\cdot)}}\left(\sum_{i,j=1}^n g^{ij}\,(0, \mathbf{u}_{x_i x_j})\right), \tag{2.6}$$

where $\mathbf{u}_t$, g^{ij}, $\mathbf{u}_{x_i x_j}$ are all evaluated at (t,x).
When $m = 1$, we denote $\mathbf{u}$ by u; let $\nu_u(t,\cdot) := \frac{(-\nabla u(t,\cdot),1)}{\sqrt{1+|\nabla u(t,\cdot)|^2}}$ be the unit normal vector to the hypersurface $\Gamma_{u(t,\cdot)}$, with positive last component. Then

$$\kappa^{\Gamma_{u(t,\cdot)}} = \mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)\nu_u = -\mathrm{div}\,(\nu_u)\,\nu_u, \tag{2.7}$$

and (2.6) becomes

$$\frac{u_t}{\sqrt{1+|\nabla u|^2}} = \mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right). \tag{2.8}$$

When $n = 1$, let $\tau(t,\cdot)$ be the unit tangent vector to the curve $\Gamma_{\mathbf{u}(t,\cdot)}$ defined as

$$\tau(t,x) := \frac{(1, \nabla \mathbf{u}(t,x))}{\sqrt{1+|\nabla \mathbf{u}(t,x)|^2}} =: (\tau_s, \tau_\eta) \in \mathbb{R} \times \mathbb{R}^m. \tag{2.9}$$

Then $g_{11}(t,x) = 1 + |\nabla \mathbf{u}(t,x)|^2$, $g^{11}(t,x) = (1 + |\nabla \mathbf{u}(t,x)|^2)^{-1}$. Equation (2.5) reads as

$$\kappa^{\Gamma_{\mathbf{u}(t,\cdot)}} = \frac{1}{(1+|\nabla \mathbf{u}|^2)^2}\Big(-\nabla \mathbf{u}\cdot\nabla^2\mathbf{u}, (1+|\nabla \mathbf{u}|^2)\nabla^2\mathbf{u} - \nabla\mathbf{u}\cdot\nabla^2\mathbf{u}\nabla\mathbf{u}\Big) = \frac{\tau_x}{\sqrt{1+|\nabla \mathbf{u}|^2}}, \tag{2.10}$$

and therefore system (2.6) becomes

$$\mathbf{u}_t + \frac{\nabla^2 \mathbf{u}}{1+|\nabla \mathbf{u}|^2} = 0. \tag{2.11}$$

Let $w : (0,\infty) \times \mathbb{R}^{n+m} \to \mathbb{R}$ be a $\mathcal{C}^2$ function with $w^2 + |\nabla w|^2 \neq 0$ and define

$$\Sigma_{w(t,\cdot)} := \{(x,y) \in \mathbb{R}^{n+m} \ : \ w(t,x,y) = 0\}.$$

Setting $\nu := \nabla w / |\nabla w|$ we have

$$\mathrm{B}^{\Sigma_{w(t,\cdot)}} \ := \ -\left(\frac{\Pi_{\nu^\perp} \nabla^2 w \Pi_{\nu^\perp}}{|\nabla w|}\right) \nu, \tag{2.12}$$

$$\kappa^{\Sigma_{w(t,\cdot)}} \ := \ -\mathrm{tr}\left(\frac{\Pi_{\nu^\perp} \nabla^2 w \Pi_{\nu^\perp}}{|\nabla w|}\right) \nu. \tag{2.13}$$

The normal velocity of $\Sigma_{w(t,\cdot)}$ is $\frac{w_t}{|\nabla w|}$ and the mean curvature flow equation of $\Sigma_{w(t,\cdot)}$ reads as

$$\frac{w_t}{|\nabla w|} = -\mathrm{tr}\left(\frac{\Pi_{\nu^\perp} \nabla^2 w \Pi_{\nu^\perp}}{|\nabla w|}\right) \nu. \tag{2.14}$$

3 The general procedure $\mathbf{F} \to \widehat{F}$

Let $n, m \geq 1$ and $\mathbf{F} : [0,+\infty) \times \mathbb{R}^n_x \times \mathbb{R}^m_y \times \mathrm{M}^{m\times n}_P \times \mathcal{T}^{n\times n\times m}_{\mathcal{X}} \to \mathbb{R}^m$ be continuous. Consider a system of partial differential equations of the form

$$\mathbf{u}_t + \mathbf{F}(t, x, \mathbf{u}, \nabla \mathbf{u}, \nabla^2 \mathbf{u}) = 0, \tag{3.1}$$

for a $\mathcal{C}^2$ vector-valued function $\mathbf{u} : (0,T) \times \mathbb{R}^n_x \to \mathbb{R}^m_y$. When $m = 1$ the function $\mathbf{F}$ will be denoted by F. We now introduce the scalar level set equation associated with (3.1).

Definition 3.1. *We define the function $\widehat{F}$ as*

$$\mathrm{dom}(\widehat{F}) := [0,+\infty) \times \mathbb{R}^n_x \times \mathbb{R}^m_y \times \mathbb{R}^n_{q_\mathrm{h}} \times (\mathbb{R}^m_{q_\mathrm{v}} \setminus \{0\}) \times \mathrm{Sym}^{n+m}_M,$$

$$\widehat{F}(t,x,y,q,M) := -\inf_{(P,\mathcal{X}) \in \Lambda(q,M)} \mathbf{F}(t,x,y,P,\mathcal{X}) \cdot q_\mathrm{v}, \tag{3.2}$$

where

$$q = (q_\mathrm{h}, q_\mathrm{v}) \in \mathbb{R}^n \times \mathbb{R}^m_0, \qquad M = \begin{pmatrix} M_\mathrm{hh} & M_\mathrm{hv} \\ M_\mathrm{vh} & M_\mathrm{vv} \end{pmatrix} \in \mathrm{Sym}^{n+m},$$

$$\Lambda(q,M) := \{(P,\mathcal{X}) \in \mathrm{M}^{m\times n} \times \mathcal{T}^{n\times n\times m} : q_\mathrm{h} + q_\mathrm{v} P = 0,$$

$$M_\mathrm{hh} + M_\mathrm{hv} P + (M_\mathrm{hv} P)^* + P^* M_\mathrm{vv} P + q_\mathrm{v} \mathcal{X} \preceq \mathrm{O}_n\}.$$

The scalar partial differential equation

$$w_t(t,x,y) + \widehat{F}(t,x,y,\nabla w(t,x,y), \nabla^2 w(t,x,y)) = 0, \tag{3.3}$$

for $w : (0,T) \times \mathbb{R}^{n+m} \to \mathbb{R}$, is called the level set equation associated with system (3.1).

Remark 3.2. Note that $\widehat{F} : \mathrm{dom}(\widehat{F}) \to (-\infty, +\infty]$ and q_v is not allowed to vanish.

Let us explain the geometric meaning of Definition 3.1.

Remark 3.3. Equation (3.3) says that w_t equals the minimum of the projections on ∇w of the velocities $(0, \mathbf{u}_t) \in \mathbb{R}^n_x \times \mathbb{R}^m_y$ of all regular evolutions $(x, \mathbf{u}(t,x))$ of *graph-type*, with $\mathbf{u}$ solution of (3.1), which are tangent (first constraint in $\Lambda(q,M)$) to the zero-level set of $w(t,\cdot)$ from one side (second constraint in $\Lambda(q,M)$ involving the second fundamental forms of $\Sigma_{w(t,\cdot)}$ and of $\Gamma_{\mathbf{u}(t,\cdot)}$). This can be obtained by formally differentiating the equation $w(t,x,\mathbf{u}(t,x)) = 0$ as follows.
Let $w : (0,T) \times \mathbb{R}^{n+m} \to \mathbb{R}$ be a smooth function with $w^2 + |\nabla w|^2 > 0$ and suppose that the zero-level set $\Sigma_{w(t,\cdot)}$ of $w(t,\cdot)$ is locally represented as the smooth graph in the y-directions as

$$y = \mathbf{u}(t,x), \qquad t \geq 0,\, x \in \mathbb{R}^n,\, y \in \mathbb{R}^m,$$

with $\mathbf{u}$ solution of (3.1). Differentiating, we have

$$0 = \frac{d}{dt} w(t,x,\mathbf{u}(t,x)) = w_t + w_y \cdot \mathbf{u}_t \tag{3.4}$$

$$0 = \big(w(t,x,\mathbf{u}(t,x))\big)_x = w_x + w_y \nabla \mathbf{u} \tag{3.5}$$

$$\mathrm{O}_n = \big(w(t,x,\mathbf{u}(t,x))\big)_{xx} = w_{xx} + w_{xy}\nabla\mathbf{u} + (w_{xy}\nabla\mathbf{u})^* + \nabla\mathbf{u}^* w_{yy}\nabla\mathbf{u} + w_y \nabla^2\mathbf{u}, \tag{3.6}$$

where the right hand sides are evaluated at $(t,x,\mathbf{u}(t,x))$.
Condition (3.5) (compare the equality in the Definition of $\Lambda(q,M)$) implies that the tangent space (of dimension $(n+m-1)$) to $\Sigma_{w(t,\cdot)}$ contains the tangent space (of dimension n) to $\Gamma_{\mathbf{u}(t,\cdot)}$.
Condition (3.6) (compare the inequality in the Definition of $\Lambda(q,M)$) is equivalent to the following equality between the second fundamental form of $\Sigma_{w(t,\cdot)}$ at $(x,\mathbf{u}(t,x))$, and the second fundamental form of $\Gamma_{u(t,\cdot)}$ at $(x,\mathbf{u}(t,x))$. Setting $\phi(t,x) := (x,\mathbf{u}(t,x))$, we have

$$\begin{aligned}
-\mathrm{B}^{\Sigma_{w(t,\cdot)}}_{(x,y)}[\phi_{x_i},\phi_{x_j}] \cdot \nu(t,x,y) &= \frac{1}{|\nabla w|}\left(w_{xx} + w_{xy}\nabla\mathbf{u} + (w_{xy}\nabla\mathbf{u})^* + \nabla\mathbf{u}^* w_{yy}\nabla\mathbf{u}\right)_{ij} \\
&= -\nu(t,x,y)\cdot\big((0,\mathbf{u}(t,x))\big)_{x_i x_j} \\
&= -\mathrm{B}^{\Gamma_{\mathbf{u}(t,\cdot)}}_{(x,\mathbf{u}(t,x))}[\phi_{x_i},\phi_{x_j}] \cdot \nu(t,x,y),
\end{aligned}$$

where $\nu(t,x,y)$, being normal to $\Sigma_{w(t,\cdot)}$, is one of the normal vectors to $\Gamma_{\mathbf{u}(t,\cdot)}$ at $(x,\mathbf{u}(t,x))$, and the quantities on the right hand side in the first equality involving w (resp. $\mathbf{u}$) must be evaluated at (t,x,y) (resp. at (t,x)).
Notice that if $m=1$ then $\nu = \nu_u$ and

$$\begin{aligned}
\mathrm{B}^{\Gamma_{u(t,\cdot)}}_{(x,u(t,x))}[\phi_{x_i},\phi_{x_j}] \cdot \nu(t,x,y) &= -\frac{\Pi_{\nu^\perp}\begin{pmatrix} -\nabla^2 u & \mathrm{O}_{n\times 1} \\ \mathrm{O}_{1\times n} & 0 \end{pmatrix}\Pi_{\nu^\perp}}{\sqrt{1+|\nabla u|^2}}[\phi_{x_i},\phi_{x_j}] \\
&= \frac{1}{\sqrt{1+|\nabla u|^2}}\,\phi_{x_i}\begin{pmatrix} \nabla^2 u & \mathrm{O}_{n\times 1} \\ \mathrm{O}_{1\times n} & 0 \end{pmatrix}\phi_{x_j}.
\end{aligned}$$

Example 3.4 (The codimension 1 case: consistency with [12] and [21]). Let $m=1$. The constraint $q_{\mathrm{h}} + q_{\mathrm{v}}P = 0$ in $\Lambda(q,M)$ can be solved for $P = -\dfrac{q_{\mathrm{h}}}{q_{\mathrm{v}}} \in \mathrm{M}^{1\times n}$. If in addition F is degenerate elliptic then

$$q_{\mathrm{v}}F\left(t,x,y,-\frac{q_{\mathrm{h}}}{q_{\mathrm{v}}},\mathcal{X}\right) \geq q_{\mathrm{v}}F\left(t,x,y,-\frac{q_{\mathrm{h}}}{q_{\mathrm{v}}},\frac{M_{\mathrm{hh}} + M_{\mathrm{hv}}P + (M_{\mathrm{hv}}P)^* + P^*M_{\mathrm{vv}}P}{q_{\mathrm{v}}}\right).$$

Therefore,

$$\begin{aligned}\widehat{F}(t,x,y,q,M) &= -\inf_{\mathcal{X}\,:\,M_{\mathrm{hh}}-2M_{\mathrm{hv}}\otimes\frac{q_{\mathrm{h}}}{q_{\mathrm{v}}}+M_{\mathrm{vv}}\frac{q_{\mathrm{h}}}{q_{\mathrm{v}}}\otimes\frac{q_{\mathrm{h}}}{q_{\mathrm{v}}}+q_{\mathrm{v}}\mathcal{X}\le 0} q_{\mathrm{v}}F\left(t,x,y,-\frac{q_{\mathrm{h}}}{q_{\mathrm{v}}},\mathcal{X}\right)\\ &= -q_{\mathrm{v}}F\left(t,x,y,-\frac{q_{\mathrm{h}}}{q_{\mathrm{v}}},-\frac{{q_{\mathrm{v}}}^2M_{\mathrm{hh}}-2q_{\mathrm{v}}M_{\mathrm{hv}}\otimes q_{\mathrm{h}}+M_{\mathrm{vv}}\,(q_{\mathrm{h}}\otimes q_{\mathrm{h}})}{{q_{\mathrm{v}}}^3}\right). \qquad (3.7)\end{aligned}$$

If $F(\mathcal{X})=-\mathrm{tr}(\mathcal{X})$, $\mathcal{X}\in\mathcal{T}^{n\times n\times 1}$, then (3.3) is the level set heat equation studied by Evans [12] (see Corollary 5.4 (i) with $f\equiv 0$). If F does not depend on $\nabla^2 u$ then (3.7) coincides essentially with the level set equation studied by Giga and Sato [21], i.e.

$$\widehat{F}(t,x,y,q)=-q_{\mathrm{v}}F(t,x,y,-q_{\mathrm{h}}/q_{\mathrm{v}}),\quad q_{\mathrm{v}}\neq 0.$$

Remark 3.5. In Definition 3.1 we assume that the domain of $\widehat{F}$ is the whole of $[0,+\infty)\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}^m_0\times\mathrm{Sym}^{n+m}$. In Section 5.4 we will consider a system of PDEs corresponding to the parametric curvature flow of planar curves (n=1, m=2), for which $\mathbf{F}$ is undefined at $P=0$.

Proposition 3.6. *The function $\widehat{F}$ defined in* (3.2) *verifies the following properties:*

(i) $\widehat{F}$ is geometric, i.e.

$$\widehat{F}(t,x,y,\lambda q,\lambda M+\sigma q\otimes q)=\lambda\widehat{F}(t,x,y,q,M), \qquad (3.8)$$

for all $\lambda>0$, $\sigma>0$, $(t,x,y,q,M)\in\mathrm{dom}(\widehat{F})$;

(ii) $\widehat{F}$ is degenerate elliptic.

Proof. (i) follows from Definition 3.1. (ii) holds since

$$M_{\mathrm{hh}}+M_{\mathrm{hv}}P+(M_{\mathrm{hv}}P)^*+P^*M_{\mathrm{vv}}P=(\mathrm{Id}_n,P^*)M(\mathrm{Id}_n,P^*)^*,$$

and

$$(\mathrm{Id}_n,P^*)M(\mathrm{Id}_n,P^*)^*\succeq \mathrm{O}_n \qquad \text{whenever} \quad M\succeq \mathrm{O}_{n+m},\ P\in\mathrm{M}^{m\times n}.$$

Therefore

$$\Lambda(q,M)\subseteq\Lambda(q,N) \qquad \text{whenever} \quad M-N\succeq \mathrm{O}_{n+m}.$$

□

Remark 3.7. It is not difficult to show that $\widehat{F}$ in general is not upper semicontinuous. In many cases $\widehat{F}$ fails to be bounded on bounded sets, and in general does not satisfy any continuity condition at $(q,M)=(0,\mathrm{O}_{n+m})$.

4 First order systems

Let $\mathbf{u}:[0,T]\times\mathbb{R}^n\to\mathbb{R}^m$ be a $\mathcal{C}^1$ solution of

$$\mathbf{u}_t+\mathbf{F}(t,x,\mathbf{u},\nabla\mathbf{u})=0 \qquad \text{in } (0,T]\times\mathbb{R}^n, \qquad (4.1)$$

with $\mathbf{F}:[0,T]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathrm{M}^{m\times n}=:D\to\mathbb{R}^m$ continuous and Lipschitz in $(x,\mathbf{u})$. Then the function $\widehat{F}:[0,T]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}^m_0=:\widehat{D}\to(-\infty,+\infty]$ is given by

$$\widehat{F}(t,x,y,q)=-\inf\{\mathbf{F}(t,x,y,P)\cdot q_{\mathrm{v}}:P\in\mathrm{M}^{m\times n},q_{\mathrm{h}}+q_{\mathrm{v}}P=0\}.$$

We assume that $\widehat{F}$ is real valued and can be extended in a continuous way on the whole of $[0,T]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}^m$. Let us further assume that there exists a modulus of continuity m_1 such that

$$|\widehat{F}(t,x,y,q)-\widehat{F}(t,x',y',q)|\le m_1((|x-x'|+|y-y'|)(1+|q|)),$$

and that for every $C>0$ there exists a modulus m_C such that

$$|\widehat{F}(t,x,y,q)-\widehat{F}(t,x,y,q')|\le m_C(|q-q'|),\qquad \text{for } |q|,|q'|\le C.$$

Let $w_0:\mathbb{R}^{n+m}\to\mathbb{R}$ be uniformly continuous and $w:[0,T]\times\mathbb{R}^{n+m}\to\mathbb{R}$ be the unique uniformly continuous viscosity solution [9] of

$$\begin{cases} w_t+\widehat{F}(t,x,y,\nabla w)=0 & \text{in } (0,T]\times\mathbb{R}^{n+m}\\ w(0,x,y)=w_0(x,y) & (x,y)\in\mathbb{R}^{n+m}. \end{cases} \tag{4.2}$$

Remark 4.1. Let $n=1$. Then

(i) $\widehat{F}(t,x,y,q_{\mathrm{h}},q_{\mathrm{v}})=-\inf\{\mathbf{F}(t,x,y,-\frac{q_{\mathrm{h}}}{|q_{\mathrm{v}}|^2}q_{\mathrm{v}}+P')\cdot q_{\mathrm{v}}:\ P'\in q_{\mathrm{v}}^{\perp}\}$;

(ii) if $\mathbf{F}(t,x,y,P)=a(t,x,y,P)P+\mathbf{b}(t,x,y,P)$, with $a\in\mathrm{L}^\infty(D)$, $\mathbf{b}\in\mathrm{L}^\infty(D;\mathbb{R}^m)$, then $\widehat{F}$ is bounded;

(iii) let $A\in\mathrm{M}^{m\times m}$ be a diagonal matrix which is not a multiple of the identity, and set $\mathbf{F}(P)=AP$. Then $\widehat{F}$ is not real valued (cfr. Example 4.8).

Proposition 4.2. *Let $\mathbf{u}_0\in\mathcal{C}^1(\mathbb{R}^n,\mathbb{R}^m)$ satisfying $\Gamma_{\mathbf{u}_0}\subseteq\Sigma_{w_0}$. Assume that $\mathbf{u}:(0,T)\times\mathbb{R}^n\to\mathbb{R}^m$ is a $\mathcal{C}^1$ solution of (4.1) with $\mathbf{u}(0,\cdot)=\mathbf{u}_0(\cdot)$. Assume further that $\widehat{F}(t,x,y,0,0)=0$ and $y\to\widehat{F}(t,x,y,q)$ is L-Lipschitz. Then $\Gamma_{\mathbf{u}(t,\cdot)}\subseteq\Sigma_{w(t,\cdot)}$ for any $t\in[0,T)$.*

Proof. Let us prove that the function $z(t,x,y):=e^{Lt}|y-\mathbf{u}(t,x)|$ is a supersolution of (4.2), with $w_0(x,y)=|y-\mathbf{u}_0(x)|$. If this is true, being 0 a solution, we have $0\le w(x,y,t)\le z(x,y,t)$ by the maximum principle, and the assertion follows. We have

$$z_y=e^{Lt}\frac{y-\mathbf{u}(t,x)}{|y-\mathbf{u}(t,x)|},\qquad z_x+z_y\nabla\mathbf{u}=0. \tag{4.3}$$

Hence, using the Definition of $\widehat{F}$,

$$z_t=Lz+z_y\mathbf{F}(t,x,\mathbf{u},\nabla\mathbf{u})\ge Lz-\widehat{F}(t,x,\mathbf{u},z_x,z_y)\ge-\widehat{F}(t,x,y,z_x,z_y).$$

□

Remark 4.3. Let $n=1$ and τ be defined as in (2.9). If the function $\mathbf{u}$ satisfies (4.1), then $\Gamma_{\mathbf{u}(t,\cdot)}$ evolves with the following geometric law:

$$V^{\perp}(t,x,y)=-\Pi_{N_{\Gamma_{\mathbf{u}(t,\cdot)}}(x,y)}(0,\mathbf{F}(t,x,y,\tau_\eta/\tau_s)),\quad t\in[0,+\infty),\ (x,y)\in\Gamma_{\mathbf{u}(t,\cdot)}.$$

4.1 The quasi-linear case

Let $\mathbf{f}:[0,+\infty)\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n$ and $\mathbf{g}:[0,+\infty)\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^m$ be continuous and Lipschitz in (x,y). Consider system (4.1) with

$$\mathbf{F}(t,x,y,P)=P\mathbf{f}(t,x,y)-\mathbf{g}(t,x,y)\in\mathbb{R}^m,\qquad (t,x,y,P)\in D. \tag{4.4}$$

Then

$$\widehat{F}(t,x,y,q) = \mathbf{f}(t,x,y)\cdot q_{\rm h} + \mathbf{g}(t,x,y)\cdot q_{\rm v}, \qquad (t,x,y,q)\in \widehat{D}, \tag{4.5}$$

so that the associated level set equation is the linear transport equation

$$w_t + \mathbf{f}(t,x,y)\cdot w_x + \mathbf{g}(t,x,y)\cdot w_y = 0. \tag{4.6}$$

Let $w_0 : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be bounded and uniformly continuous. Then there exists a unique bounded and uniformly continuous viscosity solution of (4.6) with $w(0,\cdot) = w_0(\cdot)$.

Proposition 4.4. *In the same hypotheses of Proposition 4.2 on* $\mathbf{u}_0$, $\mathbf{u}$, *and assuming* $\mathbf{u} \in \mathrm{Lip}([0,T]\times \mathbb{R}^n, \mathbb{R}^m)$ *and* $\mathbf{g} = 0$, *we get*

$$\Sigma_{w(t,\cdot)} = \Gamma_{\mathbf{u}(t,\cdot)}, \quad t\in[0,T].$$

Proof. Let $z(x,y,t) := e^{-Lt}|y-\mathbf{u}(t,x)|$. We claim that z is a subsolution of (3.3) for some $L>0$. Indeed, recalling (4.3), we have

$$z_t = -Lz + z_y \nabla\mathbf{u}\,\mathbf{f}(t,x,\mathbf{u}) = -Lz - z_x\mathbf{f}(t,x,\mathbf{u}) \le -z_x\mathbf{f}(t,x,y) = -\widehat{F}(t,x,y,z_x,z_y),$$

for some $L>0$ depending on $\|\nabla\mathbf{u}\|_\infty$ on $(0,T)$ and on the Lipschitz constant of $\mathbf{f}$. As a consequence we get $z \le w$. The thesis now follows from Proposition 4.2. □

Using the method of characteristics we get the following representation of solutions.

Proposition 4.5. *Assume that* $\mathbf{g} = 0$ *and* $\mathbf{f} = \mathbf{f}(t,y) \in \mathcal{C}^1([0,T]\times\mathbb{R}^m,\mathbb{R}^n)$; *moreover assume that*

$$w_0(x,y) = \phi(y - \mathbf{u}_0(x)), \quad x\in\mathbb{R}^n, y\in\mathbb{R}^m,$$

with $\phi \in \mathcal{C}^2(\mathbb{R}^m)$, $\phi^{-1}(0) = \{0\}$ *and* $\mathbf{u}_0 \in \mathcal{C}^2(\mathbb{R}^n,\mathbb{R}^m)$. *Then*

$$w(t,x,y) = \phi\left(y - \mathbf{u}_0\left(x - \int_0^t \mathbf{f}(s,y)\,ds\right)\right). \tag{4.7}$$

In particular if $\mathbf{f} = \mathbf{f}(y)$ *then*

$$\begin{aligned}\Sigma_{w(t,\cdot)} &= \{(x,y) : y - \mathbf{u}_0(x - \mathbf{f}(y)t) = 0\} \\ &= \bigcup_{z\in\mathbb{R}^n} (z + t\mathbf{f}(\mathbf{u}_0(z)), \mathbf{u}_0(z))\end{aligned}$$

is a m*-codimensional manifold of class* $\mathcal{C}^2$ *in* $\mathbb{R}^{n+m}$ *for any* $t\in[0,T]$.

Remark 4.6. Following the approach of [15], one may regularize equation (4.6) as follows

$$w_t + \mathbf{f}(t,x,y)\cdot w_x + \mathbf{g}(t,x,y)\cdot w_y = C|\nabla w|\mathrm{div}_y\left(\frac{w_y}{|w_y|}\right), \qquad C>0. \tag{4.8}$$

As in [15], it is reasonable to expect that the solution of (4.8) does not depend on C, for C large enough, and that its zero-level set corresponds to the entropy solution of $u_t + \nabla\mathbf{u}\,\mathbf{f}(t,x,\mathbf{u}) = \mathbf{g}(t,x,\mathbf{u})$.

We also investigate systems of quasilinear first-order partial differential equations of the form

$$\mathbf{F}(t,x,y,P) = \sum_{j=1}^n B_j(t,x,y)Pe_j - \mathbf{g}(t,x,y) \in \mathbb{R}^m, \qquad (t,x,y,P)\in D, \tag{4.9}$$

with $B_j : [0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathrm{M}^{m\times m}$ $(j = 1, ..., n)$ and $\mathbf{g} : [0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ given. From Definition 3.1 and writing $P_j := Pe_j = -\frac{q_{\mathrm{h}_j}}{|q_{\mathrm{v}}|^2} q_{\mathrm{v}} + r_j$, with $r_j \in \mathrm{M}^{m\times 1}$, $r_j \in q_{\mathrm{v}}^{\perp}$, we get

$$\widehat{F}(t,x,y,q) = - \inf_{\substack{P_j \in \mathrm{M}^{m\times 1} \\ q_{\mathrm{h}_j} + q_{\mathrm{v}} P_j = 0, \\ j=1,...,n}} \left(\sum_{j=1}^{n} (q_{\mathrm{v}} B_j(t,x,y)) P_j - q_{\mathrm{v}} \cdot \mathbf{g}(t,x,y) \right)$$

$$= \sum_{j=1}^{n} q_{\mathrm{h}_j} \frac{q_{\mathrm{v}} B_j(t,x,y) q_{\mathrm{v}}^*}{|q_{\mathrm{v}}|^2} + q_{\mathrm{v}} \cdot \mathbf{g}(t,x,y) - \inf_{\substack{r_j \in \mathrm{M}^{m\times 1} \\ r_j \in q_{\mathrm{v}}^{\perp}, \\ j=1,...,n}} \sum_{j=1}^{n} (q_{\mathrm{v}} B_j(t,x,y)) r_j$$

$$= \begin{cases} \displaystyle\sum_{j=1}^{n} q_{\mathrm{h}_j} \frac{q_{\mathrm{v}} B_j(t,x,y) q_{\mathrm{v}}^*}{|q_{\mathrm{v}}|^2} + q_{\mathrm{v}} \cdot \mathbf{g}(t,x,y) & \begin{array}{l} \text{if } q_{\mathrm{v}} B_j(t,x,y) \cdot r = 0 \\ \forall\, r \in q_{\mathrm{v}}^{\perp} \\ \forall\, j = 1,...,n \end{array} \\ +\infty & \text{otherwise} \end{cases} \tag{4.10}$$

Remark 4.7. The level set equation associated with (4.9) has the form

$$w_t + \sum_{j=1}^{n} w_{x_j} \frac{w_y B_j(t,x,y) w_y^*}{|w_y|^2} + w_y \cdot \mathbf{g}(t,x,y) = 0. \tag{4.11}$$

provided that w_y is an eigenvector of B_j^* for all $j = 1, ..., n$.

Example 4.8 (Wave equation). Let us consider the wave equation

$$v_{tt} - v_{xx} = 0 \qquad v : [0,T) \times \mathbb{R} \to \mathbb{R}.$$

By setting $u^1 := v_t$ and $u^2 := v_x$ it follows $n = 1$, $m = 2$, and

$$\mathbf{u}_t + \mathbf{F}(\nabla \mathbf{u}) = 0, \qquad \mathbf{u} : [0,T) \times \mathbb{R} \to \mathbb{R}^2$$

where

$$\mathbf{F}(P) = AP, \qquad A := \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \qquad P \in \mathrm{M}^{2\times 1}.$$

Then (4.10) becomes

$$\widehat{F}(q) = \begin{cases} q_{\mathrm{h}} \dfrac{q_{\mathrm{v}} A q_{\mathrm{v}}^*}{|q_{\mathrm{v}}|^2} & \text{if } q_{\mathrm{v}} A \cdot r = 0 \quad \forall\, r \in q_{\mathrm{v}}^{\perp} \\ +\infty & \text{otherwise} \end{cases} \qquad q \in \mathbb{R} \times \mathbb{R}_0^2.$$

The eigenvalues of A are -1, 1, and if $q_{\mathrm{v}} \neq 0$ is a λ-eigenvector of A then $\dfrac{q_{\mathrm{v}} A q_{\mathrm{v}}^*}{|q_{\mathrm{v}}|^2} = \lambda$. In addition the condition $q_{\mathrm{v}} A \cdot r = 0$ for any $r \in q_{\mathrm{v}}^{\perp}$ in (4.10) becomes

$$\begin{cases} q_{\mathrm{v}1} r_2 + q_{\mathrm{v}2} r_1 = 0, \\ q_{\mathrm{v}1} r_1 + q_{\mathrm{v}2} r_2 = 0, \end{cases}$$

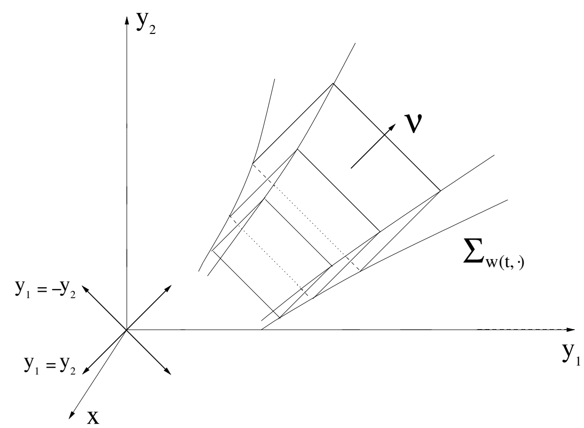

Figure 2: level set of a solution of (3.3) with $\widehat{F}$ as in (4.12) ($n = 1$, $m = 2$). The slices of the level set, at fixed x, are rectangles with sides parallel to the vectors indicated at the origin.

the nonzero solutions of which are $q_{\mathrm{v}1} = \pm q_{\mathrm{v}2}$. We conclude

$$\widehat{F}(q) = \begin{cases} -q_{\mathrm{h}} & \text{if } q_{\mathrm{v}_1} = q_{\mathrm{v}_2} \\ q_{\mathrm{h}} & \text{if } q_{\mathrm{v}_1} = -q_{\mathrm{v}_2} \\ +\infty & \text{otherwise,} \end{cases} \qquad q = (q_{\mathrm{h}}, q_{\mathrm{v}}) \in \mathrm{dom}(\widehat{F}) = \mathbb{R} \times \mathbb{R}^2_0. \tag{4.12}$$

Observe that $\widehat{F}$ is discontinuous where it is finite and $\widehat{F}$ is lower semicontinuous. If w is a solution of (3.3) with $\widehat{F}$ given in (4.12) then the normal velocity of $\Sigma_{w(t,\cdot)}$ takes values only in two directions, as shown in Figure 2.

5 Systems of second order PDEs

In this section we compute the level set equation associated with the mean curvature flow of a graph in arbitrary codimension (Theorem 5.1), with a system of 1-dimensional quasi-linear equations, with a system of reaction-diffusion equations (Theorem 5.3), and with the parametric curvature flow of a planar curve (formula (5.25)).

5.1 Mean curvature flow in arbitrary codimension

Consider a graph $\Gamma_{\mathbf{u}(t,\cdot)}$ of $\mathbf{u} : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^m$ the normal velocity of which is equal to the mean curvature vector, i.e. $\mathbf{u}$ solves (2.6). Then $\mathbf{F}$ reads as

$$\mathbf{F}(P, \mathcal{X}) = -\sum_{i,j=1}^{n} \widetilde{g}^{ij}(P)\mathcal{X}_{ij} \in \mathbb{R}^m, \qquad (P, \mathcal{X}) \in \mathrm{M}^{m\times n} \times \mathcal{T}^{n\times n\times m}, \tag{5.1}$$

where $\widetilde{g}^{ij}(P)$ is the inverse matrix of $\widetilde{g}_{ij}(P) := \delta_{ij} + Pe_i \cdot Pe_j$ for any $P \in \mathrm{M}^{m\times n}$ and $i, j = 1, \ldots, n$. Notice that $g^{ij}(t,x) = \widetilde{g}^{ij}(\nabla \mathbf{u}(t,x))$.

Theorem 5.1. *Let* $\mathbf{F}$ *be defined as in* (5.1). *Then*

$$\widehat{F}(q, M) = -\sum_{i=1}^{n} \lambda_i(\Pi_{q^\perp} M \Pi_{q^\perp}), \qquad (q, M) \in \mathbb{R}^n \times \mathbb{R}^m_0 \times \mathrm{Sym}^{n+m}, \tag{5.2}$$

where $\lambda_1(\Pi_{q^\perp} M \Pi_{q^\perp}) \ldots, \lambda_{n+m-1}(\Pi_{q^\perp} M \Pi_{q^\perp})$ *are the eigenvalues of* $M_{|q^\perp}$ *in increasing order.*

Proof. From Definition 3.1 we have, for any $(q,M) \in \mathbb{R}^n \times \mathbb{R}^m_0 \times \mathrm{Sym}^{n+m}$,

$$\widehat{F}(q,M) = -\inf_{(P,\mathcal{X})\in\Lambda(q,M)} \sum_{i,j=1}^{n} \widetilde{g}^{ij}(P) \sum_{\alpha=1}^{m} (-q_{\mathrm{h}_\alpha} \mathcal{X}^\alpha_{ij}).$$

Applying Lemma 2.1 with $A = -q_{\mathrm{v}}\mathcal{X}$, $B = M_{\mathrm{hh}} + M_{\mathrm{hv}}P + (M_{\mathrm{hv}}P)^* + P^* M_{\mathrm{vv}}P$ and $G = \widetilde{g}^{ij}(P)$, yields

$$\widehat{F}(q,M) = -\inf_{P\,:\, q_{\mathrm{h}}+q_{\mathrm{v}}P=0} \sum_{i,j=1}^{n} \widetilde{g}^{ij}(P)\left[M_{\mathrm{hh}} + M_{\mathrm{hv}}P + (M_{\mathrm{hv}}P)^* + P^* M_{\mathrm{vv}}P\right]_{ji}.$$

Set $v_k(P) := (e_k, Pe_k)$ for any $k = 1,\dots,n$. Then, for any $(q,M) \in \mathbb{R}^n \times \mathbb{R}^m_0 \times \mathrm{Sym}^{n+m}$,

$$\begin{aligned}
\widehat{F}(q,M) &= -\inf_{\substack{P\,:\, v_k(P)\in q^\perp \\ \forall k=1,\dots,n}} \sum_{i,j=1}^{n} \widetilde{g}^{ij}(P) M v_i(P)\cdot v_j(P) \\
&= -\inf_{\substack{\{\eta_1,\dots,\eta_n\}\text{ orthonormal} \\ \text{basis of } q^\perp}} \sum_{i=1}^{n} M\eta_i(P)\cdot\eta_i(P) \\
&= -\sum_{i=1}^{n} \lambda_i(\Pi_{q^\perp} M \Pi_{q^\perp}).
\end{aligned}$$

□

Note that, in the case $m = 1$, equation (5.2) reads as

$$\widehat{F}(q,M) = -\mathrm{tr}\left[\Pi_{q^\perp} M \Pi_{q^\perp}\right], \quad q \in \mathbb{R}^n \times \mathbb{R}_0,\ M \in \mathrm{Sym}^{n+1},$$

whereas, in the case $n = 1$ we have

$$\widehat{F}(q,M) = -\min\{\lambda_1,\dots,\lambda_{m-1}\}, \qquad q \in \mathbb{R}\times\mathbb{R}^m_0,\ M \in \mathrm{Sym}^{1+m},$$

where $\lambda_1,\dots,\lambda_{m-1}$ are the eigenvalues of $M_{|q^\perp}$. We remark that, following a suggestion of De Giorgi [11], equation (5.2) has been considered in [1] in order to construct weak solutions of the mean curvature flow in arbitrary codimension.

5.2 The quasi-linear case in dimension $n = 1$

Let us assume $n = 1$, $f : [0,\infty)\times\mathbb{R}\times\mathbb{R}^m\times\mathrm{M}^{m\times 1}\times\mathcal{T}^{1\times1\times m}\to[0,+\infty)$ Lipschitz in (x,y), and $\mathbf{F}$ in (1.1) given by

$$\mathbf{F}(t,x,y,P,\mathcal{X}) = -f(t,x,y,P)\mathcal{X}, \tag{5.3}$$

for any $(t,x,y,P,\mathcal{X}) \in [0,\infty)\times\mathbb{R}\times\mathbb{R}^m\times\mathrm{M}^{m\times1}\times\mathcal{T}^{1\times1\times m}$. From Definition 3.1 we get

$$\widehat{F}(t,x,y,q,M) = -\inf_{(1,P^*)\in q^\perp} f(t,x,y,P)(M_{\mathrm{hh}} + 2M_{\mathrm{hv}}P + P^* M_{\mathrm{vv}}P).$$

The function $\widehat{F}$ is linear in M and 0-homogeneous in q. Note that, if f satisfies

$$0 \le f(t,x,y,P) \le \frac{C}{1+|P|^\alpha} \qquad \alpha \ge 2,\ C > 0, \tag{5.4}$$

then $\widehat{F}$ is bounded since

$$-\widehat{F}(t,x,y,q,M) \quad = \quad \inf_{(1,P^*)\in q^\perp} \left\{ \frac{M(1,P^*)^*\cdot(1,P^*)}{1+|P|^2}(1+|P|^2)f(t,x,y,P) \right\}.$$

Systems of the form (5.4) have been considered by Slepcev in [28], in order to extend the result of Ambrosio and Soner in the case $n=1$, $m>1$.

5.3 Reaction-diffusion systems

Let $\mathbf{u}:\mathbb{R}^n\to\mathbb{R}^m$ be a solution of the system

$$\mathbf{u}_t-\Delta\mathbf{u}+\mathbf{f}(\mathbf{u})=0, \qquad (t,x)\in[0,+\infty)\times\mathbb{R}^n, \tag{5.5}$$

with a suitable smooth $\mathbf{f}:\mathbb{R}^m\to\mathbb{R}^m$, see for instance [30]. Thus

$$\mathbf{F}(y,\mathcal{X})=-\mathrm{tr}(\mathcal{X})+\mathbf{f}(y), \qquad y\in\mathbb{R}^m,\ \mathcal{X}\in\mathcal{T}^{n\times n\times m}. \tag{5.6}$$

We want to compute the corresponding function $\widehat{F}$. To this purpose, for any $y\in\mathbb{R}^m$ and $(q,M)\in(\mathbb{R}^n\times\mathbb{R}^m_0)\times\mathrm{Sym}^{n+m}$, set

$$\begin{aligned}
\mathcal{P}(q,M) &:= \Pi_{q_\mathrm{v}^\perp} M_{\mathrm{vv}} \Pi_{q_\mathrm{v}^\perp} \in \mathrm{Sym}^m,\\
\mathcal{Q}(q,M) &:= \left(M_{\mathrm{hv}}-\frac{1}{|q_\mathrm{v}|^2}\,(q_\mathrm{h}\otimes q_\mathrm{v})\,M_{\mathrm{vv}}\right)\Pi_{q_\mathrm{v}^\perp}\in \mathrm{M}^{n\times m},\\
C(y,q,M) &:= \mathrm{tr}\left(M_{\mathrm{hh}}-\frac{2}{|q_\mathrm{v}|^2}M_{\mathrm{hv}}\,(q_\mathrm{v}\otimes q_\mathrm{h})+\frac{1}{|q_\mathrm{v}|^4}(q_\mathrm{h}\otimes q_\mathrm{v})\,M_{\mathrm{vv}}\,(q_\mathrm{v}\otimes q_\mathrm{h})\right)+\mathbf{f}(y)\cdot q_\mathrm{v}\in\mathbb{R},
\end{aligned} \tag{5.7}$$

where $q_\mathrm{v}^\perp\subset\mathbb{R}^m$. Define also

$$Z:=\Big\{(q,M)\in(\mathbb{R}^n\times\mathbb{R}^m_0)\times\mathrm{Sym}^{n+m}:\ \mathcal{P}(q,M)\succeq \mathrm{O}_m,\ \mathrm{Ker}(\mathcal{P}(q,M))\subseteq\mathrm{Ker}(\mathcal{Q}(q,M))\Big\}.$$

Remark 5.2. If $m=1$ then $\mathcal{P}(q,M)=0$, $\mathcal{Q}(q,M)=\mathrm{O}_{n\times 1}$, and hence

$$Z=(\mathbb{R}^n\times\mathbb{R}_0)\times\mathrm{Sym}^{n+1}.$$

If $m>1$, Z is a proper subspace of $(\mathbb{R}^n\times\mathbb{R}^m_0)\times\mathrm{Sym}^{n+m}$, and for any $(q,M)\in Z$, the matrix $\mathcal{P}(q,M)$ is invertible on $\mathrm{Im}\,(\mathcal{Q}^*(q,M))$ since $\mathrm{Im}\,(\mathcal{Q}^*(q,M))\subseteq(\mathrm{Ker}\,\mathcal{P}(q,M))^\perp$. Accordingly, we will write $\mathcal{P}(q,M)^{-1}\mathcal{Q}(q,M)^*$.

Theorem 5.3. *Let* $\mathbf{F}$ *be defined as in* (5.6)*. Then*

$$\widehat{F}(y,q,M)=\begin{cases} -C(y,q,M)+\mathrm{tr}\left[\mathcal{Q}(q,M)\mathcal{P}(q,M)^{-1}\mathcal{Q}(q,M)^*\right] & \text{if } (y,q,M)\in\mathbb{R}^m\times Z\\ +\infty & \text{otherwise.}\end{cases} \tag{5.8}$$

Proof. According to Definition 3.1 we must solve the minimum problem

$$\begin{aligned}
\widehat{F}(y,q,M) &= -\inf_{\Lambda(q,M)}\mathrm{tr}_{\mathbb{R}^n}[-q_{\mathrm{h}_\alpha}\mathcal{X}^\alpha]+\mathbf{f}(y)\cdot q_\mathrm{v}\\
&= -\inf_{P\,:\,q_\mathrm{h}+q_\mathrm{v}P=0}\mathrm{tr}[M_{\mathrm{hh}}+M_{\mathrm{hv}}P+(M_{\mathrm{hv}}P)^*+P^*M_{\mathrm{vv}}P]+\mathbf{f}(y)\cdot q_\mathrm{v}\\
&= -\mathrm{tr}(M_{\mathrm{hh}})-\mathbf{f}(y)\cdot q_\mathrm{v}-\inf_{P\,:\,q_\mathrm{h}+q_\mathrm{v}P=0}\mathrm{tr}[M_{\mathrm{hv}}P+(M_{\mathrm{hv}}P)^*+P^*M_{\mathrm{vv}}P].
\end{aligned} \tag{5.9}$$

We now observe that

$$P \in \Lambda(q,M) \quad \Longleftrightarrow \quad P = -\frac{1}{|q_{\rm v}|^2} q_{\rm v} \otimes q_{\rm h} + P', \quad P' \in {\rm M}^{m\times n}, \quad q_{\rm v} P' = {\rm O}_{1\times m}.$$

Hence (5.9) implies

$$\widehat{F}(y,q,M) = -C(y,q,M) - \inf\left\{h(q,M,P') \;:\; P' \in {\rm M}^{m\times n},\; q_{\rm v}P' = {\rm O}_{1\times m}\right\},$$

where

$$h(q,M,P') := {\rm tr}\left[2\left(M_{\rm hv} - \frac{1}{|q_{\rm v}|^2} q_{\rm h} \otimes q_{\rm v} M_{\rm vv}\right)P' + (P')^* M_{\rm vv} P'\right].$$

Now, if $(q,M) \notin Z$, the infimum of h is $-\infty$, since it is achieved at any P' consisting of n column vectors all equal to $\xi \in \mathbb{R}^m$ which satisfies

$$q_{\rm v} \cdot \xi = 0, \quad M_{\rm vv}\xi = 0, \quad \left(M_{\rm hv} - \frac{1}{|q_{\rm v}|^2} q_{\rm h} \otimes q_{\rm v} M_{\rm vv}\right)\xi \neq 0.$$

If $(q,M) \in Z$ then

$$\widehat{F}(y,q,M) = -C(y,q,M) - \inf\left\{{\rm tr}\left[2\mathcal{Q}(q,M)P' + (P')^*\mathcal{P}(q,M)P'\right] \;:\; P' \in {\rm M}^{m\times n}\right\},$$

and the conclusion follows since the minimum is achieved at $P' = -\mathcal{P}(q,M)^{-1}\mathcal{Q}(q,M)^*$. □

Corollary 5.4. *(i) Let $m = 1$ and F be defined as in* (5.6)*. Then*

$$\widehat{F}(y,q,M) = -\frac{{q_{\rm v}}^2 {\rm tr}(M_{\rm hh}) - 2q_{\rm v} M_{\rm vh} \cdot q_{\rm h} + |q_{\rm h}|^2 M_{\rm vv}}{{q_{\rm v}}^2} - f(y) q_{\rm v},$$

for any $(y,q,M) \in \mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}_0) \times {\rm Sym}^{n+1}$.

(ii) Let $n = 1$ and F be defined as in (5.6)*. Then*

$$\widehat{F}(y,q,M) = \begin{cases} -c(q,M) + \mathcal{Q}(q,M)\mathcal{P}(q,M)^{-1}\mathcal{Q}(q,M)^* & \text{if } (y,q,M) \in \mathbb{R}^m \times Z \\ +\infty & \text{otherwise,} \end{cases}$$

where

$$c(q,M) := M_{\rm hh} - 2\frac{q_{\rm h}}{|q_{\rm v}|^2} M_{\rm hv} q_{\rm v} + \frac{|q_{\rm h}|^2}{|q_{\rm v}|^4} q_{\rm v} M_{\rm vv} q_{\rm v} + \mathbf{f}(y) \cdot q_{\rm v}.$$

Proof. The assertions follow from Theorem 5.3 and Remark 5.2. □

Remark 5.5. From (i) of Corollary 5.4, it follows that if $m = 1$ and $f = 0$, the level set equation associated with equation (5.5) according to Definition 3.1 coincides with the level set heat equation studied by Evans in [12].

Proposition 5.6. *Let $n \geq 1$ and $m > 1$. The level set equation associated with* (5.5) *is given by*

$$w_t = {\rm tr}(w_{xx} - w_{xy} w_{yy}^{-1} w_{xy}^*) + \frac{\sum_{i=1}^n \left(w_{x_i} - w_y w_{yy}^{-1} w_{x_i y}^*\right)^2}{w_y w_{yy}^{-1} w_y^*} + \mathbf{f}(y) \cdot w_y, \tag{5.10}$$

provided that $w_{yy} \succ {\rm O}_m$ during the evolution.

Proof. We want to solve explicitly the minimum problem (5.9). Setting $P_i := Pe_i$ and introducing the Lagrange multipliers $\lambda_1, \dots, \lambda_n$, we must solve the linear systems

$$\begin{pmatrix} q_{\rm v} & 0 \\ M_{\rm vv} & q_{\rm v}^* \end{pmatrix} \begin{pmatrix} P_i \\ \lambda_i \end{pmatrix} = \begin{pmatrix} -q_{{\rm h}_i} \\ -M_{{\rm h}_i{\rm v}}^* \end{pmatrix}, \qquad i = 1, \dots, n. \tag{5.11}$$

The solution of (5.11) is given by

$$\begin{cases} \lambda_i \;=\; \dfrac{\det \begin{pmatrix} q_{\rm v} & -q_{{\rm h}_i} \\ M_{\rm vv} & -M_{{\rm h}_i{\rm v}}^* \end{pmatrix}}{\det \begin{pmatrix} q_{\rm v} & 0 \\ M_{\rm vv} & q_{\rm v}^* \end{pmatrix}} \;=\; \dfrac{q_{{\rm h}_i} - q_{\rm v} M_{\rm vv}^{-1} M_{{\rm h}_i{\rm v}}^*}{q_{\rm v} M_{\rm vv}^{-1} q_{\rm v}^*}, \\[2ex] P_i \;=\; -M_{\rm vv}^{-1}\left(\lambda_i q_{\rm v}^* + M_{{\rm h}_i{\rm v}}^*\right) \;=\; \dfrac{q_{\rm v} M_{\rm vv}^{-1} M_{{\rm h}_i{\rm v}}^* - q_{{\rm h}_i}}{q_{\rm v} M_{\rm vv}^{-1} q_{\rm v}^*}\, M_{\rm vv}^{-1} q_{\rm v}^* - M_{\rm vv}^{-1} M_{{\rm h}_i{\rm v}}^*, \end{cases} \tag{5.12}$$

provided that $\det M_{\rm vv} \neq 0$. Here we have used the formula

$$\det \begin{pmatrix} a & c \\ R & b^* \end{pmatrix} = (-1)^m (\det R)\left(c - aR^{-1}b^*\right),$$

where $c \in \mathbb{R}$, $a, b \in \mathbb{R}^m$ (considered as row vectors) and $R \in {\rm Sym}^m$, with $\det R \neq 0$. Now observe that P_i in (5.12) is a minimum point only if $M_{\rm vv}$ is positive definite. □

When $\mathbf{f}$ is a Lipschitz function, it is possible to identify explicit classical sub- and supersolution of the corresponding level set equation.

Proposition 5.7. *Assume that* $\mathbf{f}$ *is a Lipschitz function with constant* $L > 0$*, and let* $\mathbf{u} : (0,T)\times\mathbb{R}^n \to \mathbb{R}^m$ *be a classical solution of* (5.5)*. Then, the function* $z^+(t,x,y) := e^{Lt}|y - \mathbf{u}(t,x)|^2$ *is a classical supersolution of* (1.3)*, with* $\widehat{F}$ *as in* (5.8)*. Similarly, the function* $z^-(t,x,y) := e^{-Lt}|y - \mathbf{u}(t,x)|^2$ *is a classical subsolution of* (1.3)*.*

Proof. Let us consider $z := \frac{1}{2}z^+$. Recalling the definition of $\mathcal{P}$ in (5.7), we have

$$z_{yy} = e^{Lt}{\rm Id}_m \qquad \text{and} \qquad \mathcal{P}(\nabla z, \nabla^2 z) = e^{Lt}\Pi_{(z_y)^\perp},$$

and, from

$$z_{xy} = -e^{Lt}\nabla\mathbf{u}^* \tag{5.13}$$

and the definition of $\mathcal{Q}$ in (5.7),

$$\mathcal{Q}(\nabla z, \nabla^2 z) = -e^{Lt}\nabla\mathbf{u}^*\Pi_{(z_y)^\perp}.$$

Notice that $\mathcal{P}(\nabla z, \nabla^2 z) \succeq {\rm O}_m$ and ${\rm Ker}\left(\mathcal{P}(\nabla z, \nabla^2 z)\right) \subseteq {\rm Ker}\left(\mathcal{Q}(\nabla z, \nabla^2 z)\right)$, which implies that $\widehat{F}(y, \nabla z, \nabla^2 z)$ is real-valued and is given by (5.8). Hence,

$$[\mathcal{Q}\mathcal{P}^{-1}\mathcal{Q}^*](\nabla z, \nabla^2 z) = e^{Lt}(\nabla\mathbf{u})^*\Pi_{(z_y)^\perp}\nabla\mathbf{u},$$

and from

$$z_x = -e^{Lt}(y-u)\nabla\mathbf{u} = -z_y\nabla\mathbf{u} \tag{5.14}$$

we get

$${\rm tr}\{[\mathcal{Q}\mathcal{P}^{-1}\mathcal{Q}^*](\nabla z, \nabla^2 z)\} = {\rm tr}(\nabla\mathbf{u}^* z_{yy}\nabla\mathbf{u}) - e^{Lt}\frac{|z_x|^2}{|z_y|^2}. \tag{5.15}$$

Using (5.13) and (5.14) yields

$$z_{xy}z_y^* = e^{Lt}z_x^* \tag{5.16}$$

and thus

$$\begin{aligned} C(y,\nabla z,\nabla^2 z) &= \mathrm{tr}\Big(z_{xx} - \frac{2}{|z_y|^2} z_{xy}(z_y\otimes z_x)\Big) + e^{Lt}\frac{|z_x|^2}{|z_y|^2} + \mathbf{f}(y)\cdot z_y \\ &= \mathrm{tr}(z_{xx}) - e^{Lt}\frac{|z_x|^2}{|z_y|^2} + \mathbf{f}(y)\cdot z_y \end{aligned} \tag{5.17}$$

Furthermore, differentiating (5.14) and using (5.13) yields

$$z_{xx} = (\nabla \mathbf{u})^* z_{yy}\nabla\mathbf{u} - z_y\nabla^2\mathbf{u} \tag{5.18}$$

and from (5.8) and (5.15) we get

$$\begin{aligned} \widehat{F}(y,\nabla z,\nabla^2 z) &= -C(y,\nabla z,\nabla^2 z) + \mathrm{tr}\left(\mathcal{Q}(\nabla z,\nabla^2 z)\mathcal{P}(\nabla z,\nabla^2 z)^{-1}\mathcal{Q}(\nabla z,\nabla^2 z)^*\right) \\ &= -\mathrm{tr}\left(z_{xx}\right) - \mathbf{f}(y)\cdot z_y + \mathrm{tr}(\nabla\mathbf{u}^* z_{yy}\nabla\mathbf{u}) = \mathrm{tr}(z_y\nabla^2\mathbf{u}) - \mathbf{f}(y)\cdot z_y \\ &= \Delta\mathbf{u}\cdot z_y - \mathbf{f}(y)\cdot z_y = -\mathbf{F}(\mathbf{u},\nabla^2\mathbf{u})\cdot z_y + (\mathbf{f}(\mathbf{u}) - \mathbf{f}(y))\cdot z_y. \end{aligned} \tag{5.19}$$

Therefore, we get

$$\begin{aligned} z_t &= Lz + \mathbf{F}(\mathbf{u},\nabla^2\mathbf{u})\cdot z_y \\ &= Lz - \widehat{F}(y,\nabla z,\nabla^2 z) + (\mathbf{f}(\mathbf{u}) - \mathbf{f}(y))\cdot z_y \\ &\geq -\widehat{F}(y,\nabla z,\nabla^2 z), \end{aligned}$$

since $\mathbf{f}$ is L-Lipschitz. Similarly, one proceeds for z^-. □

Remark 5.8. We observe that, being $z_{yy}\succ O$, one could get (5.19) directly from Proposition 5.6 using (5.18), (5.14) and (5.16).

Notice that, if equation (1.3), with $\widehat{F}$ as in (5.8), admits existence of viscosity solutions satisfying a comparison principle, then Proposition 5.7 implies that we can recover the (graph of) classical solutions of (5.5) from the zero sub-level set of solutions of (1.3). Moreover, Proposition 5.7 could be considered as a first step towards the asymptotic analysis of solutions of the scaled Ginzburg–Landau systems, and towards a proof of their convergence to the mean curvature flow in arbitrary codimension, alternative to the energy–based approach followed in [2, 5].

5.4 Parametric curvature flow of planar curves

Parametric curvature flow of planar curves describes the evolution of curves by curvature, possibly with self-intersections which are not considered as singularities of the flow. The computations of this section show that such a flow can be described using the level set method. As we shall see, it turns out that the corresponding level set equation is rather involved (see (5.25)). As already observed in Remark

3.5, this is a case where the domain of $\widehat{F}$ is not the whole of $[0,+\infty) \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}_0^m \times \mathrm{Sym}^{1+m}$. Consider the following system of differential equations

$$\mathbf{u}_t - \frac{\nabla^2 \mathbf{u}}{|\nabla \mathbf{u}|^2} = 0, \tag{5.20}$$

for a function $\mathbf{u} : [0,T) \times \mathbb{R} \to \mathbb{R}^m$. In this case $\mathbf{F}$ reads as

$$\mathbf{F}(P, \mathcal{X}) = -\frac{\mathcal{X}}{|P|^2}, \qquad (P, \mathcal{X}) \in \left(\mathrm{M}^{m\times 1} \setminus \{\mathrm{O}_{m\times 1}\}\right) \times \mathcal{T}^{1\times 1\times m}.$$

Equation (5.20) says that the family $\{\mathbf{u}(t,\cdot)\}_t$ of curves in $\mathbb{R}^m$ evolves by curvature (see [7], [23]). Recall that, in this case, we have

$$\widehat{F}(q, M) = -\inf_{(1,P^*)\in q^\perp} \left\{ \frac{M(1,P^*)^* \cdot (1,P^*)}{|P|^2} \right\}, \qquad (q, M) \in \mathbb{R} \times \mathbb{R}_0^m \times \mathrm{Sym}^m, \tag{5.21}$$

which is singular on the m-dimensional subspace $\{q_{\mathrm{h}} = 0\} \subset \mathbb{R}^{1+m}$. Hence, it seems reasonable to consider $\widehat{F}$ defined as in (5.21) but restricted to the domain

$$\mathrm{dom}(\widehat{F}) = \mathbb{R}_0 \times \mathbb{R}_0^m \times \mathrm{Sym}^{1+m} =: \widehat{D}_0.$$

Remark 5.9. With computations similar to the case of the one-dimensional heat equation one checks that the graph $\Gamma_{\mathbf{u}(t,\cdot)}$ of $x \in \mathbb{R} \mapsto \mathbf{u}(t,x) \in \mathbb{R}^m$ evolves in $\mathbb{R}^{m+1}$ with the geometric law

$$\mathrm{V}^\perp = \frac{\kappa}{1-\tau_s^2}. \tag{5.22}$$

Following the approach of [12], it would be interesting to consider the solution of the following regularized version of (5.22),

$$\mathrm{V}^\perp = \frac{\kappa}{1-\tau_s^2 + \varepsilon^2 \tau_s^2},$$

corresponding to the regularized system

$$\mathbf{u}_t - \frac{\nabla^2 \mathbf{u}}{\varepsilon^2 + |\nabla \mathbf{u}|^2} = 0, \qquad \text{in } (0,+\infty) \times \mathbb{R},$$

and to the regularized level set equation

$$w_t + \widehat{F}^\varepsilon(\nabla w, \nabla^2 w) = 0, \qquad \text{in } (0,+\infty) \times \mathbb{R}^{1+m}, \tag{5.23}$$

with

$$\widehat{F}^\varepsilon(q, M) = -\inf_{(1,P^*)\in q^\perp} \left\{ \frac{M(1,P^*)^* \cdot (1,P^*)}{1+|P|^2} \frac{1+|P|^2}{\varepsilon^2+|P|^2} \right\},$$

for any $(q, M) \in \mathbb{R} \times \mathbb{R}_0^m \times \mathrm{Sym}^m$, and compare the limit as $\varepsilon \to 0$ of any solution w^ε of (5.23) with the evolution by curvature of $\{\mathbf{u}(t,\cdot)\}_t$.

We explicitly compute $\widehat{F}$ only for $m = 1$ and $m = 2$; the case $m = 2$ corresponds to the flow of planar curves.

5.4.1 The case $m = 1$

From Definition 3.1, we have for any $(q, M) \in \widehat{D}_0$

$$\begin{aligned}\widehat{F}(q,M) &= -\inf_{P\,:\,q_{\rm h}+q_{\rm v}P=0} \frac{M_{\rm hh}+2M_{\rm hv}P+P^*M_{\rm vv}P}{|P|^2}\\ &= -\frac{M_{\rm hh}{q_{\rm v}}^2-2M_{\rm hv}q_{\rm h}q_{\rm v}+M_{\rm vv}{q_{\rm h}}^2}{{q_{\rm h}}^2}. \end{aligned} \tag{5.24}$$

Equation (5.24) is the level set heat equation, exchanging the role of $q_{\rm h}$ and $q_{\rm v}$ (compare with Corollary 5.4 (i) with $f \equiv 0$). Indeed, if we assume that the function $y = u(t, x)$ is invertible for any $t \in [0, T)$, and we let $v(t, y) := u^{-1}(t, \cdot)(y)$ its inverse, then v solves the heat equation.

5.4.2 The case $m = 2$

To simplify notations we set $P = (P_1, P_2)$, $q_{\rm v} = (q_1, q_2)$,

$$M_{\rm vv} = \begin{pmatrix} a & c \\ c & b \end{pmatrix}, \qquad M_{\rm hv} = \begin{pmatrix} d & e \end{pmatrix}, \qquad M_{\rm hh} = f.$$

From Definition 3.1, we have for any $(q, M) \in \widehat{D}_0$

$$\widehat{F}(q,M) = -\inf_{(P_1,P_2):q_1P_1+q_2P_2+q_{\rm h}=0} \frac{aP_1^2+2cP_1P_2+bP_2^2+2dP_1+2eP_2+f}{P_1^2+P_2^2}.$$

Notice that $\widehat{F}(q_{\rm h}, q_1, q_2, M) = \widehat{F}(q_{\rm h}, q_2, q_1, M)$. Since $q_{\rm v} \neq 0$, without loss of generality, we can assume $q_2 \neq 0$ and set $P_2 = -\frac{q_{\rm h}+q_1P_1}{q_2}$. Then

$$\widehat{F}(q,M) = -\inf_{P_1\in\mathbb{R}} h(q,M,P_1),$$

where

$$h(q,M,P_1) := \frac{A'P_1^2+B'P_1+C'}{AP_1^2+BP_1+C}$$

and

$$\begin{aligned} A &:= q_1^2+q_2^2 > 0, & A' &:= aq_2^2-2cq_1q_2+bq_1^2,\\ B &:= 2q_{\rm h}q_1, & B' &:= 2(bq_{\rm h}q_1-cq_{\rm h}q_2+dq_2^2-eq_1q_2),\\ C &:= {q_{\rm h}}^2 > 0, & C' &:= b{q_{\rm h}}^2-2eq_{\rm h}q_2+fq_2^2. \end{aligned}$$

Notice that $AP_1^2 + BP_1 + C > 0$. Now, we have

$$\frac{d}{dP_1}h(q,M,P_1) = \frac{(A'B-AB')P_1^2+2(A'C-AC')P_1+B'C-BC'}{(AP_1^2+BP_1+C)^2}.$$

Since $E := (CA' - AC')^2 - (A'B - AB')(B'C - BC') \geq 0$, letting

$$P_\pm := \frac{AC'-A'C\pm\sqrt{E}}{A'B-AB'},$$

we conclude

$$\widehat{F}(q,M) = -\min\left\{\frac{A'}{A},\, h(q,M,P_-),\, h(q,M,P_+)\right\}, \qquad (q,M)\in\widehat{D}_0. \tag{5.25}$$

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