Convergence of a semidiscrete scheme for a forward-backward parabolic equation

Giovanni Bellettini [*] Carina Geldhauser [†] Matteo Novaga [‡]

Abstract

We study the convergence of a semidiscrete scheme for the forward-backward parabolic equation $u_t = (W'(u_x))_x$ with periodic boundary conditions in one space dimension, where W is a standard double-well potential. We characterize the equation satisfied by the limit of the discretized solutions as the grid size goes to zero. Using an approximation argument, we show that it is possible to flow initial data $\overline{u}$ having regions where $\overline{u}_x$ falls within the concave region $\{W'' < 0\}$ of W, where the backward character of the equation manifests. It turns out that the limit equation depends on the way we approximate $\overline{u}$ in its unstable region.

1 Introduction

In this paper we are interested in the existence of solutions to the gradient flow of the nonconvex and nonconcave functional

$$F(u) := \int_{\mathbb{T}} W(u_x)\ dx, \qquad W(p) := \frac{1}{4}(1-p^2)^2, \tag{1.1}$$

where $\mathbb{T}$ is the one-dimensional torus. The formal L^2-gradient flow of (1.1) leads to the forward-backward parabolic equation

$$u_t = (W'(u_x))_x \qquad \text{in } \mathbb{T} \times [0, +\infty), \tag{1.2}$$

that we couple with the initial condition

$$u(0) = \overline{u}.$$

As (1.2) is not well-posed due to the nonconvexity of W, it may fail to admit local in time classical solutions, at least for a large class of initial data $\overline{u}$. A typical source of instability is, for example, the case when there are intervals $I \subset \mathbb{T}$ for which

$$\overline{u}_x(x) \in (p^-, p^+), \qquad x \in I, \tag{1.3}$$

[*]Dipartimento di Matematica, Università di Roma Tor Vergata, via della Ricerca Scientifica 1, 00133 Roma, Italy, and LNF-INFN, via E. Fermi 40, 00044 Frascati, Italy. e-mail: `Giovanni.Bellettini@lnf.infn.it`

[†]Institut für Angewandte Mathematik, Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany. email: `geldhauser@iam.uni-bonn.de`

[‡]Dipartimento di Matematica, Università di Padova, via Trieste 63, 35121 Padova, Italy. email: `novaga@math.unipd.it`

where

$$(p^-, p^+) = \{W'' < 0\} \tag{1.4}$$

is the concave region of W (in our case $p^- = -1/\sqrt{3}$ and $p^+ = 1/\sqrt{3}$). Indeed, under these conditions the backward character of the equation manifests and instabilities, such as the quick formation of microstructures, are expected, making the subsequent evolution difficult to describe.

It is therefore natural to approximate (1.2) with a regularized equation, and indeed different regularizations have been proposed in the literature. We recall in particular [14], where the author considers the regularization

$$u_t = \varepsilon u_{txx} + (W'(u_x))_x \,,$$

and proves convergence as $\varepsilon \to 0^+$ to a measure-valued solution to (1.2). In [10] (see also [16, 17] and references therein) further properties of such limit solutions are discussed.

We also mention that in [7] a fourth order regularization for another forward-backward parabolic equation (the Perona-Malik equation) is suggested, which in our case reads as

$$u_t = -\varepsilon u_{xxxx} + (W'(u_x))_x \,. \tag{1.5}$$

The dynamics of this regularization as $\varepsilon \to 0^+$ is quite involved and was studied in [15, 3, 2] (notice that setting $u_x = v$, equation (1.5) becomes the Cahn-Hilliard equation by differentiation). In particular, in [2] the authors are able to pass to the limit in (1.5) as $\varepsilon \to 0^+$, under the assumption that the initial data (which now depend also on ϵ) converge to $\overline{u}$ in a suitable energetic sense.

Another possible way to regularize (1.2) is by approximation with a semidiscrete scheme (see [12]), and this is the approach that we shall adopt in this paper. More precisely, let us consider the semidiscrete scheme

$$\begin{cases} \dfrac{du^h}{dt} = D_h^+ \, W'(D_h^- u^h) & \text{in } \mathbb{T} \times [0, +\infty), \\ \\ u^h(0) \, = \, \overline{u}^h & \text{on } \mathbb{T}, \end{cases} \tag{1.6}$$

where $h > 0$ denotes the grid size on the torus $\mathbb{T}$, $D_h^\pm$ are the difference quotients defined in (3.2), and $\overline{u}^h$ are the piecewise linear discrete initial data, with $\overline{u}^h \to \overline{u}$ in $L^\infty(\mathbb{T})$ as $h \to 0^+$. The spatial discretization acts as a regularization of the ill-posed problem (1.2), and regular solutions u^h to (1.6) indeed exist for all times. Moreover, it has been shown in [12] that one can pass to the (full) limit in such solutions, as $h \to 0^+$, when the gradients of $\overline{u}$ and $\overline{u}^h$ lie in the region $\{W'' \geq 0\}$ (namely, the complement of the interval $(-1/\sqrt{3}, 1/\sqrt{3})$ appearing in (1.4), compare Figure 1), with possible jumps from one connected component to another[1] (see also [4, 13]).

The aim of this paper is to show convergence of (1.6) for a larger class of initial data, namely those satisfying

$$|\overline{u}_x(x)| \leq M^+ := 2/\sqrt{3}, \qquad x \in \mathbb{T}, \tag{1.7}$$

[1] The solution found in [12] is such that $W'(u_x(\cdot, t))$ is continuous at the points where there is a jump from one connected component to another, namely where $u_x(\cdot, t)$ has a discontinuity, and these points, differently to what is expected [3] from the limits of (1.5), do not move in the horizontal direction.

(see (2.1) for the definition[2] of M^+. In particular, we allow for initial data $\overline{u}$ satisfying (1.3). We believe this to be a significant improvement with respect to the previous results, for the above mentioned reason that these data quickly lead to instabilities and formation of microstructures.

One of the main ideas of the present paper is to consider the initial datum $\overline{u}$ endowed with an auxiliary function $\overline{\varrho} \in L^\infty(\mathbb{T}; [0,1])$ measuring the percentage of mesh intervals where $\overline{u}^h$ is decreasing in a neighborhood of the point $x \in \mathbb{T}$, see Definition 4.1. The functions $\overline{u}$ and $\overline{\varrho}$ are not independent, since they must satisfy the compatibility condition (4.1). The reason for introducing $\overline{\varrho}$ is that it allows to construct a sequence $(\overline{u}^h)$ (depending on $\overline{\varrho}$) and converging to $\overline{u}$ as $h \to 0^+$, with the crucial property that the discrete gradients $D_h^- \overline{u}^h$ always belong to $\{W'' \geq 0\}$ (see Section 4.2). This requirement is important since, under assumption (1.7), such a property is preserved by the semidiscrete scheme (1.6), as stated in (3.10). On the other hand, it is clear that this amounts to a restriction on the sequence of approximating initial data $\overline{u}^h$ that we are able to consider. We observe however that the requirement $D_h^- \overline{u}^h(x) \in \{W'' \geq 0\}$ for any $x \in \mathbb{T}$, is rather natural: indeed, given any initial condition $\overline{u}$, numerical experiments [11, 3] give evidence that the slope of the discrete solution, due to the quick formation of microstructures, takes values in this region after a very short time.

Summarizing, we can say that our initial conditions can be represented by a pair $(\overline{u}, \overline{\varrho})$ verifying (4.1). Observe that, given a function $\overline{u}$ satisfying (1.7), there always exists a function $\overline{\varrho}$ satisfying the required assumptions. We also notice that the function $\overline{\varrho}$ is not uniquely determined in general.

Under these hypotheses, we show in Theorem 6.7 that the sequence (u^h) of solutions to the semidiscrete scheme has enough compactness properties to pass to the limit. This crucial compactness cannot, however, be obtained for the sequence $(D_h^- u^h)$ of discrete gradients. Indeed, due to the instability of (1.2), u^h have oscillations which are typically of order h when $D_h^- u^h$ belongs to the nonconvex region of W, so that there is no hope to have strong convergence of $D_h^- u^h$ as $h \to 0^+$. Rather, it is possible to obtain a compactness property for $W'(D_h^- u^h)$: indeed, in Proposition 5.3 and Corollary 5.4 we show a uniform Hölder estimate for the sequence $\left(W'\left(D_h^- u^h\right)\right)$. However, these estimates are not enough for passing to the limit in the nonlinear term of the equation; the crucial point is then to gain a compactness property for the *averaged discrete gradients*

$$D_{n_i h}^- u^h \tag{1.8}$$

on an intermediate grid of size $n_i h$, where n_i is a suitable positive integer related to the values of $\overline{\varrho}$: see Definition 4.6 for the details. Therefore, we can say that introducing the function $\overline{\varrho}$ allows to identify an intermediate (or mesoscopic) scale, which in turn permits to obtain a compactness for the averaged gradients (1.8). Our conclusion (Theorems 6.9 and 6.14) is that the function $u := \lim_h u^h$ solves distributionally the limit problem

$$u_t = \left(W'\left(q\left(u_x, \overline{\varrho}\right)\right)\right)_x \tag{1.9}$$

[2]As we shall see, assumption (1.7) is necessary in order to approximate $\overline{u}$ with discrete initial data $\overline{u}^h$ such that the solutions u^h have discrete gradients $D_h^- u^h$ lying everywhere in the region $\{W'' \geq 0\}$ for all times $t \geq 0$.

where the map q, which depends on $\overline{\varrho}$, is defined in Definition 3.6, and is related to the two stable branches of the local inverse of W'. No extraction of a subsequence is necessary in Theorems 6.9 and 6.14 (see Remark 6.10). We observe also that this result can be considered as a characterization of a (nonunique) selection among the infinitely many Young measure solutions to (1.2), obtained through the semidiscrete scheme (1.6).
The case when $\overline{u}$ is such that $\overline{u}_x(x) \in \{W'' \geq 0\}$ for any $x \in \mathbb{T}$ can be covered by taking either $\overline{\varrho} \equiv 0$ or $\overline{\varrho} \equiv 1$ in $\mathbb{T}$, and equation (1.9) reduces to equation (1.2) (well posed, in this situation)[3]. The particular case considered in [12] corresponds to a choice of $\overline{\varrho} \in L^\infty(\mathbb{T}; \{0,1\})$, which is possible only if the initial datum $\overline{u}$ avoids the nonconvex region (p^-, p^+) of W.

One may expect that a generic sequence of initial data $\overline{u}^h$ defines, up to subsequences, a limit function $\overline{u}$ and a function $\overline{\varrho} \in L^\infty(\mathbb{T}; [0,1])$, such that the corresponding function $u := \lim_h u^h$ is a solution of (1.9) with initial datum $\overline{u}$. Such an extension would require significant modifications to our proofs, and goes beyond the scope of this paper. However, we believe that it is already interesting to consider a specific choice of $\overline{\varrho}$, since it gives rise to a well-defined solution to a limit evolution problem, for a large class of initial data.
We stress that our result shows that the evolution law obtained as a limit of the semidiscrete scheme is not unique in general, and depends on the sequence $(\overline{u}^h)$ of initial data chosen to approximate $\overline{u}$. We also notice that there does *not* seem to be a choice of $\overline{\varrho}$ which reproduces the solution numerically observed in [3], for the limit of equation (1.5) as $\epsilon \to 0$.
Up to minor modifications, the results of this paper still hold if we replace W in (1.1) with any smooth enough double-well potential with similar qualitative (in particular convexity/concavity) properties.

The plan of the paper is the following. In Section 2 we introduce the map T, leading to the definition of the function

$$q(p, \sigma)$$

used in (1.9), compare also (4.2). In Section 3 we introduce the semidiscrete scheme (1.6) for equation (1.2). In Section 4 we define the class of initial data for which we are able to show convergence of the semidiscrete scheme. In Section 5 we prove the apriori estimates which give compactness of the discrete solutions. Finally, in Section 6 we pass to the limit in the discretized equation (1.6) and characterize the limit evolution law. For simplicity of presentation, the proof will be given first for a piecewise constant function $\overline{\varrho}$ taking rational values in $[0,1]$, and then for a general $\overline{\varrho}$.

2 Unstable slopes and the map T

In this section we introduce a map T which will be of crucial importance in our notion of solution to equation (1.2).
Recall our notation: we have $W'(p) = p^3 - p$, and the concave region of W is the interval

$$(p^-, p^+) := \left\{ p \in \mathbb{R} : W''(p) < 0 \right\},$$

[3] Notice that also in this case, which is the simplest possible, since there are infinitely many choices of $\overline{\varrho}$ (different from $\overline{\varrho} \equiv 0$ or $\overline{\varrho} \equiv 1$), one gets infinitely many possibile different limit equations.

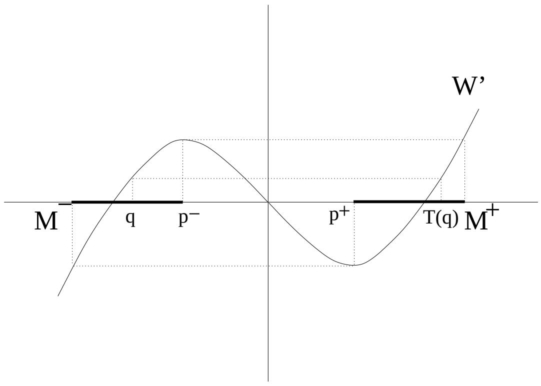

Figure 1: Graph of the function W'. In bold the set $[M^-,M^+]\setminus(p^-,p^+)$. The point $\mathrm{T}(q)$ for $q\in[M^-,M^+]\setminus(p^-,p^+)$.

which represents the set of "unstable" slopes. In our specific case we have

$$-p^- = p^+ = \frac{1}{\sqrt{3}}.$$

If $q\in(p^-,p^+)$ then $W'^{-1}(W'(q))$ consists of three distinct elements (one belonging to the unstable branch of the local inverse of W', and the other two belonging to the stable branches), and if $q\in\{p^-,p^+\}$ then $W'^{-1}(W'(q))$ reduces to two distinct elements, that we write as follows:

$$W'^{-1}\big(W'(p^-)\big)=\{p^-,M^+\},\qquad W'^{-1}\big(W'(p^+)\big)=\{p^+,M^-\},\tag{2.1}$$

where $p^- < M^+$ and $M^- < p^+$, see Figure 1. With our choice of W, we have $W'(p^-) = -W'(p^+) = \frac{2}{3\sqrt{3}}$,

$$-M^- = M^+ = \frac{2}{\sqrt{3}}.$$

We now introduce the map T on the disconnected set $[M^-,M^+]\setminus(p^-,p^+)$ corresponding to the two stable branches of the local inverse of W'.

Definition 2.1 (**The function** T)**.** *We define the continuous function*

$$\mathrm{T}:[M^-,M^+]\setminus(p^-,p^+)\to[M^-,M^+]\setminus(p^-,p^+)$$

as follows: given $q\in[M^-,M^+]\setminus(p^-,p^+)$, $\mathrm{T}(q)$ *is the unique point in* $[M^-,M^+]\setminus(p^-,p^+)$ *different from* q*, such that*

$$W'(q)=W'(\mathrm{T}(q)).$$

The map T satisfies

$$\mathrm{T}\circ\mathrm{T}=\mathrm{id}.$$

Furthermore, it is strictly increasing, smooth on the interior of its domain,

$$q\in(p^+,M^+)\Rightarrow\mathrm{T}(q)\in(M^-,p^-),\qquad \mathrm{T}(p^+)=M^-,\ \mathrm{T}(M^+)=p^-,$$

$$q\in(M^-,p^-)\Rightarrow\mathrm{T}(q)\in(p^+,M^+),\qquad \mathrm{T}(M^-)=p^+,\ \mathrm{T}(p^-)=M^+,$$

and $\lim_{q\downarrow M^-} T'(q) = +\infty$, $\lim_{q\uparrow p^-} T'(q) = 0$.

Definition 2.2 (**Local unstable region of** u)**.** *Given a Lipschitz function* $u : \mathbb{T} \to \mathbb{R}$ *and a point* $x \in \mathbb{T}$ *where* u *is differentiable, we write*

$$x \in \Sigma_L(u)$$

if $u_x(x) \in (p^-, p^+)$*. We call* $\Sigma_L(u)$ *the local unstable region of* u*.*

3 Semidiscrete scheme

In what follows, given $h > 0$ sufficiently small, we assume $\mathbb{T}$ to be discretized with a grid of N_h subintervals of equal length h.
$PL_h(\mathbb{T})$ is the N_h-dimensional vector subspace of $\mathrm{Lip}(\mathbb{T})$ of all piecewise linear functions defined on the grid. $PC_h(\mathbb{T})$ is the N-dimensional vector subspace of $L^2(\mathbb{T})$ of all left-continuous piecewise constant functions on the grid.
Given $u \in PL_h(\mathbb{T})$ (resp. $u \in PC_h(\mathbb{T})$) we denote with $u_1, \ldots, u_{N_h}$ the coordinates of u with respect to the basis of the hat (resp. flat) functions, and $u \in PL_h(\mathbb{T})$ will be identified with $(u_1, \ldots, u_{N_h}) \in \mathbb{R}^{N_h}$, where $u_i := u(ih)$, $i = 1, \ldots, N_h$, and $u_0 := u_{N_h}$. $PL_h(\mathbb{T})$ is endowed with the norm

$$\|u\|^2_{PL_h(\mathbb{T})} := h \sum_{i=1}^{N_h} (u_i)^2. \tag{3.1}$$

Notice that on $PL_h(\mathbb{T})$ the norms $\|\cdot\|_{PL_h(\mathbb{T})}$ and $\|\cdot\|_{L^2(\mathbb{T})}$ are equivalent, since a direct computation gives

$$\|u\|_{L^2(\mathbb{T})} \le \|u\|_{PL_h(\mathbb{T})} \le \sqrt{\frac{3}{2}}\, \|u\|_{L^2(\mathbb{T})}, \qquad u \in PL_h(\mathbb{T}).$$

Moreover these two norms coincide on $PC_h(\mathbb{T})$.
We define the linear maps $D_h^{\pm} : PL_h(\mathbb{T}) \to PC_h(\mathbb{T})$ as

$$(D_h^- u)_i = \frac{1}{h}(u_i - u_{i-1}), \qquad (D_h^+ u)_i = \frac{1}{h}(u_{i+1} - u_i), \qquad i \in \{1, \ldots, N_h\}, \tag{3.2}$$

where $u_{N_h+1} := u_1$. Notice that $(D_h^- u)_{i+1} = (D_h^+ u)_i$. In addition, the following discrete integration by parts formula holds for functions $u, v \in PL_h(\mathbb{T})$ (hence with $u_0 = u_{N_h}$ and $v_0 = v_{N_h}$):

$$\sum_{i=1}^{N_h} (D_h^- u)_i v_i = -\sum_{i=1}^{N_h} u_i (D_h^+ v)_i. \tag{3.3}$$

It is clear that if $u \in PL_h(\mathbb{T})$ and $i \in \{1, \ldots, N_h\}$, then $(D_h^- u)_i = u_x$ in the interval $((i-1)h, ih)$.

3.1 The semidiscrete scheme: the system of ODEs

The restriction F_h of F to $PL_h(\mathbb{T})$ is a smooth function of N variables and reads as

$$F_h(u) = h\sum_{i=1}^{N_h} W\left((D_h^- u)_i\right) = h\sum_{i=1}^{N_h} W\left(\frac{u_i - u_{i-1}}{h}\right), \qquad u \in PL_h(\mathbb{T}).$$

The $L^2(\mathbb{T})$-gradient flow of F_h (or equivalently the gradient flow of F_h with respect to the scalar product producing the norm $\|\cdot\|_{PL_h(\mathbb{T})}$ in (3.1)) on $PL_h(\mathbb{T})$ is expressed by

$$\frac{du_i}{dt} = -\frac{1}{h}\frac{\partial F_h}{\partial u_i} = \frac{1}{h}\left\{W'\left(\frac{u_{i+1}-u_i}{h}\right) - W'\left(\frac{u_i - u_{i-1}}{h}\right)\right\} = \left(D_h^+ W'(D_h^- u)\right)_i$$

for any $i \in \{1, \ldots, N_h\}$, with the periodicity conditions $u_0 = u_{N_h}$, $u_1 = u_{N_h+1}$.
As already said in the introduction, we will be interested in the asymptotic limit, as $h \downarrow 0$, of space-periodic solutions u^h to the system of ordinary differential equations (1.6) with initial condition $\overline{u}^h$ (allowing initial conditions depending on h will be crucial).
The following two theorems taken from [12, Theorem 2.1 and Prop. 2.3] are the starting point of our analysis. Given a function $v : \mathbb{T} \times [0, +\infty) \to \mathbb{R}$ and $t \in [0, +\infty)$, we let $v(t)$ be the function on $\mathbb{T}$ defined as $v(t)(x) := v(x, t)$.

Theorem 3.1 (Properties of u^h). *Let $\overline{u}^h \in PL_h(\mathbb{T})$ and suppose that*

$$\sup_h \|D_h^- \ \overline{u}^h\|_{L^\infty(\mathbb{T})} =: C < +\infty. \tag{3.4}$$

Then there exists a unique solution $u^h \in \mathcal{C}^\infty([0, +\infty); PL_h(\mathbb{T}))$ of (1.6) and

$$\|D_h^- u^h(t)\|_{L^\infty(\mathbb{T})} \leq \max\Big(C, M^+\Big), \qquad t \in [0, +\infty). \tag{3.5}$$

Moreover

$$\frac{d}{dt}F_h(u^h(t)) = -\left\|\frac{du^h}{dt}\right\|^2_{PL_h(\mathbb{T})} \leq 0, \qquad t \in (0, +\infty). \tag{3.6}$$

Note in particular[4] that $u^h \in AC_2\left([0, +\infty); L^2(\mathbb{T})\right) \subset \mathcal{C}^{\frac{1}{2}}\left([0, +\infty); L^2(\mathbb{T})\right)$, and

$$\int_{\mathbb{T}} u^h(x, t)\, dx = \int_{\mathbb{T}} \overline{u}^h(x)\, dx, \qquad t \in [0, +\infty). \tag{3.7}$$

From (3.5) it follows that the Lipschitz constant of the solution u^h is conserved, provided it is larger[5] than M^+.

Remark 3.2 (Uniform L^∞-bound). Despite the maximum and minimum principles on u^h in general are not valid, from (3.7) and (3.5) it follows that, if (3.4) holds, then

$$\sup_h \|u^h\|_{L^\infty(\mathbb{T}\times(0,+\infty))} < +\infty. \tag{3.8}$$

[4] $AC_2([0, +\infty); L^2(\mathbb{T}))$ denotes the space of absolutely continuous functions from $[0, +\infty)$ to $L^2(\mathbb{T})$ with derivative in L^2, see [1].

[5] Small Lipschitz constants are not preserved, due to the formation of microstructures in correspondence of the concave region of W.

Theorem 3.3 (Preserving avoidance of $\Sigma_L(u^h)$). *If $\overline{u}^h \in PL_h(\mathbb{T})$ satisfies*

$$p^+ \leq \left|(D_h^- \overline{u}^h)_j\right| \leq M^+, \qquad j \in \{1, \dots, N_h\}, \tag{3.9}$$

and if $i \in \{1, \dots, N_h\}$, then

$$\begin{aligned} p^+ \leq (D_h^- \overline{u}^h)_i \leq M^+ \quad &\Rightarrow \quad p^+ \leq (D_h^- u^h(t))_i \leq M^+, \quad t \in [0, +\infty), \\ M^- \leq (D_h^- \overline{u}^h)_i \leq p^- \quad &\Rightarrow \quad M^- \leq (D_h^- u^h(t))_i \leq p^-, \quad t \in [0, +\infty). \end{aligned} \tag{3.10}$$

In particular

$$p^+ \leq \left|(D_h^- u^h(t))_j\right| \leq M^+, \qquad j \in \{1, \dots, N_h\},\ t \in [0, +\infty). \tag{3.11}$$

Therefore, if the initial data $\overline{u}^h$ have slopes which avoid[6] the concave region (p^-, p^+) of W then, as a consequence of (3.11), the same property is shared by the discrete solutions u^h, namely

$$\Sigma_L(\overline{u}^h) = \emptyset \quad \Rightarrow \quad \Sigma_L(u_h(t)) = \emptyset, \qquad t \geq 0. \tag{3.12}$$

We will make repeated use of this fact in the sequel (for instance in the proof of Theorem 6.7).

3.2 A constrained problem on slopes

In this section we formalize the idea of expressing a "macroscopic" gradient (denoted below by p) at a given point $x \in \mathbb{T}$ with a percentage σ of, say, negative "microscopic" gradient (indicated below by q, see also (3.16)) and a remaining percentage $1-\sigma$ of positive "microscopic" gradient (indicated below by $\mathrm{T}(q)$, where the map T is introduced in Definition 2.1). We stress that, by construction, q and $\mathrm{T}(q)$ are required not to lie in the concave region (p^-, p^+) of W. At the end of the procedure, the main concept will be the map $q(p, \sigma)$ in Definition 3.6, which will be crucial in order to "prepare" the initial datum $\overline{u}$ using an approximating sequence $(\overline{u}^h)$. We also anticipate here that in order to prepare $\overline{u}$, we will need to constrain the values of p (see (3.15)), and this will be source of a restriction on our initial datum.

Let $p \in \mathbb{R}$ and $\sigma \in [0, 1]$ be given. Let us consider the following problem: find q such that

$$q \in [M^-, M^+] \setminus (p^-, p^+) \tag{3.13}$$

and

$$\sigma q + (1 - \sigma)\mathrm{T}(q) = p. \tag{3.14}$$

It is not difficult to prove that if (3.14) admits a solution q, then the set $\{q, \mathrm{T}(q)\}$ is unique; namely, if q_1 and q_2 are two solutions of (3.14), i.e.,

$$p = \sigma q_1 + (1 - \sigma)\mathrm{T}(q_1) = \sigma q_2 + (1 - \sigma)\mathrm{T}(q_2),$$

[6] The functions $\overline{u}^h$ will be approximations of $\overline{u}$ which, on the contrary, has in general a *nonempty* local unstable region.

then either $q_2 = q_1$ or $q_2 = \mathrm{T}(q_1)$. Indeed, if $\sigma \in \{0,1\}$ this assertion is obvious. Assume then $\sigma \in (0,1)$. Since q_1 and $\mathrm{T}(q_1)$ belong to different connected components of $[M^-, M^+] \setminus (p^-, p^+)$ (and the same holds for q_2 and $\mathrm{T}(q_2)$), up to exchanging q_1 with $\mathrm{T}(q_1)$ and q_2 with $\mathrm{T}(q_2)$ we can assume that $q_1, q_2 \in [p^+, M^+]$. Assume by contradiction that $q_1 \neq q_2$. Supposing without loss of generality that $q_1 < q_2$ we get, recalling that T is increasing,

$$0 > q_1 - q_2 = \frac{1-\sigma}{\sigma}(\mathrm{T}(q_2) - \mathrm{T}(q_1)) > 0,$$

which is absurd.
We now come to the existence of a solution to equations (3.13) and (3.14), noticing that they are not always solvable: for instance, if $\sigma = 1$, then $q = p$ which implies that the problem does not have a solution unless $p \in [M^-, M^+] \setminus (p^-, p^+)$. Similarly, if $\sigma = 0$, then $\mathrm{T}(q) = p$ which again implies that the problem has not a solution, unless $p \in [M^-, M^+] \setminus (p^-, p^+)$.

Example 3.4. Let $\sigma = 1/2$. Then equation (3.14) gives, for q as in (3.13),

$$\frac{1}{2\sqrt{3}} = \frac{p^- + M^+}{2} \geq \frac{q + \mathrm{T}(q)}{2} = p \geq \frac{p^+ + M^-}{2} = -\frac{1}{2\sqrt{3}},$$

which constraints p to $|p| \leq \frac{1}{2\sqrt{3}}$.

Definition 3.5 (The set-valued map G). *Given $\sigma \in [0,1]$, we set*

$$G(\sigma) := [\sigma M^- + (1-\sigma)p^+, \sigma p^- + (1-\sigma)M^+].$$

For $q \in [p^-, M^-]$, as a consequence of the fact that the function

$$q \in [p^-, M^-] \to \sigma q + (1-\sigma)\mathrm{T}(q)$$

is increasing, we deduce (generalizing Example 3.4) that the inclusion

$$p \in G(\sigma) \tag{3.15}$$

implies that (3.13), (3.14) admit a unique solution $\{q, \mathrm{T}(q)\}$.

We are now in a position to give the following definition[7].

Definition 3.6 (The function q). *Let $p \in G(\sigma)$ and let $\{q, \mathrm{T}(q)\}$ be the solution to* (3.13) *and* (3.14). *We set*

$$q(p,\sigma) := \min\{q, \mathrm{T}(q)\} < 0. \tag{3.16}$$

Observe that $q(p,1) = p$ and $q(p,0) = \mathrm{T}(p)$.

[7]Our results still hold with obvious modifications if in place of min we take max in (3.16). For instance, substitute the sentence prc-percentage of negative slopes with prc-percentage of positive slopes in Definition 4.1.

Remark 3.7 (**Regularity of the function** $q(\cdot,\sigma)$**).** For all $\sigma \in [0,1]$, the function $q(\cdot,\sigma) : G(\sigma) \to [M^-, p^-]$ satisfies

$$q(\cdot,\sigma) \in \mathcal{C}^\infty(\mathrm{int}(G(\sigma))) \cap \mathcal{C}^0(G(\sigma))$$

and

$$q_p(\cdot,\sigma) > 0 \qquad \text{in } \mathrm{int}(G(\sigma)),$$

where $\mathrm{int}(G(\sigma)) = (\sigma M^- + (1-\sigma)p^+, \sigma p^- + (1-\sigma)M^+)$. In addition, differentiating (3.14) with respect to p and taking $\sigma \in (0,1)$ yields $q_p(p,\sigma) = \frac{1}{\sigma + (1-\sigma)\mathrm{T}'(q(p,\sigma))}$, and

$$\text{for } \sigma \neq 1 \qquad \lim_{p \downarrow \min G(\sigma)} q_p(p,\sigma) = 0,$$

$$\text{for } \sigma \neq 0 \qquad \lim_{p \uparrow \max G(\sigma)} q_p(p,\sigma) = \frac{1}{\sigma},$$

and $\lim_{p \uparrow \max G(1)} q_p(p,1) = +\infty$.

4 Admissible initial data

The instability of (1.2) implies that the limit of (u^h) as $h \to 0^+$ could depend on the sequence $(\overline{u}^h)$ chosen in (1.6) for approximating $\overline{u}$ in $L^\infty(\mathbb{T})$. In this section we specify which kind of sequences of initial data we will consider. Let us start with the following definition, which fixes our compatible initial data $\overline{u}$. $\mathbb{Q} \subset \mathbb{R}$ denotes the set of rational numbers.

Definition 4.1 (**The class** $\mathcal{D}$ **and the percentage** $\overline{\varrho}$**).** *We say that a function* $\overline{u}$ *belong to the class* $\mathcal{D}$ *if the following properties hold:*

(i) $\overline{u} \in \mathrm{Lip}(\mathbb{T})$*;*

(ii) there exists a function $\overline{\varrho} \in L^\infty(\mathbb{T};[0,1])$ *such that* $\overline{\varrho}(\mathbb{T})$ *is a finite subset of* $\mathbb{Q}$ *and* $\overline{\varrho}^{-1}(\alpha)$ *is a (nontrivial) subinterval of* $\mathbb{T}$ *for all* $\alpha \in \overline{\varrho}(\mathbb{T})$*, and*

$$\overline{u}_x(x) \in G(\overline{\varrho}(x)), \qquad \text{a.e. } x \in \mathbb{T}. \tag{4.1}$$

The function $\overline{\varrho}$ *will be called a piecewise constant rational percentage (pcr-percentage for short) of negative slopes for* $\overline{u}$*.*

Notice that from (4.1) it follows that $\mathrm{lip}(\overline{u}) \leq M^+$.

Remark 4.2. From our definitions it follows that, given $\overline{u} \in \mathcal{D}$ and a pcr-percentage $\overline{\varrho}$ of negative slopes for $\overline{u}$, for almost every $x \in \mathbb{T}$ we have

$$q(\overline{u}_x(x), \overline{\varrho}(x)) < 0, \qquad \overline{\varrho}(x) q\left(\overline{u}_x(x), \overline{\varrho}(x)\right) + (1-\overline{\varrho}(x))\mathrm{T}\left(q(\overline{u}_x(x), \overline{\varrho}(x))\right) = \overline{u}_x(x). \tag{4.2}$$

Example 4.3. Let $\overline{u} \in \mathrm{Lip}(\mathbb{T})$ satisfy the following property: there exists a finite set of points of $\mathbb{T}$ such that if I is a connected component of the complement of this set, then $\overline{u}_{|\overline{I}} \in \mathcal{C}^1(\overline{I})$, and

$$\text{either} \quad \overline{u}_x(I) \subset (p^-, p^+) \quad \text{or} \quad \overline{u}_x(I) \subset [M^-, p^-] \quad \text{or} \quad \overline{u}_x(I) \subset [p^+, M^+]. \tag{4.3}$$

Then $\overline{u} \in \mathcal{D}$. Indeed, it is enough to choose

$$\overline{\varrho} : \mathbb{T} \to \left\{0, \frac{1}{2}, 1\right\}$$

as follows: if $x \in \mathbb{T}$ is a differentiability point of u,

$$\overline{\varrho}(x) := \begin{cases} 1 & \text{if } \overline{u}_x(x) \subset [M^-, p^-], \\ \frac{1}{2} & \text{if } \overline{u}_x(x) \subset (p^-, p^+), \\ 0 & \text{if } \overline{u}_x(x) \subset [p^+, M^+]. \end{cases}$$

Then, remembering that $G(0) = [p^+, M^+]$, $G(\frac{1}{2}) = \frac{1}{2}[M^- + p^+, p^- + M^+]$, and $G(1) = [M^-, p^-]$, it is immediate to see that, in view of (4.3), condition (4.1) is satisfied. Note that, in this case, $\mathrm{lip}\Big(\overline{u}_{|\Sigma_L(\overline{u})}\Big) \leq \frac{p^- + M^+}{2}$.

Remark 4.4. The condition in (ii) on the structure of a prc-percentage $\overline{\varrho}$ is a technical assumption used in the proof of Theorem 6.9. Such an assumption will be removed in Theorem 6.14. We recall also that condition (4.1) is needed to construct the function $q(p, \sigma)$ illustrated in Section 3.2, applied with the choice $p = \overline{u}(x)$ and $\sigma = \overline{\varrho}(x)$.
Notice that

$$\{v \in \mathcal{C}^1(\mathbb{T}) : \mathrm{lip}(v) \leq M^+\} \subset \mathcal{D} \subset \{v \in \mathrm{Lip}(\mathbb{T}) : \mathrm{lip}(v) \leq M^+\},$$

with dense inclusions with respect to the $L^\infty(\mathbb{T})$-norm.

4.1 Admissible sequences of initial data

We now specify the approximating sequences $(\overline{u}^h)$ that we will be consider in the semidiscrete scheme (1.6).

Definition 4.5 (Admissible sequences). *Let $\overline{u} \in \mathcal{D}$. We say that the sequence $(\overline{u}^h)$ is admissible for $\overline{u}$ if the following properties hold:*

- $\overline{u}^h \in PL_h(\mathbb{T})$;
- $\mathrm{lip}(\overline{u}^h) \leq M^+$;
- $\Sigma_L(\overline{u}^h) = \emptyset$;
- $\lim_{h \to 0^+} \|\overline{u}^h - \overline{u}\|_{L^\infty(\mathbb{T})} = 0.$

Observe that if $(\overline{u}^h)$ is admissible for $\overline{u}$, then

$$\sup_h \Big(\|\overline{u}^h\|_{L^\infty(\mathbb{T})} + F_h(\overline{u}^h) \Big) < +\infty. \tag{4.4}$$

Given $\overline{u} \in \mathcal{D}$, we now construct an admissible sequence $(\overline{u}^h)$ for $\overline{u}$. The idea for constructing $(\overline{u}^h)$ is to keep the values of $\overline{u}$ out of $\Sigma_L(\overline{u})$, and to take a suitable approximation of $\overline{u}$ in $\Sigma_L(\overline{u})$, in particular ensuring that the avoidance condition $\Sigma_L(\overline{u}^h) = \emptyset$ is satisfied. To do that, we will make use of the existence of a pcr-percentage of negative slopes for $\overline{u}$.
Let us first introduce the following "averaged discrete gradient", where the average is taken over a scale of order nh.

Definition 4.6 (Averaged discrete gradients). *Let $u \in PL_h(\mathbb{T})$ and $n \in \mathbb{N}$. We set*

$$(D^-_{nh}u)_i := \frac{1}{n} \sum_{j=i-n+1}^{i} (D^-_h u)_j = \frac{u_i - u_{i-n}}{nh}, \qquad i \in \{1, \dots, N_h\}. \tag{4.5}$$

For example, if $n = 2$, equation (4.5) is nothing else than the discrete derivative in (3.2) (left), on a grid of size $2h$.
Of course, Definition 4.6 makes sense also for $n = 1$; however, the general case $n \in \mathbb{N}$ will be necessary in the proof of Theorem 6.7.

4.2 Construction of an admissible sequence for $\overline{u} \in \mathcal{D}$.

Let $\overline{u} \in \mathcal{D}$ and let $\overline{\varrho}$ be a pcr-percentage of negative slopes for $\overline{u}$. We have a set $\{\xi_1, \dots, \xi_M\} \subset \mathbb{T}$ of points with $\xi_j < \xi_{j+1}$ so that

$$\overline{\varrho}(x) = \sigma_j = \frac{m_j}{n_j} \in \mathbb{Q}, \qquad x \in (\xi_j, \xi_{j+1}),\ j \in \{1, \dots, M\},$$

where we let $\xi_{M+1} := \xi_1$.
We want to define $\overline{u}^h$ in $[\xi_j, \xi_{j+1}]$: we partition $[\xi_j, \xi_{j+1}]$ into subintervals of length $n_j h$ indicized by the index k (if the partition does not coincide exactly with $[\xi_j, \xi_{j+1}]$ then there may remain at most two small intervals adjacent to ξ_j and ξ_{j+1}, where we will set $\overline{u}^h := \overline{u}$).
In each of these intervals of length $n_j h$, say $[kn_j h, (k+1)n_j h]$, the averaged slope of $\overline{u}$ is given by

$$\overline{p}_k := (D^-_{n_j h}\overline{u})_{(k+1)n_j}.$$

We want that:

- $\overline{u}^h$ has the same averaged slope $\overline{p}_k$ of $\overline{u}$ in $[kn_j h, (k+1)n_j h]$,
- $D^-_h \overline{u}^h$ belongs to the region where W is convex, namely it avoids the concave region of W,
- $\overline{u}^h$ is decreasing with slope $\overline{q}_k$ on the first m_j subintervals of $[kn_j h, (k+1)n_j h]$, and increasing with slope $\mathrm{T}(\overline{q}_k)$ on the remaining $n_j - m_j$ subintervals, where

$$\overline{q}_k := q\,(\overline{p}_k, \sigma_j) < 0.$$

Being the avergaed slope of $\overline{u}^h$ equal to $\overline{p}_k$, it follows that $D_h^- \overline{u}^h$ must be equal to $\overline{q}_k$ on m_j subintervals, and $\mathrm{T}(\overline{q}_k)$ on the remaining subintervals. Therefore we set $D_h^- \overline{u}^h = \overline{q}_k$ on the first m_j subintervals, and $D_h^- \overline{u}^h = \mathrm{T}(\overline{q}_k)$ on the remaining $n_j - m_j$ subintervals.
The definition is the following:

Definition 4.7 (Construction of $(\overline{u}^h)$). *Let $j \in \{1, \dots, M\}$ and let $x \in [\xi_j, \xi_{j+1}]$ be a grid point of $\mathbb{T}$. We write $x = (kn_j + m)h$ for some $k, m \in \mathbb{N}$ with $m < n_j$. We define $\overline{u}^h(x)$ as follows:*

- *if $\{kn_jh, (k+1)n_jh\} \not\subset [\xi_j, \xi_{j+1}]$, we set*

$$\overline{u}^h(x) := \overline{u}(x);$$

- *if $\{kn_jh, (k+1)n_jh\} \subset [\xi_j, \xi_{j+1}]$ and $m \le m_j$, we set*

$$\overline{u}^h(x) := \overline{u}(kn_jh) + m\overline{q}_k h\,, \tag{4.6}$$

- *if $\{kn_jh, (k+1)n_jh\} \subset [\xi_j, \xi_{j+1}]$ and $m > m_j$, we set*

$$\overline{u}^h(x) := \overline{u}(kn_jh) + m_j\overline{q}_k h - (m - m_j)\mathrm{T}(\overline{q}_k)h\,. \tag{4.7}$$

The function $\overline{u}^h$ is then extended piecewise linearly out of the nodes of the grid.

Notice the signs on the right hand sides of (4.6) and (4.7): remember that $\overline{q}_k$ is negative by construction, while $\mathrm{T}(\overline{q}_k)$ is positive, hence $\overline{u}^h$ is decreasing up to m_j, and then increases. By construction, we have the following result.

Lemma 4.8. *Let $\overline{u} \in \mathcal{D}$. Then the sequence $(\overline{u}^h)$ of Definition 4.7 is admissible for $\overline{u}$.*

5 A priori estimates

In order to pass to the limit in equation (1.6), we need to find several a priori estimates on approximate solutions u^h, under restriction (3.9) on the inital datum $\overline{u}^h$.

Proposition 5.1. *Let $\overline{u}^h \in PL_h(\mathbb{T})$ and suppose that condition (3.9) holds. Let u^h be the corresponding solution given by Theorem 3.1. Then*

$$\frac{d}{dt}\left(\left\|\frac{d}{dt}u^h(t)\right\|_{PL_h(\mathbb{T})}\right) \le 0, \tag{5.1}$$

and

$$\frac{d}{dt}\left(\left\|\frac{d}{dt}u^h(t)\right\|_{L^\infty(\mathbb{T})}\right) \le 0. \tag{5.2}$$

Proof. Set for notational simplicity $w := \dfrac{du^h}{dt}$. We have, for $i \in \{1, \dots, N_h\}$ and using (1.6),

$$\begin{aligned} \frac{dw_i}{dt} =& \frac{d}{dt}\Big(D_h^+ W'\left((D_h^- u^h)_i\right)\Big) = D^+ \frac{d}{dt}\Big(W'\left((D_h^- u^h)_i\right)\Big) \\ =& D_h^+ \Big(W''\left((D_h^- u^h)_i\right) D_h^- w_i\Big). \end{aligned} \tag{5.3}$$

From inequalities (3.11) it follows

$$W''\left((D_h^- u^h(t))_i\right) \geq 0, \qquad t \in (0, +\infty). \tag{5.4}$$

Remembering the definition (3.1) of the $\|\cdot\|_{PL_h(\mathbb{T})}$ norm, and using the discrete integration by parts formula (3.3), we have

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2}\left\|\frac{d}{dt}u^h(t)\right\|^2_{PL_h(\mathbb{T})}\right) =& \frac{d}{dt}\left(\frac{1}{2}\sum_{i=1}^{N_h}(w_i)^2\right) \\ =& h\sum_{i=1}^{N_h} w_i D_h^+\Big(W''\left((D_h^- u)_i\right)(D_h^- w)_i\Big) \\ =& -h\sum_{i=1}^{N_h}(D_h^- w)_i W''\left((D_h^- u)_i\right)(D_h^- w)_i \leq 0, \end{aligned}$$

which proves (5.1).
Let us now prove (5.2). Let $ih \in \mathbb{T}$ be a grid point where W takes its maximum. Then

$$(D_h^- w)_i \geq 0, \qquad (D_h^- w)_{i+1} \leq 0.$$

Recalling (5.4) we deduce

$$W''\left((D_h^- u^h)_i\right)(D_h^- w)_i \geq 0, \qquad W''\left((D_h^- u^h)_{i+1}\right)(D_h^- w)_{i+1} \leq 0.$$

From the above inequalities and (5.3) we obtain

$$\frac{dw_i}{dt} = D_h^+\Big(W''\left((D_h^- u^h)_i\right)(D_h^- w)_i\Big) \leq 0.$$

A similar proof holds for a minimum point. Then (5.2) follows. □

Corollary 5.2 (**Convexity of $F_h(u^h)$**)**.** *The function $t \in [0, +\infty) \to F_h(u^h(t))$ is convex and nonincreasing.*

Proof. It is a consequence of (3.6) and (5.1). □

The next result shows that $W'(D_h^- u^h)$ are 1/2-Hölder continuous in space. This fact will allow to use the Ascoli-Arzelà theorem for passing to the limit in the nonlinear term $W'(D_h^- u^h)$ in (1.6).

Proposition 5.3 (**Hölder continuity of $W'(D_h^- u^h)$**)**.** *Let $\overline{u}^h \in PL_h(\mathbb{T})$ and suppose that* (3.9) *holds. Let u^h be the corresponding solution given by Theorem 3.1. Then for all $k, l \in \{1, \dots, N_h\}$ and $t \in (0, +\infty)$ we have*

$$\left|W'\left((D_h^- u^h(t))_l\right) - W'\left((D_h^- u^h(t))_k\right)\right| \le \sqrt{|l-k|h}\, \left\|\frac{du^h}{dt}(t)\right\|_{PL_h(\mathbb{T})}. \tag{5.5}$$

Proof. Without loss of generality we assume $l > k$, and we write for notational simplicity

$$u = u^h.$$

We have, using the equality $(D_h^+ u)_i = (D_h^- u)_{i+1}$,

$$\begin{aligned}
&\left|W'\Big((D_h^- u(t))_l\Big) - W'\Big((D_h^- u(t))_k\Big)\right| \\
= &\left|W'\Big((D_h^- u(t))_l\Big) - W'\Big((D_h^- u(t))_{l-1}\Big) + \ \dots \ + W'\Big((D_h^- u(t))_{k+1}\Big) - W'\Big((D_h^- u(t))_k\Big)\right| \\
\le &\sum_{i=k}^{l-1} \left|W'\Big((D_h^- u(t))_{i+1}\Big) - W'\Big((D_h^- u(t))_i\Big)\right| \\
\le &\sum_{i=k}^{l-1} \left|W'\Big((D_h^+ u(t))_i\Big) - W'\Big((D_h^- u(t))_i\Big)\right|,
\end{aligned}$$

where we stress that any difference appearing in the last line involves the two operators D_h^- and D_h^+. Then, using Hölder's inequality and the definition (3.1) of $PL_h(\mathbb{T})$-norm, it follows

$$\begin{aligned}
&\left|W'\Big((D_h^- u(t))_l\Big) - W'\Big((D_h^- u(t))_k\Big)\right| \\
\le \ &\sqrt{(l-k)h}\ \left(h \sum_{i=k}^{l-1} \left|\frac{1}{h}\Big(W'\left((D_h^+ u(t))_i\right) - W'\left((D_h^- u(t))_i\right)\Big)\right|^2\right)^{1/2} \\
= \ &\sqrt{(l-k)h}\ \left(h \sum_{i=k}^{l-1} \left|D_h^+ W'\left((D_h^- u(t))_i\right)\right|^2\right)^{1/2} \\
\le \ &\sqrt{(l-k)h}\ \left(h \sum_{i=k}^{l-1} \left|\frac{du_i}{dt}(t)\right|^2\right)^{1/2} = \sqrt{(l-k)h}\ \left\|\frac{du}{dt}(t)\right\|_{PL_h(\mathbb{T})},
\end{aligned}$$

which gives the desired result. □

The Hölder space constant of $W'(D_h^- u^h)$ is uniform with respect to time, since the quantity $\left\|\frac{du^h}{dt}\right\|_{PL_h(\mathbb{T})}$ on the right hand side of (5.5) is nonincreasing in time by inequality (5.1). More precisely, the following result holds.

Corollary 5.4. *Let $\overline{u}^h \in PL_h(\mathbb{T})$ and suppose that* (3.9) *holds. Let u^h be the corresponding solution given by Theorem 3.1 and let $\tau > 0$. Then for all $k, l \in \{1, \dots, N_h\}$ and $t > \tau$ we have*

$$\left|W'\left((D_h^- u^h(t))_l\right) - W'\left((D_h^- u^h(t))_k\right)\right| \le \sqrt{|l-k|h}\, \sqrt{\frac{F_h(\overline{u}^h)}{\tau}}. \tag{5.6}$$

Proof. From estimate (5.5) and using formula (3.6) we have

$$\left|W'\Big(\big(D_h^- u^h(t)\big)_k\Big) - W'\Big(\big(D_h^- u^h(t)\big)_l\Big)\right| \le \sqrt{|l-k|h}\ \left|\frac{d}{dt}F_h(u^h(t))\right|. \tag{5.7}$$

If $f : [0,+\infty) \to (0,+\infty)$ is a convex nonincreasing differentiable function and $\tau > 0$, we have $f'(\tau) \ge \frac{f(\tau)-f(0)}{\tau} \ge -\frac{f(0)}{\tau}$, hence $\dfrac{f(0)}{\tau} \ge |f'(\tau)|$. Then (5.6) follows from Corollary 5.2 and the latter inequality. □

6 Main results and the limit problem

Set

$$\mathcal{X} := AC_2\Big([0,+\infty); L^2(\mathbb{T})\Big) \cap L^\infty\Big((0,+\infty); \mathrm{Lip}_{M^+}(\mathbb{T})\Big) \cap L^\infty\Big(\mathbb{T}\times(0,+\infty)\Big), \tag{6.1}$$

where $\mathrm{Lip}_{M^+}(\mathbb{T}) := \{u \in \mathrm{Lip}(\mathbb{T}) : \mathrm{lip}(u) \le M^+\}$ is endowed with the topology of uniform convergence.

Remark 6.1. Let $(\overline{u}^h)$ satisfy (3.4) and assume in addition that (4.4) holds. Let u^h be the solution to (1.6). Then

$$u^h \in \mathcal{X}. \tag{6.2}$$

A first compactness result on the sequence (u^h) is given by the following proposition.

Proposition 6.2 (Weak compactness). *Let $(\overline{u}^h)$ satisfy (3.4) and (4.4). Let u^h be the solution to (1.6). Then there exist a function*

$$u \in \mathcal{X} \quad \text{with} \quad u(0) = \overline{u},$$

and a (not relabelled) subsequence (u^h) such that

(i) $\lim_{h\to 0^+} u^h = u$ *weakly in* $H^1_{\mathrm{loc}}([0,+\infty); L^2(\mathbb{T}))$,

(ii) $\lim_{h\to 0^+} u^h = u$ *weakly* in* $L^\infty_{\mathrm{loc}}((0,+\infty); \mathrm{Lip}_{M^+}(\mathbb{T}))$,

and

$$\int_{\mathbb{T}} u(x,t)\, dx = \int_{\mathbb{T}} \overline{u}(x)\, dx, \qquad t \ge 0. \tag{6.3}$$

Moreover there exists a constant $C_1 > 0$ such that

$$|u_x| \le C_1 \qquad \text{in } \mathbb{T}\times[0,+\infty).$$

Proof. From (3.6) and (4.4) we have that, for any $T > 0$, the sequence (u^h) is bounded in $H^1([0,T]; L^2(\mathbb{T}))$. Therefore, passing to a (not relabelled) subsequence, we have conclusion (i) for a function $u \in AC_2([0,+\infty); L^2(\mathbb{T}))$. Using (3.5) we have assertion (ii) and the inclusion $u \in L^\infty((0,+\infty); \mathrm{Lip}_{M^+}(\mathbb{T}))$. In particular $u^h(\cdot,t) \to u(\cdot,t)$ uniformly for any $t \ge 0$, which implies $u(\cdot,0) = \lim_{h\to 0^+} \overline{u}^h(\cdot) = \overline{u}(\cdot)$.
The inclusion $u \in L^\infty(\mathbb{T}\times(0,+\infty))$ follows from (3.8), hence $u \in \mathcal{X}$. Eventually, equality (6.3) follows from (3.7). □

Before observing another compactness property of the sequence (u^h), we need the following result, which is independent of the evolution equation (1.2), being a property of the function space $\mathcal{X}$ only.

Proposition 6.3. *Let $v \in \mathcal{X}$. Then there exists a constant $C_2 > 0$ such that*

$$|v(x,t) - v(y,s)| \le C_2 \left(|x-y| + |t-s|^{1/3} \right), \qquad x, y \in \mathbb{T},\ s,t \in (0,+\infty). \tag{6.4}$$

Proof. Let $I \subseteq \mathbb{T}$ be an interval of length $|I|$. Fix $x, y \in I$ and $s, t \in (0,\infty)$. Since $v \in L^\infty((0,+\infty); \mathrm{Lip}_{M^+}(\mathbb{T}))$, it follows that $v(\cdot,t)$ is Lipschitz continuous uniformly with respect to t, therefore

$$\begin{aligned}
|v(x,t) - v(y,s)| &= \frac{1}{|I|} \int_I |v(x,t) - v(y,s)|\ dz \\
&\le \frac{1}{|I|} \int_I \big(|v(x,t) - v(z,t)| + |v(z,t) - v(z,s)| + |v(z,s) - v(y,s)| \big)\ dz \\
&\le \frac{1}{|I|} \int_I |v(z,t) - v(z,s)|\ dz + C|I|,
\end{aligned}$$

where $C > 0$ is a suitable constant. Hence, using twice Hölder's inequality and the assumption $v \in AC_2([0,+\infty); L^2(\mathbb{T}))$ we deduce

$$\begin{aligned}
|v(x,t) - v(y,s)| &\le \frac{1}{|I|^{1/2}} \left(\int_{\mathbb{T}} |v(z,t) - v(z,s)|^2\ dz \right)^{1/2} + C|I| \\
&\le \frac{\sqrt{|t-s|}}{|I|^{1/2}} \left(\int_{\mathbb{T}\times[0,+\infty)} (v_t)^2\ dzdt \right)^{1/2} + C|I| \\
&\le \kappa \left(\frac{\sqrt{|t-s|}}{|I|^{1/2}} + |I| \right),
\end{aligned}$$

where $\kappa > 0$ is a suitable constant.
Since $|x-y| \le |I| \le 1$, we have

$$|v(x,t) - v(y,s)| \le \kappa \min_{\lambda \in [|x-y|,1]} \left(\frac{\sqrt{|t-s|}}{\lambda^{1/2}} + \lambda \right) \le C_2 \left(|t-s|^{1/3} + |x-y| \right) \tag{6.5}$$

for a constant $C_2 > 0$. □

Corollary 6.4 (Hölder continuity and uniform convergence of (u^h)). *Let $(\overline{u}^h)$, (u^h) and u be as in Proposition 6.2. Then there exists a constant $C_3 > 0$ such that*

$$|u^h(x,t) - u^h(y,s)| \le C_3 \left(|x-y| + |t-s|^{1/3} \right), \qquad x, y \in \mathbb{T},\ s,t \in (0,+\infty),\ h \in \mathbb{N}. \tag{6.6}$$

Moreover there exists a (not relabelled) subsequence (u^h) such that

$$u^h \to u \text{ uniformly on compact subsets of } \mathbb{T} \times [0,+\infty). \tag{6.7}$$

Proof. The proof of (6.6) is the same as the proof of (6.4), recalling (3.5) and (3.6). The uniform convergence of (u^h) then follows from (3.8), (6.6), and the Ascoli-Arzelà theorem. □

6.1 Compactness of gradients

In order to pass to the limit in problem (1.6) we need some compactness properties on the discrete gradients of the sequence of solutions to (1.6). We cannot hope to have simply a compactness of the sequence $(D_h^- u^h)$ of "microscopic" gradients, because of their oscillations (microstructures) in the local unstable region. Therefore, we will take a suitable average of these gradients giving raise to a sort of "macroscopic" gradient $D_{nh}^- u^h$, $n \in \mathbb{N}$. We will gain a uniform Hölder's estimate for the sequence $\left(W'\left(D_h^- u^h\right)\right)$.
We start with the following result, which shows that, for functions in $\mathcal{X}$, Hölder continuity in time can be obtained from Hölder continuity in space.

Lemma 6.5. *Let $v \in \mathcal{X}$. Suppose that there exist $C > 0$ and $\alpha \in (0,1)$ such that*

$$|v_x(x,t) - v_x(y,t)| \leq C|x-y|^\alpha, \qquad x, y \in \mathbb{T},\ t > 0. \tag{6.8}$$

Then there exists a constant $C_4 > 0$ such that

$$|v_x(x,t) - v_x(x,s)| \leq C_4 |t-s|^{\frac{\alpha}{3(\alpha+1)}}, \qquad x \in \mathbb{T},\ s, t > 0.$$

Proof. For $\delta > 0$ we have

$$\begin{aligned} v(x+\delta,t) - v(x,t) &= \int_x^{x+\delta} v_x(y,t)\, dy = v_x(x,t)\,\delta + \int_x^{x+\delta} (v_x(y,t) - v_x(x,t))\, dy, \\ v(x+\delta,s) - v(x,s) &= \int_x^{x+\delta} v_x(y,s)\, dy = v_x(x,s)\,\delta + \int_x^{x+\delta} (v_x(y,s) - v_x(x,s))\, dy. \end{aligned}$$

Subtracting the two equations we have

$$\begin{aligned} v_x(x,t) - v_x(x,s) =& \frac{1}{\delta}\Big(v(x+\delta,t) - v(x+\delta,s) + v(x,s) - v(x,t)\Big) \\ &+ \frac{1}{\delta}\int_x^{x+\delta} \Big[(v_x(x,t) - v_x(y,t)) + (v_x(y,s) - v_x(x,s))\Big]\, dy =: \mathrm{I} + \mathrm{II}. \end{aligned}$$

Recalling (6.4) we have

$$\mathrm{I} \leq \frac{2C_2}{\delta}\ |t-s|^{1/3}.$$

Moreover, from hypothesis (6.8) it follows

$$\mathrm{II} \leq 2C\delta^\alpha.$$

Therefore, for $\kappa := 2\max(C_2, C)$ we have

$$|v_x(x,t) - v_x(x,s)| \leq \kappa\left(\frac{|t-s|^{\frac{1}{3}}}{\delta} + \delta^\alpha\right). \tag{6.9}$$

The thesis follows by minimizing the right hand side of (6.9) among all $\delta > 0$. $\square$

Lemma 6.5 has a the following discrete version, the proof of which is omitted being similar to the previous proof.

Lemma 6.6. *Let $v \in \mathcal{X}$. Suppose that there exist $C > 0$ and $\alpha \in (0,1)$ such that*

$$|D_h^- v(hi,t) - D_h^- v(hj,t)| \le C\,(h|i-j|)^\alpha\,, \qquad i,j \in \{1,\dots,N_h\},\ t>0.$$

Then there exists a constant $C_5 > 0$ such that

$$|D_h^- v(hi,t) - D_h^- v(hj,s)| \le C_5 |t-s|^{\frac{\alpha}{3(\alpha+1)}}, \qquad i,j \in \{1,\dots,N_h\},\ s,t>0.$$

We are now in a position to prove the following compactess result for the averaged discrete gradients.

Theorem 6.7 (Compactness of discrete gradients). *Let $\overline{u} \in \mathcal{D}$ and let $\overline{\varrho}$ be a pcr-percentage of negative slopes for $\overline{u}$. Accordingly, let $M \in \mathbb{N}$ and $\{\xi_1,\dots,\xi_M\} \subset \mathbb{T}$ be so that $\overline{\varrho}$ takes a constant rational value σ_i on each (ξ_i,ξ_{i+1}) and (4.1) holds. Finally, let $(\overline{u}^h)$ be the admissible sequence for $\overline{u}$ constructed in Definition 4.7 and let u^h be the solution of (1.6). Then there exist a function*

$$u \in \mathcal{X} \quad \text{with} \quad u(0) = \overline{u},$$

and a (not relabelled) subsequence (u^h) converging to u uniformly on compact subsets of $\mathbb{T} \times [0,+\infty)$, such that

$$\lim_{h\to 0^+} D_{n_i h}^- u^h = u_x \qquad \text{uniformly on compact subsets of } (\xi_i,\xi_{i+1}) \times (0,+\infty), \tag{6.10}$$

for all $i \in \{1,\dots,M\}$.

Proof. Let u and (u^h) be the function and the subsequence given by Proposition 6.2 respectively. Given $\tau > 0$, from (5.6) we have for any $l,k \in \{1,\dots,N_h\}$ and $t > \tau$,

$$\left|W'\left((D_h^- u^h(t))_l\right) - W'\left((D_h^- u^h(t))_k\right)\right| \le \sqrt{\frac{F_h(\overline{u}^h)}{\tau}}\sqrt{h|l-k|}. \tag{6.11}$$

We would like to deduce from (6.11) an estimate for $|(D_h^- u^h(t))_l - (D_h^- u^h(t))_k|$, by locally "inverting"[8] W' on the interval $(W'(p^+), W'(p^-))$. In general, such an estimate is not possible; however, we are able to prove an Hölder estimate for the averaged discrete gradients (see (6.18) below), and it is precisely here that $\overline{\varrho}$ takes a role: actually, estimate (6.18) is the motivation for introducing the notion of averaged gradient in Definition 4.6.
Remember that our construction of $(\overline{u}^h)$ is made so that (3.9) holds (see the third condition in Definition 4.5), and therefore (3.11) ensures that the discrete gradient of u^h avoids the concave region of W, i.e.,

$$(D_h^- u^h(t))_l \notin (p^-,p^+), \qquad l \in \{1,\dots,N_h\}, \qquad t \in [0,+\infty).$$

From (6.11) we deduce that there exists a constant $C_1(\tau) > 0$ such that, for $t > \tau$, either

$$|(D_h^- u^h(t))_l - \mathrm{T}\left((D_h^- u^h(t))_k\right)| \le C_1(\tau)(h|l-k|)^{1/4}, \tag{6.12}$$

[8] For any $y \in (W'(p^+), W'(p^-))$, ${W'}^{-1}(y)$ consists of three elements, two of them in $[M^-,p^-] \cup [p^+,M^+]$ and the other one in (p^-,p^+).

or

$$|(D_h^- u^h(t))_l - (D_h^- u^h(t))_k| \le C_1(\tau)(h|l-k|)^{1/4}, \tag{6.13}$$

where the exponent $1/4$ is due to the fact that the local inverses of W' are $1/2$-Hölder continuous, since

$$W'''(p^+) = -W'''(p^-) > 0.$$

Let $t > \tau$; observe that if $(D_h^- u^h(t))_k < 0$ then

$$(6.13) \text{ holds when } (D_h^- u^h(t))_l < 0 \quad \text{and} \quad (6.12) \text{ holds when } (D_h^- u^h(t))_l > 0. \tag{6.14}$$

Similarly, if $(D_h^- u^h(t))_k > 0$ then

$$(6.13) \text{ holds when } (D_h^- u^h(t))_l > 0 \quad \text{and} \quad (6.12) \text{ holds when } (D_h^- u^h(t))_l < 0. \tag{6.15}$$

Let now K be a compact subset of (ξ_i, ξ_{i+1}). Denote, as usual, by $\sigma_i = \frac{n_i}{m_i} \in \mathbb{Q}$ the value of $\overline{\varrho}$ on (ξ_i, ξ_{i+1}). Possibly reducing $h > 0$, we may assume that $\mathrm{dist}(K, \{\xi_i, \xi_{i+1}\}) > n_i h$. Recall also that by Definition 4.6

$$\left(D_{n_i h}^- u^h(t)\right)_k = \frac{1}{n_i} \sum_{l=k-n_i+1}^{k} \left(D_h^- u^h(t)\right)_l. \tag{6.16}$$

By our choice of the admissible sequence $(\overline{u}^h)$ in Section 4.2 and remembering (3.10), it follows that among the terms $(D_h^- u^h(t))_l$ on the right-hand side of (6.16), m_i are negative, while $(n_i - m_i)$ are positive. Using (6.13), (6.14) and (6.15) we can then replace in (6.16) the terms $(D_h^- u^h(t))_l$ with m_i times the term $(D_h^- u^h(t))_k$ and with $(n_i - m_i)$ times the term $\mathrm{T}((D_h^- u^h(t))_k)$ at the expenses of an error of order $h^{1/4}$, and we obtain[9] for $t > \tau$,

$$\begin{aligned} &\left| (D_{n_i h}^- u^h(t))_k - \frac{m_i (D_h^- u^h(t))_k + (n_i - m_i)\mathrm{T}\left((D_h^- u^h(t))_k\right)}{n_i} \right| \le C_2(\tau) h^{1/4} \quad \text{if } (D_h^- u^h(t))_k < 0 \\ &\left| (D_{n_i h}^- u^h(t))_k - \frac{(n_i - m_i)(D_h^- u^h(t))_k + m_i\mathrm{T}\left((D_h^- u^h(t))_k\right)}{n_i} \right| \le C_2(\tau) h^{1/4} \quad \text{if } (D_h^- u^h(t))_k > 0, \end{aligned} \tag{6.17}$$

for a suitable constant $C_2(\tau) > 0$, for all $k \in \mathbb{N}$ such that $kh \in K$. From (6.12) and (6.13) we finally get our desired estimate on the averaged gradients:

$$|(D_{n_i h}^- u^h(t))_l - (D_{n_i h}^- u^h(t))_k| \le C_3(\tau)(h|l-k|)^{1/4}, \qquad t > \tau. \tag{6.18}$$

for all $l, k \in \mathbb{N}$ such that $lh, kh \in K$, for a suitable constant $C_3(\tau)$.

We now want to apply the Ascoli-Arzelà Theorem; notice however that $D_{n_i h}^- u^h(t) \in PC_h((\xi_i, \xi_{i+1}))$, so that $D_{n_i h}^- u^h(t)$ is not continuous in general. Define the sequence

$$(p_h) \subset PL_h((\xi_i, \xi_{i+1}) \times (0, +\infty))$$

[9] To understand the meaning of inequalities (6.17), assume that the right hand sides vanish. Then, remembering (3.14), it follows that the first inequality of (6.17) says that $D_h u^h(t) = q(D_{n_i h}^- u^h(t), \sigma_i)$, while the second inequality says that $\mathrm{T}(D_h u^h(t)) = q(D_{n_i h}^- u^h(t), \sigma_i)$.

so that $p_h(t)$ is the linear interpolation of the values of $D^-_{n_ih}u^h(t)$ on the grid of size h, for any $t \in (0,+\infty)$. From (6.18), Lemma 6.6 and from the Ascoli-Arzelà Theorem (invading (ξ_i, ξ_{i+1}) with a sequence of compact subsets) it follows that (p_h) has a (not relabelled) subsequence converging to some

$$p \in \mathcal{C}((\xi_i, \xi_{i+1}) \times (0,+\infty))$$

uniformly on compact subsets of $(\xi_i, \xi_{i+1}) \times (0,+\infty)$ as $h \to 0^+$. Hence $(D^-_{n_ih}u^h)$ also converges to p uniformly on compact subsets of $(\xi_i, \xi_{i+1}) \times (0,+\infty)$ as $h \to 0^+$, and therefore

$$p = u_x.$$

□

The origin of the term $q(u_x, \overline{\varrho})$ in equation (1.9) is then contained in the following result.

Corollary 6.8. *Under the assumptions of Theorem 6.7 we have that (u^h) has a (not relabelled) subsequence such that*

$$\lim_{h\to 0^+} W'\left(D^-_h u^h\right) = W'\left(q(u_x, \overline{\varrho})\right)$$

uniformly on compact subsets of $(\mathbb{T} \setminus \{\xi_1, \ldots, \xi_M\}) \times (0,+\infty)$.

Proof. Fix $i \in \{1, \ldots, M\}$ and a compact set $K \subset (\xi_i, \xi_{i+1})$. Recalling the notation in the proof of Theorem 6.7, and also (6.17), we have

$$\begin{cases} \left| q\left(D^-_{n_ih}u^h(t), \sigma_i\right) - D^-_h u^h(t)\right| \le C(\tau)h^{1/4} & \text{if } D^-_h u^h(t) < 0, \\ \\ \left| q\left(D^-_{n_ih}u^h(t), \sigma_i\right) - \mathrm{T}(D^-_h u^h(t), \sigma_i)\right| \le C(\tau)h^{1/4} & \text{if } D^-_h u^h(t) > 0. \end{cases}$$

for all $t > \tau > 0$ and a constant $C(\tau) > 0$. The assertion then follows from Theorem 6.7 and the continuity of W'. □

6.2 Limit evolution law

Theorem 6.7 is the crucial result which allows to pass to the limit as $h \to 0^+$ in the semidiscrete scheme (1.6).

Theorem 6.9. *Let $\overline{u}$, $\overline{\varrho}$, $(\overline{u}^h)$, (u^h) and $u \in \mathcal{X}$ be as in Theorem 6.7. Then u is a distributional solution of the following problem:*

$$u_t = \Big(W'\left(q\left(u_x, \overline{\varrho}\right)\right)\Big)_x \qquad \text{in } \mathbb{T} \times (0,+\infty). \tag{6.19}$$

Proof. Let $\varphi \in \mathcal{C}^1_0(\mathbb{T} \times (0,+\infty))$. Define $\varphi^h \in \mathcal{C}^1((0,+\infty); PL_h(\mathbb{T}))$ as the function having the same values as φ on the nodes of the grid. We multiply the first equation in (1.6) by φ^h. After an integration by parts we get

$$\int_0^\infty \sum_{i=1}^{N_h} u^h_i \, \varphi^h_{i\,t} \, dt = \int_0^\infty \int_{\mathbb{T}} W'(D^-_h u^h)\, D^-_h \varphi^h \, dx\, dt.$$

Since (u^h) converges to u uniformly (see (6.7)) and φ_t^h converges uniformly to φ_t as $h \to 0^+$, we have

$$\lim_{h\to 0^+} \int_0^{\infty} \sum_{i=1}^{N_h} u_i^h \, \varphi_{i\,t}^h \; dt = \int_0^{\infty} \int_{\mathbb{T}} u\varphi_t \; dxdt. \tag{6.20}$$

Using Corollary 6.8 we get

$$\lim_{h\to 0^+} \int_0^{\infty} \int_{\mathbb{T}} W'(D_h^- u^h)\, D_h^- \varphi^h \; dx\, dt = \int_0^{\infty} \int_{\mathbb{T}} W'(q(u_x, \overline{\varrho}))\, \varphi_x \; dx\, dt. \tag{6.21}$$

From (6.20) and (6.21) we get

$$\int_0^{\infty} \int_{\mathbb{T}} u\, \varphi_t \; dxdt = \int_0^{\infty} \int_{\mathbb{T}} W'(q(u_x, \overline{\varrho}))\, \varphi_x \; dx\, dt,$$

which is the distributional formulation of (6.19). $\square$

Concerning Theorem 6.9, some observations are in order.

Remark 6.10. Since $q(\cdot, \sigma)$ is increasing, equation (6.19) is well-posed and corresponds to the gradient flow in $L^2(\mathbb{T})$ of the convex functional

$$v \to F_{\overline{\varrho}}(v) := \int_{\mathbb{T}} f(v_x, x) \; dx, \tag{6.22}$$

where the Carathéodory integrand f is given by

$$f(p, x) := \int_0^p W'(q(s, \overline{\varrho}(x))) \; ds. \tag{6.23}$$

Notice that we can assume that f is defined on $\mathbb{R} \times \mathbb{T}$, by extending q to a continuous function on the whole of $\mathbb{R} \times [0, 1]$, increasing in the first variable. For instance, we may set

$$\begin{aligned} q(p, \sigma) &:= p^- \qquad && \text{if } p > \sigma p^- + (1-\sigma)M^+ \\ q(p, \sigma) &:= M^- && \text{if } p < \sigma M^- + (1-\sigma)p^+. \end{aligned}$$

By the results in [5], it then follows that (6.19) admits a unique solution for any initial datum $\overline{u} \in \mathcal{X} \subset L^2(\mathbb{T})$. Therefore, we can remove the extraction of a subsequence in Theorems 6.7 and 6.9.

Let us also observe that $F_{\overline{\varrho}}$ has a one-parameter family of minimizers given by

$$x \in \mathbb{T} \to \int_0^x \Big(\sigma(s)p + (1-\sigma(s))\mathrm{T}(p)\Big) \; ds + \lambda,$$

where $p \in [M^-, p^-]$ is uniquely determined by the condition

$$\int_{\mathbb{T}} \Big(\sigma(s)p + (1-\sigma(s))\mathrm{T}(p)\Big) \; ds = 0$$

and $\lambda \in \mathbb{R}$.

Remark 6.11. Recalling (6.1), so that $u_t(\cdot,t) \in L^2(\mathbb{T})$ for almost every $t \in (0,+\infty)$, we have

$$\Big(W'\big(q\big(u_x(\cdot,t)\big),\overline{\varrho}(\cdot)\big)\Big)_x \in L^2(\mathbb{T}) \qquad \text{for a.e. } t \in (0,+\infty).$$

This implies in particular that $W'\big(q(u_x(\cdot,t)),\overline{\varrho}(\cdot)\big)$ is continuous on $\mathbb{T}$ for almost every $t \in (0,+\infty)$, and the following interior conditions hold:

$$W'(q(u_x(\xi_i^+,t),\sigma_i)) = W'(q(u_x(\xi_i^-,t),\sigma_{i-1})), \qquad t \in (0,+\infty),\ i \in \{1,\ldots,M\}, \tag{6.24}$$

where $u_x(\xi_i^\pm,t)$ are the left and right limits of $u_x(\cdot,t)$ at the point ξ_i. From Remark 3.7 it also follows that if $\overline{u}_x(x) \in \mathrm{int}(G(\sigma_i))$ for any $x \in (\xi_i,\xi_{i+1})$, then

$$u \in \mathcal{C}^\infty((\xi_i,\xi_{i+1}) \times (0,+\infty)).$$

Remark 6.12. If $\Sigma_L(\overline{u}) = \emptyset$ it is natural to choose (among the infinitely many possible choices of $\overline{\varrho}$) either $\overline{\varrho} \equiv 0$ or $\overline{\varrho} \equiv 1$ in $\mathbb{T}$. Then equation (6.19) reduces to equation (1.2) which, in this case, is well-posed. If $\overline{u}$ and $\overline{\varrho}$ are as in Example 4.3, the regions of $\Sigma_L(u(t))$ evolve under (6.19) with $\overline{\varrho} = 1/2$. In particular in those regions we have $u_t \neq 0$ (unless u is linear). This is a different evolution with respect to the one numerically observed in [3] obtained as a limit of (1.5) as $\epsilon \to 0$.

Remark 6.13. A solution to (6.19) typically decreases the functional (6.22) (see Remark 6.10) but does not necessarily decreases the functional (1.1).

6.3 The case of a general $\overline{\varrho}$

In this last section we generalize Theorem 6.9, removing the assumption of $\overline{\varrho}$ being piecewise constant with rational values.

Theorem 6.14. *Let $\overline{u} \in \mathrm{Lip}(\mathbb{T})$ and assume that there exists $\overline{\varrho} \in L^\infty(\mathbb{T};[0,1])$ such that* (4.1) *holds. Then there exists an admissible sequence $(\overline{u}^h)$ for $\overline{u}$ such that the corresponding sequence (u^h) of solutions to* (1.6) *converges uniformly on compact subsets of $\mathbb{T} \times [0,+\infty)$ to a function $u \in \mathcal{X}$ which is a distributional solution of* (6.19).

Proof. Let $(\overline{\varrho}_n) \subset L^\infty(\mathbb{T};\mathbb{Q}\cap[0,1])$ be a sequence of prc-percentages for $\overline{u}$ pointwise converging to $\overline{\varrho}$ as $n \to +\infty$, and let $\overline{u}_n^h$ be the admissible sequence for $\overline{u}$ constructed in Section 4.2 with $\overline{\varrho}$ replaced by $\overline{\varrho}_n$. Let also u_n^h be the solutions to (1.6) with $\overline{\varrho}$ replaced by $\overline{\varrho}_n$ and initial condition $\overline{u}_n^h$. From Theorem 6.9 it follows that (u_n^h) converges to a function $u_n \in \mathcal{X}$ solving distributionally (6.19) with $\overline{\varrho}$ replaced by $\overline{\varrho}_n$.

We will show that (u_n) converges to a distributional solution u of (6.19) as $n \to +\infty$. Indeed, let f_n be the function defined in (6.23), with $\overline{\varrho}$ replaced by $\overline{\varrho}_n$. By the pointwise convergence of $\overline{\varrho}_n$ to $\overline{\varrho}$, for almost every $x \in \mathbb{T}$ we have that $f_n(p,x) \to f(p,x)$ as $n \to +\infty$, locally uniformly in $p \in \mathbb{R}$. Therefore, the convex functional $F_{\overline{\varrho}_n}$ defined in (6.22) Γ-converges to $F_{\overline{\varrho}}$ in $L^2(\mathbb{T})$. Since these functionals are convex, by the results in [8] (see also [6, Th. 2.17]) it follows that the gradient flow u_n of $F_{\overline{\varrho}_n}$ converges in $\mathcal{C}^0([0,+\infty);L^2(\mathbb{T}))$ to the gradient flow u of $F_{\overline{\varrho}}$, which is the unique solution to (6.19) with initial datum $\overline{u}$. As the functions u_n are uniformly bounded in $\mathcal{X}$, it follows that $u \in \mathcal{X}$ and (u_n) converges to u uniformly on compact subsets of $\mathbb{T} \times [0,+\infty)$.

By a diagonal argument, we can eventually find a sequence (h_n) converging to 0^+ such that (u_n^h) converges to u uniformly on compact subsets of $\mathbb{T} \times [0, +\infty)$, thus concluding the proof. $\square$

Remark 6.15 (Long time behavior). Since (6.19) is the gradient flow of the convex functional $F_{\overline{\varrho}}$ in $L^2(\mathbb{T})$, by [5] (see also [1]) it follows that there exists

$$\lim_{t\to+\infty} u(t),$$

where the limit is taken in $L^2(\mathbb{T})$, and hence in $L^\infty(\mathbb{T})$. Moreover, recalling (3.7), such a limit is the global minimizer of $F_{\overline{\varrho}}$ in $L^2(\mathbb{T})$ with the choice

$$\lambda = \int_{\mathbb{T}} \overline{u}\, dx.$$

References

[1] L. Ambrosio, N. Gigli and G. Savarè. Gradient Flows in Metric Spaces and in the Wesserstein Space of Probability Measures. ETH Lectures Notes, Birkhäuser, Basel, 2005.

[2] G. Bellettini, L. Bertini, M. Mariani and M. Novaga. Convergence of the one-dimensional Cahn-Hilliard equation. *SIAM J. Math. Anal.*, to appear.

[3] G. Bellettini, G. Fusco and N. Guglielmi. A concept of solution for forward-backward equations of the form $u_t = \frac{1}{2}(\phi'(u_x))_x$ and numerical experiments for the singular perturbation $u_t = -\epsilon^2 u_{xxxx} + \frac{1}{2}(\phi'(u_x))_x$, *Discrete Contin. Dyn. Syst.*, 16:783–842, 2006.

[4] G. Bellettini, M. Novaga, M. Paolini and C. Tornese. Convergence of discrete schemes for the Perona-Malik equation. *J. Differential Equations*, 245: 892–924, 2008.

[5] H. Brezis. Operateurs Maximaux Monotones. North-Holland Publishing Co., Amsterdam, 1973.

[6] S. Daneri and G. Savarè. Lecture Notes on Gradient Flows and Optimal Trasport, *Seminaires et Congres, SMF*, to appear, `arxiv:1009.3737v1`.

[7] E. De Giorgi. Conjectures concerning some evolution problems. *Duke Math. J.*, 81(2):255–268, 1996.

[8] M. Degiovanni, A. Marino, M. Tosques. Evolution equations with lack of convexity. *Nonlinear Anal.*, 9(12):1401–1443, 1985.

[9] S. Demoulini. Young measure solutions for a nonlinear parabolic equation of forward-backward type. *SIAM J. Math. Anal.*, 27(2):376–403, 1996.

[10] L. C. Evans and M. Portilheiro. Irreversibility and hysteresis for a forward-backward diffusion equation. *Math. Models Methods Appl. Sci.*, 14(11):1599–1620, 2004.

[11] F. Fierro, R. Goglione and M. Paolini, Numerical simulations of mean curvature flow in presence of a nonconvex anisotropy, *Math. Models Methods Appl. Sci.*, 8:573–601, 1998.

[12] C. Geldhauser and M. Novaga. A semidiscrete scheme for a one-dimensional Cahn-Hilliard equation. *Interfaces Free Boundaries*, 13:327–339, 2011.

[13] M. Ghisi and M. Gobbino. Gradient estimates for the Perona-Malik equation. *Math. Ann.*, 337(3):557–590, 2007.

[14] P. I. Plotnikov. Passage to the limit with respect to viscosity in an equation with a variable direction of parabolicity. *Differentsialnye Uravneniya*, 30(4):665–674, 734, 1994.

[15] M. Slemrod. Dynamics of measure valued solutions to a backward-forward heat equation. *J. Dynam. Differential Equations*, 3(1):1–28, 1991.

[16] F. Smarrazzo and A. Tesei. Degenerate regularization of forward-backward parabolic equations. Part I: the regularized problem. *Arch. Rational Mech. Anal.*, 204:85–139, 2012.

[17] F. Smarrazzo and A. Tesei. Degenerate regularization of forward-backward parabolic equations: the vanishing viscosity limit. *Math. Ann.*, to appear.