L^1–GRADIENT FLOW OF CONVEX FUNCTIONALS

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ABSTRACT. We are interested in the gradient flow of a general first order convex functional with respect to the L^1-topology. By means of an implicit minimization scheme, we show existence of a global limit solution, which satisfies an energy-dissipation estimate, and solves a non-linear and non-local gradient flow equation, under the assumption of strong convexity of the energy. Under a monotonicity assumption we can also prove uniqueness of the limit solution, even though this remains an open question in full generality. We also consider a geometric evolution corresponding to the L^1-gradient flow of the anisotropic perimeter. When the initial set is convex, we show that the limit solution is monotone for the inclusion, convex and unique until it reaches the Cheeger set of the initial datum. Eventually, we show with some examples that uniqueness cannot be expected in general in the geometric case.

CONTENTS

1. INTRODUCTION

We consider the functional

$$\Phi(u) := \int_\Omega F(Du) \qquad u \in BV(\Omega),$$

where Ω is a bounded, connected, open subset of $\mathbb{R}^d$, and $F : \mathbb{R}^d \to [0, +\infty]$ is a convex function with $F(\xi) \geq c(|\xi| - 1)$ for some $c > 0$. We also assume that the set $\{\xi : F(\xi) < +\infty\}$ has non-empty interior and contains 0.

We are interested in the gradient flow of Φ with respect to the $L^1(\Omega)$-topology, with either homogeneous Neumann, or Dirichlet boundary conditions; in the latter case the functional has to be relaxed, with an appropriate boundary integral, if the function F has linear growth, and we leave the details to the reader.

In order to show existence of a solution, we follow the general approach in [11] (see also the comprehensive reference [3]), which is known as the *minimizing movement scheme* and applies to functionals on metric spaces, under general assumptions. However, most of the theory developed in [3] does not apply to our setting, since the Banach space $L^1(\Omega)$ does not satisfy the Radon-Nikodým property (see [3, Remark 1.4.6]). In particular, we cannot derive uniqueness of gradient flow solution from general results, and we are able to prove it only in some special cases.

For this reason, the are few results in the literature concerning L^1-gradient flows. In [10] the author considers the L^1-gradient flow of a second order functional related to the Willmore energy, and studies in detail rotationally symmetric solutions. We also mention [17] where the authors, motivated by a model of delamination between elastic bodies, study a monotone geometric flow by means of a minimizing movement scheme reminiscent to the one in Section 5. They show existence of a limit solution and discuss some examples.

The plan of the paper is the following: in Section 2 we introduce the minimizing movements and we show convergence of the discrete solutions to a limit solution. We also show a general dissipation estimate from which we derive, under the assumption of strong convexity of F, a gradient flow equation satisfied by the limit solution.

In Section 3 we analyze the case when the initial datum is a subsolution (see Definition 3.1). In such case the limit solution is non-decreasing in time and it is indeed unique.

In Section 4 we consider the simplest possible functional, that is, the Dirichlet energy. In this particular case we can show a stronger uniqueness result, namely that the limit gradient flow equation always admits a unique solution.

Finally, in Section 5 we consider the geometric evolution corresponding to the L^1-gradient flow of the anisotropic perimeter. Even if we are not able to characterize the limit flow as we do in the case of functions, when the initial set is convex, we can prove that the evolution is unique, monotone for the inclusion, and remains convex until it reaches the Cheeger set of the initial set. In two dimensions we also show that it stays convex until it becomes a Wulff Shape, and then shrinks to a point in finite time. Simple examples show that the geometric evolution is in general non-unique, after reaching the Cheeger set.

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2. Existence of solutions

2.1. **Minimizing movements.** Following [11], we introduce the L^1-minimizing movement scheme. Given $u^0 \in L^1(\Omega)$, we let u^n, for $n \geq 1$, be a minimizer of

$$\min_u \Phi(u) + \frac{1}{2\tau}\left(\int_\Omega |u - u^{n-1}|dx\right)^2. \tag{2.1}$$

If F has superlinear growth, then $u^n \in W^{1,1}(\Omega)$. Assuming in addition that F is strictly convex, we deduce that if u' is another solution, $Du' = Du^n$ a.e., and $u' - u^n$ is a constant. As a consequence, any other solution is of the form $u^n + c$ where c is a minimizer of $\|u^n - u^{n-1} - c\|_1$, that is, a median value of $u^n - u^{n-1}$. Notice that, by convexity, the set of median values is an interval. If $u^{n-1} \in W^{1,1}(\Omega)$ (which is true for $n \geq 2$, and which will we assume for $n = 1$), then, since $u^n - u^{n-1} \in W^{1,1}(\Omega)$ and Ω is connected, it has a unique median value, hence we have the following result.

Lemma 2.1. *Assume that F is stricly convex with superlinear growth, and that $\Phi(u^0) < +\infty$. Then for any $n \geq 1$, there is a unique minimizer to (2.1).*

Remark 2.2. In case F is not strictly convex or the growth is not superlinear, the uniqueness is not guaranteed. However, in that case,

(1) by strong convexity in $u \mapsto \|u - u^{n-1}\|_1$ of the energy, one easily sees that given any two minimizers u, u' of (2.1), $\|u - u^{n-1}\|_1 = \|u' - u^{n-1}\|_1$;
(2) one can easily build measurable selections of the solutions $\tau \mapsto u^n$ as τ varies, as follows. A first observation is that for any $p \in [1, d/(d-1)]$, if the energy of u in (2.1) is finite, then $u \in L^p(\Omega)$, by Sobolev's embedding and using that $\Phi(u)$ controls the total variation of u. Then, given $p \in (1, d/d-1)$, for $\varepsilon > 0$, one can consider the unique minimizer $u^\varepsilon(\tau)$ of the strictly convex energy:
$$\Phi(u) + \frac{1}{2\tau}\left(\int_\Omega |u - u^{n-1}|dx\right)^2 + \varepsilon \int_\Omega |u|^p dx$$
and one easily shows that $\tau \mapsto u^\varepsilon(\tau)$ is continuous (in $L^1(\Omega)$, as well as $L^p(\Omega)$). Sending $\varepsilon \to 0$ we find that $u^\varepsilon(\tau) \to u(\tau)$, the solution of (2.1) with minimal L^p norm, which is thus a measurable selection.

We can now define $u_\tau(t) := u^{\lfloor t/\tau \rfloor}$ where $\lfloor \cdot \rfloor$ is the integer part, and we show the following theorem (whose proof is classical).

Theorem 2.3. *Assume that $\Phi(u^0) < +\infty$. Then, there exists $u \in C^0([0, +\infty); L^1(\Omega))$ and a subsequence $\tau_k \to 0$ such that $u_{\tau_k} \to u$ in $L^\infty([0, T]; L^1(\Omega))$, for all $T > 0$, and*
$$\|u(s) - u(t)\|_1 \leq \sqrt{2\Phi(u^0)}\sqrt{|t - s|}$$
for any $t, s \in [0, T]$.

Remark 2.4. If we consider the piecewise affine interpolant $\hat{u}_\tau$ of u^n in time, defined as $u^n + (t/\tau - 1)(u^{n+1} - u^n)$ for $n\tau \leq t \leq (n+1)\tau$, rather than the piecewise constant interpolant, then the convergence is also in $C^0([0, T]; L^1(\Omega))$.

Proof. For any $0 \leq m < n$, we have
$$\begin{aligned}
\|u_\tau(n\tau) - u_\tau(m\tau)\|_1^2 &\leq \left(\sum_{k=m}^{n-1} \|u_\tau((k+1)\tau) - u_\tau(k\tau)\|_1\right)^2 \\
&\leq (n - m)\sum_{k=m}^{n-1} \|u_\tau((k+1)\tau) - u_\tau(k\tau)\|_1^2 \\
&\leq 2\tau(n - m)\sum_{k=m}^{n-1} (\Phi(u_\tau(k\tau) - \Phi(u_\tau((k+1)\tau)) \\
&= 2(\Phi(u^m) - \Phi(u^n))(n\tau - m\tau) \\
&\leq 2\Phi(u^0)(n\tau - m\tau),
\end{aligned}$$

where we used the Cauchy-Schwarz inequality and the minimality of $u_\tau(k\tau)$, and the fact that the sequence $(\Phi(u^n))_n$ is non-increasing.

We deduce in addition that $\Phi(u_\tau(t)) \le \Phi(u^0)$ for any $t > 0$, so that, thanks to the assumptions on F and together with the bound on $\|u_\tau(t) - u^0\|_1$, we find that there is a compact subset of $L^1(\Omega)$ (even $L^p(\Omega)$, for $p < d/(d-1)$) which contains $u_\tau(t)$ for any $t > 0$.

For any $t, s \ge 0$, if follows that

$$\|u_\tau(t) - u_\tau(s)\|_1 \le \sqrt{2\Phi(u^0)}\sqrt{|\lfloor t/\tau\rfloor - \lfloor s/\tau\rfloor|\,\tau} \le \sqrt{2\Phi(u^0)}\sqrt{\tau + |t-s|}.$$

The compactness and convergence is then deduced by the Ascoli-Arzelà Theorem. □

By a simple interpolation argument, we can show a slightly improved convergence for the previous theorem.

Proposition 2.5. *Let $p \in [1, d/(d-1))$. Then the subsequence $(u_{\tau_k})_k$ in Theorem 2.3 also converges to u in $L^\infty([0,T]; L^p(\Omega))$ for any $T > 0$, while the piecewise-affine interpolants $\hat{u}_{\tau_k}$ converge in $C^0([0,T]; L^p(\Omega))$.*

Proof. By construction, for $t \in [0,T]$ the norms $\|u_\tau(t)\|_{d/(d-1)}$ are uniformly bounded and for $1 < p < d/(d-1)$, there is a compact set of $L^p(\Omega)$ such that $u_\tau(t) \in C_p$.

For $0 < \epsilon < 1$, writing $|u_\tau(t) - u_\tau(s)|^p = |u_\tau(t) - u_\tau(s)|^{1-\epsilon}|u_\tau(t) - u_\tau(s)|^{p-1+\epsilon}$ and using Hölder inequality, we have

$$\|u_\tau(t) - u_\tau(s)\|_p^p \le \|u_\tau(t) - u_\tau(s)\|_1^{1-\epsilon}\left(\int_\Omega |u_\tau(t) - u_\tau(s)|^{\frac{p-1+\epsilon}{\epsilon}}\right)^\epsilon$$

hence if $(p-1)/\epsilon + 1 \le d/(d-1)$, for instance for $\epsilon = (p-1)(d-1) < 1$ (or any $\epsilon < 1$ if $d = 1$), we find that

$$\|u_\tau(t) - u_\tau(s)\|_p \le C\sqrt{\tau + |t-s|}^{\frac{d}{p} - (d-1)}$$

Hence, the convergence is also in $L^\infty([0,T], L^p(\Omega))$. □

2.2. **Euler-Lagrange equation.** The Euler-Lagrange equation for u^n minimizing (2.1) takes the form:

$$\begin{cases} -\operatorname{div} z^n + \frac{\|u^n - u^{n-1}\|_1}{\tau}\operatorname{sign}(u^n - u^{n-1}) \ni 0 \\ z^n \cdot Du^n = F(Du^n) + F^*(z^n) \end{cases} \tag{2.2}$$

where the last statement is in a weak sense if F has minimal growth 1 (Du can be a measure), otherwise we just have $z^n \in \partial F(Du^n)$.

This follows from [13, Prop. 5.6], applied in $V = L^1(\Omega)$ and $V^* = L^\infty(\Omega)$. In that case, $u \mapsto \|u - u^{n-1}\|^2/(2\tau)$ is everywhere continuous while Φ is lower semicontinuous. Hence, $\partial(\Phi(\cdot) + \|\cdot - u^{n-1}\|_1^2/(2\tau)) = \partial\Phi + \partial\|\cdot - u^{n-1}\|_1^2/(2\tau)$, where the subgradients are elements of $L^\infty(\Omega)$. So a minimizer (u^n) is characterized by

$$0 \in \partial\Phi(u^n) + \frac{\|u^n - u^{n-1}\|_1}{\tau}\operatorname{sign}(u^n - u^{n-1}) \tag{2.3}$$

where $\operatorname{sign}(t) = \{1\}$ for $t > 0$, $\{-1\}$ for $t < 0$, and $[-1, 1]$ for $t = 0$.

2.3. **Estimate of the time derivative.** The class of functionals Φ we are considering satisfy the following fundamental estimate: for any $u, v \in L^1(\Omega)$,

$$\Phi(u \wedge v) + \Phi(u \vee v) \leq \Phi(u) + \Phi(v) \tag{2.4}$$

(with equality if F has superlinear growth). Here for $x, y \in \mathbb{R}$, $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$ and the notation extends to real-valued functions. In this context, we can prove the following:

Lemma 2.6. *Let $v \in L^1(\Omega)$, $q \in L^\infty(\Omega)$ with $q \in \partial\Phi(v)$. Let u be a minimizer of:*

$$\Phi(u) + \frac{1}{2\tau}\|u - v\|_1^2.$$

Then

$$\frac{\|u - v\|_1}{\tau} \leq \|q\|_\infty.$$

Proof. The following remark is crucial: if $q \in \partial\Phi(v)$, $p \in \partial\Phi(u)$, then (denoting as usual $x^+ = x \vee 0$ and $x^- = (-x)^+$):

$$\int_\Omega (q - p)(v - u)^+ dx \geq 0 \quad \text{ and } \quad \int_\Omega (q - p)(v - u)^- dx \leq 0. \tag{2.5}$$

Indeed, one has:

$$\Phi(u \vee v) \geq \Phi(u) + \int_\Omega p(u \vee v - u)dx \quad \text{ and } \quad \Phi(u \wedge v) \geq \Phi(v) + \int_\Omega q(u \wedge v - v)dx. \tag{2.6}$$

Using that $u \vee v - u = (v - u)^+$ and $u \wedge v - v = -(v - u)^+$, the first inequality in (2.5) follows by summing the two previous inequalities and using (2.4). The second is proved similarly.

Since in the Lemma, u satisfies the equation (*cf* (2.3)):

$$\exists p \in \partial\Phi(u) \cap -\mathrm{sign}(u - v)\frac{\|u - v\|_1}{\tau},$$

we deduce from (2.5) that (here "sign" is single-valued as the integrand vanishes for $v \leq u$):

$$0 \leq \int_\Omega \left(q - \mathrm{sign}(v - u)\frac{\|u - v\|_1}{\tau}\right)(v - u)^+ dx \leq \left((\operatorname*{ess\,sup}_\Omega q) - \frac{\|u - v\|_1}{\tau}\right)\int_\Omega (v - u)^+ dx \tag{2.7}$$

so that if $\{v > u\}$ has positive measure, $\frac{\|u-v\|_1}{\tau} \leq \mathrm{ess\,sup}_\Omega\, q$. Similarly (multiplying with $-(v - u)^-$) we show that so that if $\{v < u\}$ has positive measure, $\frac{\|u-v\|_1}{\tau} \leq -\mathrm{ess\,inf}_\Omega\, q$. The thesis follows. □

We deduce immediately the following result, as a consequence of Lemma 2.6 and the Euler-Lagrange equation (2.3).

Theorem 2.7. *Let $u^0 \in L^1(\Omega)$, $\tau > 0$ and $(u^n)_{n\geq 0}$ defined by the minimizing movement scheme. Then*

i. *for any $n \geq 1$,*

$$\frac{\|u^{n+1} - u^n\|_1}{\tau} \leq \frac{\|u^n - u^{n-1}\|_1}{\tau};$$

ii. *if in addition $\partial\Phi(u^0) \neq \emptyset$, then for any $n \geq 0$,*

$$\frac{\|u^{n+1} - u^n\|_1}{\tau} \leq \|\partial\Phi^0(u^0)\|_\infty$$

where $\partial\Phi^0$ denotes the minimal norm of an element in the subgradient.

Corollary 2.8. *Let u be an evolution provided by Theorem 2.3, and assume again $\partial\Phi(u^0) \neq \emptyset$. Then u is Lipschitz in time and satisfies, for a.e. $t \geq 0$,*

$$\|\dot{u}(t)\|_1 \leq \|\partial\Phi^0(u^0)\|_\infty.$$

2.4. **Dissipation estimate.** We shall prove the following dissipation estimate.

Theorem 2.9. *Let u^0 satisfy $\Phi(u^0) < +\infty$ and let u be a limit of minimizing movements given by Theorem 2.3. Then, for any $t > 0$, $\dot{u}$ is a measure with marginal $s \mapsto |\dot{u}(s)|(\Omega)$ in $L^2(0,t)$ and there exists $q \in L^2((0,t);L^\infty(\Omega))$ with $q(s) \in -\partial\Phi(u(s))$ for a.e. $s \geq 0$ such that*

$$\Phi(u(t)) + \frac{1}{2}\int_0^t (|\dot{u}(s)|(\Omega))^2 ds + \frac{1}{2}\int_0^t \|q(s)\|_\infty^2 ds \leq \Phi(u^0). \tag{2.8}$$

Proof. We remain in the framework of Theorem 2.3, assuming that $\Phi(u^0) < +\infty$ and that u_τ, defined above converges, up to a subsequence, to a function $u \in C^{0,1/2}([0,T];L^1(\Omega))$.

As usual (see for instance [3, Sec. 3.2]), for $n\tau < t < (n+1)\tau$, we let $\tilde{u}_\tau(t)$ be a minimizer of

$$\min_u \Phi(u) + \frac{1}{2(t-n\tau)}\|u - u^n\|_1^2,$$

which satisfies the Euler-Lagrange equation

$$\partial\Phi(\tilde{u}_\tau(t)) + \frac{\|\tilde{u}_\tau(t) - u^n\|_1}{2(t-n\tau)}\mathrm{sign}(\tilde{u}_\tau(t) - u^n) \ni 0. \tag{2.9}$$

By Remark 2.2-(1), observe that even if the minimizer might be non-unique, the value of $\|\tilde{u}_\tau(t) - u^n\|_1$ is. In any case, as mentioned in Remark 2.2-(2), we assume that $t \mapsto \tilde{u}_\tau(t)$ is measurable. We also have that $\|\tilde{u}_\tau(t) - u_\tau(t)\|_1 \leq \sqrt{2\Phi(u^0)(t-\tau)}$ so that $\tilde{u}_\tau$ converges to the same limit as u_τ, also uniformly in time.

Now, for $n \geq 0$, $0 < s < \tau$, we let $h(s) = \Phi(\tilde{u}_\tau(n\tau + s)) + \|\tilde{u}_\tau(n\tau + s) - u^n\|_1^2/(2s)$, hence $h(\tau) = \Phi(u^{n+1}) + \|u^{n+1} - u^n\|_1^2/(2\tau)$ and $\lim_{s\to 0} h(s) = \Phi(u^n)$. It is standard that:

$$h'(s) \leq -\frac{\|\tilde{u}_\tau(n\tau + s) - u^n\|_1^2}{2s^2},$$

so that (using $h(\tau) = \lim_{\epsilon\to 0} h(\epsilon) + \int_\epsilon^\tau h'(s)ds$)

$$\Phi(u^{n+1}) + \frac{\|u^{n+1} - u^n\|_1^2}{2\tau} \leq \Phi(u^n) - \frac{1}{2}\int_0^\tau \frac{\|\tilde{u}_\tau(n\tau + s) - u^n\|_1^2}{s^2} ds.$$

Thanks to the Euler-Lagrange equation (2.9), we deduce:

$$\Phi(u^{n+1}) + \frac{1}{2}\int_{n\tau}^{(n+1)\tau} \|\dot{\hat{u}}_\tau(s)\|_1^2 ds + \frac{1}{2}\int_{n\tau}^{(n+1)\tau} \|q_\tau(s)\|_\infty^2 ds \leq \Phi(u^n),$$

where for all t, $q_\tau(t) \in -\partial\Phi(\tilde{u}_\tau(t))$, and $\hat{u}(t)$ is the piecewise-affine interpolant, which also converges to u up to a subsequence (in $C^0([0,T];L^p(\Omega))$ for $1 \leq p \leq d/(d-1)$, see Prop. 2.5). Summing this inequality from $n = 0$ to $\lfloor t/\tau \rfloor - 1$, for $0 < t \leq T$, we find:

$$\Phi(u_\tau(t)) + \frac{1}{2}\int_0^{t-\tau} \|\dot{\hat{u}}_\tau(s)\|_1^2 ds + \frac{1}{2}\int_0^{t-\tau} \|q_\tau(s)\|_\infty^2 ds \leq \Phi(u^0).$$

By lower-semicontinuity of the convex functions appearing in the integrals we claim that (2.8) is deduced, where q is a weak limit (in $L^2([0,T];L^{p'}(\Omega))$) of q_τ, and p' the conjugate exponent of some $p \in (1, d/(d-1))$.

The only difficulty is with the measure term. Given $\varphi \in C_c^\infty([0,T)\times\Omega)$, it is not difficult to check that for τ small enough:

$$\frac{1}{2}\int_0^{t-\tau}\|\dot{\hat{u}}_\tau(s)\|_1^2 ds \geq \int_\Omega \varphi(0,x)u^0(x)dx - \int_0^t\int_\Omega \dot\varphi \hat{u}_\tau dxds - \frac{1}{2}\int_0^t \|\varphi(s)\|_\infty^2 ds$$

so that, passing to the limit along an appropriate subsequence,

$$\int_\Omega \varphi(0)u^0 dx - \int_0^t\int_\Omega \dot\varphi u dxds - \frac{1}{2}\int_0^t\|\varphi(s)\|_\infty^2 ds \leq \liminf_{\tau\to 0}\frac{1}{2}\int_0^{t-\tau}\|\dot{\hat{u}}_\tau(s)\|_1^2 ds =: \ell.$$

In particular (using also that $u(t)\to u^0$ as $t\to 0$), we deduce immediately that the distribution $\dot u$ is a bounded Radon measure (in $[0,T)\times\Omega$), satisfying for all $t\leq T$:

$$\int_{[0,t]\times\Omega}\varphi d\dot u - \frac{1}{2}\int_0^t \|\varphi(s)\|_\infty^2 ds \leq \ell.$$

Letting $n\geq 1$, $0=t_0<t_1<\cdots<t_n=t$ and considering $m_i\geq 0$, $i=1,\dots,n$, and the supremum over all functions φ with $\varphi_{|(t_{i-1},t_i)}\in C_c^\infty([t_{i-1},t_i)\times\Omega)$ with $\|\varphi\|_{L^\infty(t_{i-1},t_i)}\leq m_i$ we deduce:

$$\sum_{i=1}^n m_i|\dot u|([t_{i-1},t_i)\times\Omega) - (t_i-t_{i-1})\frac{m_i^2}{2}\leq \ell.$$

By uniform approximation of a smooth function $\psi\in C_c^\infty([0,t);\mathbb{R}_+)$ by piecewise constant functions, we deduce that the marginal measure $|\dot u|(\Omega)$ in $(0,t)$ satisfies:

$$\int_0^t \psi(s)d(|\dot u|(\Omega))(s) - \frac{1}{2}\psi^2(s)ds \leq \ell$$

and it follows that $|\dot u|(\Omega)$ is indeed in $L^2(0,t)$, with

$$\frac{1}{2}\int_0^t (|\dot u|(\Omega))^2 ds \leq \ell.$$

Now, we check that $q(t)\in\partial\Phi(u(t))$ a.e.: given $\varphi\in C_c^\infty((0,T)\times\Omega)$, we have

$$\int_0^T \Phi(\varphi(t))dt \geq \int_0^T\Phi(\tilde u_\tau(t))dt + \int_0^T\int_\Omega q_\tau(t,x)(\tilde u_\tau(t,x)-\varphi(t,x))dxdt.$$

Since $\tilde u_\tau\to u$ in $L^\infty([0,T];L^p(\Omega))$ (using Prop. 2.5) and $q_\tau \rightharpoonup q$ in $L^2([0,T];L^{p'}(\Omega))$, we obtain that

$$\int_0^T\int_\Omega q_\tau(t,x)\tilde u_\tau(t,x)dxdt \to \int_0^T\int_\Omega q(t,x)u(t,x)dxdt.$$

It follows that

$$\int_0^T \Phi(\varphi(t))dt \geq \int_0^T \Phi(u(t))dt + \int_0^T\int_\Omega q(t,x)(u(t,x)-\varphi(t,x))dxdt.$$

We deduce that for a.e. t, $-q(t)\in\partial\Phi(u(t))$. □

A dissipation estimate like (2.8) usually implies that the flow $u(t)$ is a curve of maximal slope in the sense of [3, Def. 1.3.2], satisfying

$$\partial_t \Phi(u(t)) = -\int_\Omega q(t)u(t)\,dx \qquad \text{for a.e. } t \geq 0. \tag{2.10}$$

However, as already observed in the Introduction, the results in [3] fail to apply in the $(1,\infty)$-duality, since $L^1(\Omega)$ does not satisfy the Radon-Nikodým property.

We shall rigorously prove (2.10) in the next section, under the additional assumption that F is smooth and strongly convex.

2.5. **Strongly convex case.** In this part, we first assume that in addition there exists $\gamma > 0$ such that F is γ-convex:

$$F(\eta) \geq F(\xi) + p\cdot(\eta-\xi) + \frac{\gamma}{2}|\eta-\xi|^2$$

for any $\eta, \xi \in \mathbb{R}^d$ and $p \in \partial F(\xi)$. Then (2.6) becomes

$$\begin{aligned}
\Phi(u\vee v) &\geq \Phi(u) + \int_\Omega p(u\vee v - u)dx + \frac{\gamma}{2}\int_\Omega |D(u\vee v-u)|^2 dx \\
\Phi(u\wedge v) &\geq \Phi(v) + \int_\Omega q(u\wedge v - v)dx + \frac{\gamma}{2}\int_\Omega |D(u\wedge v-v)|^2 dx.
\end{aligned}$$

One now deduces, following the arguments in the proof of Lemma 2.6:

$$\begin{aligned}
\gamma\int_{\{v>u\}} |Dv-Du|^2 dx &\leq \int_\Omega (q-p)(v-u)^+ dx \\
\gamma\int_{\{v<u\}} |Dv-Du|^2 dx &\leq -\int_\Omega (q-p)(v-u)^- dx.
\end{aligned} \tag{2.11}$$

Summing, we find:

$$\gamma\int_\Omega |Dv-Du|^2 dx \leq \int_\Omega (q-p)(v-u)dx.$$

Using $v = u^n$, $u = u^{n+1}$ and (2.3), it follows for all $n \geq 1$:

$$\begin{aligned}
&\gamma\int_\Omega |Du^{n+1}-Du^n|^2 dx \\
&\leq \int_\Omega \left(-\operatorname{sign}(u^n-u^{n-1})\frac{\|u^n-u^{n-1}\|_1}{\tau} + \operatorname{sign}(u^{n+1}-u^n)\frac{\|u^{n+1}-u^n\|_1}{\tau}\right)(u^n-u^{n+1})dx \\
&\qquad\qquad \leq -\frac{1}{\tau}\|u^{n+1}-u^n\|_1^2 + \frac{1}{\tau}\|u^n-u^{n-1}\|_1\|u^{n+1}-u^n\|_1
\end{aligned}$$

(there is an abuse of notation here since "sign" is multivalued, however we use only that $|\text{sign}| \leq 1$ and $\operatorname{sign}(u^{n+1}-u^n)(u^n-u^{n+1}) = -|u^{n+1}-u^n|$), which we rewrite:

$$\begin{aligned}
\gamma\tau\int_\Omega \left|D\frac{u^{n+1}-u^n}{\tau}\right|^2 dx + \frac{1}{2\tau^2}(\|u^{n+1}-u^n\|_1 - \|u^n-u^{n-1}\|_1)^2 + \frac{\|u^{n+1}-u^n\|^2}{2\tau^2} \\
\leq \frac{\|u^n-u^{n-1}\|^2}{2\tau^2}.
\end{aligned} \tag{2.12}$$

Then, summing (2.12), we get the estimate:

$$\gamma \int_0^{n\tau} \|D\dot{\hat{u}}_\tau(t+\tau)\|_2^2 dt \le \frac{\|u^1 - u^0\|^2}{2\tau^2}. \tag{2.13}$$

Remark 2.10. If the initial speed is not bounded we can sum from m to n and get

$$\gamma \int_{m\tau}^{n\tau} \|D\dot{\hat{u}}_\tau(t+\tau)\|_2^2 dt \le \frac{\|u^{m+1} - u^m\|_1^2}{2\tau^2} \le \frac{1}{m\tau^2} \sum_{k=0}^{m} \|u^{k+1} - u^k\|_1^2,$$

using Theorem 2.7, i. Then, using the minimality in the energy, we obtain:

$$\gamma \int_{m\tau}^{n\tau} \|D\dot{\hat{u}}_\tau(t+\tau)\|_2^2 dt \le \frac{2}{m\tau} \Phi(u^0). \tag{2.14}$$

Recalling Theorem 2.7, ii., we are in particular able to deduce the following result:

Theorem 2.11. *Assume F is γ-convex and let u be given by Theorem 2.3. Then $\dot{u} \in L^2((t,+\infty); H^1(\Omega))$ for any $t > 0$, with*

$$\frac{\gamma}{2} \int_t^{+\infty} |D\dot{u}|^2 ds \le \frac{\Phi(u^0)}{t}.$$

If in addition $\partial\Phi(u^0) \neq \emptyset$, then

$$\gamma \int_0^{+\infty} |D\dot{u}|^2 ds \le \|\partial^0 \Phi(u^0)\|_\infty^2$$

Remark 2.12. Taking into account (2.11) when deriving (2.7), we can derive slightly more precise estimates which may be useful in case the initial speed $q^0 \in \partial\Phi(u^0)$ has a sign. Indeed, we obtain for instance that:

- If $\{u^1 > u^0\}$ has positive measure, then
$$\frac{\|u^1 - u^0\|_1}{\tau} \le \operatorname{ess\,sup}(-q^0) - \gamma \frac{\int_\Omega |D(u^1-u^0)^+|^2 dx}{\|(u^1-u^0)^+\|_1};$$
- If $\{u^1 < u^0\}$ has positive measure, then
$$\frac{\|u^1 - u^0\|_1}{\tau} \le \operatorname{ess\,sup} q^0 - \gamma \frac{\int_\Omega |D(u^1-u^0)^-|^2 dx}{\|(u^1-u^0)^-\|_1}.$$

In particular, if $q^0 \le 0$ a.e., we deduce that $u^1 \ge u^0$ a.e., but then $q^1 := -\mathrm{sign}(u^1 - u^0)\|u^1 - u^0\|^2/\tau \in \partial\Phi(u^1)$ is also non-positive and again, $u^2 \ge u^1$ a.e.: by induction we find that $u^{n+1} \ge u^n$ for all $n \ge 0$.

If F is γ-convex and C^1, we have additionally (Remark 2.10) that $\dot{u}(t) \in H^1(\Omega)$ for $t > 0$, and $q(t) = \operatorname{div} \nabla F(Du(t))$ for a.e. $t > 0$. In addition, $\partial_t \Phi(u) = \int_\Omega \nabla F(Du) \cdot D\dot{u} dx = -\int_\Omega q\dot{u} dx$ so that

$$\Phi(u(0)) - \Phi(u(t)) = \int_0^t \int_\Omega q(s)\dot{u}(s) dx ds \le \frac{1}{2} \int_0^t \|q(s)\|_\infty^2 + \|\dot{u}(s)\|_1^2$$

which combined with (2.8), yields that $q(s) \in \partial\|\cdot\|_1^2(\dot{u}(s))/2$ for a.e. $s > 0$. Hence we have:

Theorem 2.13. *Assume F is C^1 and strongly convex, and let u be a limit of minimizing movements given by Theorem 2.3, starting from u^0 with $\Phi(u^0) < +\infty$. Then, $\dot u \in L^2((t,+\infty); H^1(\Omega))$ for any $t > 0$ and satisfies the equations*

$$\begin{cases} |\operatorname{div} \nabla F(Du)| \le \|\dot u\|_1 & a.e.\ in\ \Omega \times (0,+\infty) \\ \dot u \operatorname{div} \nabla F(Du) = |\dot u| \|\dot u\|_1 & a.e.\ in\ \Omega \times (0,+\infty). \end{cases} \tag{2.15}$$

2.6. **Minimal surface energy.** The case where $\Phi(u) = \int_\Omega \sqrt{1+|Du|^2}dx$ is in between the prevous section and the last Section 5, where we introduce a geometric version of this gradient flow. In that case, we remark that if we can show that when u^0 is L-Lipschitz for some constant $L \ge 0$, u remains L-Lipschitz, then from Section 2.5 we deduce that the solution satisfies $\dot u \in H^1(\Omega)$ for positive time and that the characterization (2.15) holds. Indeed, in that case, since the gradients are all bounded by L, F is γ-convex, with $\gamma = (1+L^2)^{-3/2}$.

This is the case for instance if we consider the problem in a periodic setting ($\Omega = \mathbb{R}^d/\mathbb{Z}^d$):

Lemma 2.14. *Let $\Omega = \mathbb{R}^d/\mathbb{Z}^d$, $F(p) = \sqrt{1+|p|^2}$, v a L-Lipschitz, $L \ge 0$ function and u a minimizer of:*

$$\min_u \Phi(u) + \frac{1}{2\tau}\left(\int_\Omega |u-v|dx\right)^2.$$

Then u is L-Lipschitz, and unique.

Proof. It is enough to show it for the unique solution u_p, $p > 1$, of:

$$\min_u \Phi(u) + \frac{1}{2\tau}\left(\int_\Omega |u-v|^p dx\right)^{2/p} \tag{2.16}$$

since in the limit $p \to 1$ one recover a minimizer (hence *the* minimizer) for $p = 1$.

We first show the following: let $v > v'$, let u minimize, for some $\lambda > 0$:

$$\min_u \Phi(u) + \frac{\lambda}{p}\int_\Omega |u-v|^p dx \tag{2.17}$$

and let u' solve the same problem with v replaced with v'. Then, comparing the energy of u with the energy of $u \vee u'$, and the energy of u' with the energy of $u \wedge u'$ and summing both inequality we end up (using (2.4)) with:

$$\int_\Omega |u'-v'|^p dx - \int_\Omega |u \wedge u' - v'|^p dx \le \int_\Omega |u \vee u' - v|^p dx - \int_\Omega |u-v|^p dx,$$

that is:

$$\int_{\{u<u'\}} |u'-v'|^p - |u-v'|^p dx \le \int_{\{u<u'\}} |u'-v|^p - |u-v|^p dx.$$

One may write this:

$$\int_{\{u<u'\}} \int_{u(x)}^{u'(x)} p|t-v'(x)|^{p-2}(t-v'(x)) - p|t-v(x)|^{p-2}(t-v(x)) dx \le 0,$$

which, since $-v(x) < -v'(x)$, is not true unless $u \ge u'$ a.e.

Now, assume v is L-Lipschitz and let $u = u_p$ be the minimizer of (2.16). For $z \in \mathbb{R}^d$, $\varepsilon > 0$, let $v'(x) = v(x-z) - L|z| - \varepsilon < v(x)$ and $u'(x) = u(x-z) - L|z| - \varepsilon$ be the solution of (2.16) with v replaced with v'. The Euler-Lagrange equation for v and v' are:

$$\begin{cases} -\partial\Phi(u) + \frac{1}{\tau}\left(\int_\Omega |u-v|^p dx\right)^{2/p-1} |u-v|^{p-2}(u-v) = 0, \\ -\partial\Phi(u') + \frac{1}{\tau}\left(\int_\Omega |u'-v'|^p dx\right)^{2/p-1} |u'-v'|^{p-2}(u'-v') = 0 \end{cases}$$

hence letting $\lambda = \|u-v\|^{2-p}/\tau = \|u'-v'\|^{2-p}/\tau$, we find that u is a minimizer of (2.17) while u' is a minimizer of the same problem with v replaced with v'. We deduce that $u' \le u$. Sending $\varepsilon \to 0$, it follows that

$$u(x-z) - L|z| \le u(x) \quad \forall x \in \Omega, z \in \mathbb{R}^d$$

which shows that u is L-Lipschitz.

We now observe that if there is another solution $u' \in BV(\mathbb{R}^d/\mathbb{Z}^d)$, by convexity arguments, the absolutely continuous part of the gradient must be the same as Du, and they can differ only by a singular part. On the other hand, the regularity theory for minimal surfaces shows that $\{(x, u'(x)) : x \in \mathbb{R}^d/\mathbb{Z}^d\} \subset (\mathbb{R}^d/\mathbb{Z}^d)\times\mathbb{R}$ is $C^{1,\alpha}$ (for any $\alpha < 1$), so that if it has a vertical part, one must have $|Du| = +\infty$ somewhere, a contradiction. Hence u is the unique solution of the problem. □

Hence, we obtain the following result:

Theorem 2.15. *Let u^0 a Lipschitz function over $\Omega = \mathbb{R}^d/\mathbb{Z}^d$. Then the discrete motion converges to $u(t) \in C^0([0,T]; L^p(\Omega))$ for any $p < d/(d-1)$, with $\int_s^\infty |D\dot{u}|^2 dt \le C\Phi(u^0)/s$ for any $s > 0$, and satisfies*

$$\begin{cases} |\kappa_u(x)| \le \|\dot{u}\|_1 & a.e.\ in\ \Omega, \\ -\dot{u}\kappa_u(x) = |\dot{u}|\|\dot{u}\|_1 & a.e.\ in\ \Omega \end{cases} \quad for\ a.e.\ t \ge 0,$$

where $\kappa_u = \operatorname{div}\left(Du/\sqrt{1+|Du|^2}\right)$ a.e.

Remark 2.16. The proof of the existence of a L-Lipschitz solution on the torus when u^0 is L-Lipschitz is valid for any convex function $F(Du)$, so that the results in this section are also true for F smooth and strongly convex on any bounded subset of $\mathbb{R}^d$.

3. Monotone solutions

In this section we consider the case of Dirichlet boundary conditions ($\mathrm{dom}(\Phi) = u^0 + H_1^0(\Omega)$), and we assume that $u^0 \in BV(\Omega)$ a *subsolution* in the following sense:

Definition 3.1. *We say that $u^0 \in BV(\Omega)$ is a subsolution if for any $v \in BV(\Omega)$ with $\{v \ne u^0\} \subset\subset \Omega$, we have*

$$v \le u^0 \Rightarrow \Phi(v) \ge \Phi(u^0).$$

Lemma 3.2. *If u^0 is a subsolution then, for any $v \in BV(\Omega)$ with $\{v \ne u^0\} \subset\subset \Omega$, we have*

$$\Phi(\max\{u^0, v\}) \le \Phi(v).$$

Proof. Since u_0 is a subsolution, we know that $\Phi(\min\{u^0, v\}) \ge \Phi(u^0)$. Recalling that

$$\Phi(\min\{u^0, v\}) + \Phi(\max\{u^0, v\}) \le \Phi(v) + \Phi(u^0),$$

it follows that $\Phi(\max\{u^0, v\}) \le \Phi(v)$.

□

Replacing u^0 with $\max\{u^0, u^1\}$ in the variational problem which defines u^1, we find that $u^1 \geq u^0$ a.e. in Ω; in particular, the Euler-Lagrange equation reads:

$$\partial\Phi(u^1) + \frac{\|u^1 - u^0\|_1}{\tau}\varphi(x) = 0, \quad \varphi \in \operatorname{sign}(u^1 - u^0) \quad \text{a.e. in } \Omega.$$

Proposition 3.3. *If u^0 is a subsolution then, for any $n \geq 1$, $u^n \geq u^{n-1}$ and u^n is also a subsolution.*

Proof. This follows the proof of a similar result in [12] for mean-convex sets, see also Sec. 5.1. By Lemma 3.2, for any $v \in BV(\Omega)$ with $\{v \neq u^0\} \subset\subset \Omega$, we have

$$\Phi(\max\{u^0, v\}) \leq \Phi(v).$$

Let $v \in BV(\Omega)$ with $\{v \neq u^0\} \subset\subset \Omega$, and assume $v \leq u^1$. We have

$$\begin{aligned}\Phi(v) \geq \Phi(\max\{u^0, v\}) \geq \Phi(u^1) + \int_\Omega -\frac{\|u^1 - u^0\|_1}{\tau}\varphi(\max\{u^0, v\} - u^1)dx \\ = \Phi(u^1) + \frac{\|u^1 - u^0\|_1}{\tau}\int_{\{u^1>u^0\}}(u^1 - \max\{u^0, v\})dx \geq \Phi(u^1)\end{aligned}$$

showing that u^1 is also a subsolution, and the thesis follows by iterating the argument. $\square$

Let us set now $\lambda_1 = \|u^1 - u^0\|_1/\tau$. We observe that for any $v \geq u^0$ with $v - u^0$ with compact support, one has

$$\Phi(v) \geq \Phi(u^1) + \lambda_1 \int_\Omega -\varphi(v - u^1)dx.$$

Since $v \geq u^0$, we get

$$\begin{aligned}\int_\Omega -\varphi(v - u^1)dx = \int_{\{u^1>u^0\}} u^1 - v\,dx + \int_{\{u^1=u^0\}} -\varphi(v - u^0)dx \\ \geq \int_{\{u^1>u^0\}} u^1 - v\,dx - \int_{\{u^1=u^0\}} v - u^0\,dx = \int_\Omega u^1 - v\,dx,\end{aligned}$$

and we deduce that

$$\Phi(v) + \lambda_1 \int_\Omega v\,dx \geq \Phi(u^1) + \lambda_1 \int_\Omega u^1 dx.$$

It follows that u^1 is a solution of the obstacle problem (with Dirichlet boundary conditions)

$$\min_{v \geq u^0} \Phi(v) + \lambda_1 \int_\Omega v\,dx.$$

Notice that, if F has superlinear growth and is strictly convex, the solution is unique.[1] Observe also that, if v, v' are minimizers of the above obstacle problem for, respectively, two different non-negative parameters λ and λ', then the inequality

$$\Phi(v) + \lambda \int_\Omega v\,dx + \Phi(v') + \lambda' \int_\Omega v'\,dx \leq \Phi(v \wedge v') + \lambda \int_\Omega v \wedge v'\,dx + \Phi(v \vee v') + \lambda' \int_\Omega v \vee v'\,dx$$

shows that $(\lambda - \lambda')\int_\Omega (v - v')^+ dx \leq 0$. Hence, if $\lambda > \lambda'$ one has $v \leq v'$.

[1] When F has linear growth such a statement is unclear, we only know that, for all λ_1 but a countable number, the solution is unique, otherwise it is trapped in between a minimal and a maximal solution.

Let us now introduce, for $m \geq 0$, the volume function

$$f(m) := \min \left\{ \Phi(v) : v \geq u^0, v = u^0 \text{ on } \partial\Omega, \int_\Omega v - u^0 dx = m \right\}. \tag{3.1}$$

From now on we shall assume that F is strictly convex and superlinear. For any $\lambda \in \mathbb{R}$, we define v^λ as the solution of the obstacle problem with parameter λ, that is, the unique minimizer of

$$\min_{v \geq u^0, v = u^0 \; \partial\Omega} \Phi(v) + \lambda \int_\Omega v \, dx. \tag{3.2}$$

Observe that $\lim_{\lambda \to +\infty} v^\lambda = u^0$. Thanks to the uniqueness of the solution and the comparison principle, we observe that the domain $\mathcal{D} := \{(x,t) : x \in \Omega, u^0(x) < t < \sup_{\lambda \leq 0} v^\lambda(x)\}$ is such that for any $(x,t) \in \mathcal{D}$, there is a unique $\lambda > 0$ such that $t = v^\lambda(x)$. Indeed, since both $\sup_{\lambda' > \lambda} v^{\lambda'}$ and $\inf_{\lambda' < \lambda} v^\lambda$ are minimizers of (3.2), they must coincide for all λ. In particular, the function

$$\lambda \mapsto \int_\Omega v^\lambda - u^0 dx =: m^\lambda$$

is continuous and decreasing, going from 0 as $\lambda \to +\infty$, to some maximal value $\bar{m} \leq +\infty$ as $\lambda \to +\infty$[2]. One can check easily that for any m, one has $f(m) = \Phi(v^\lambda)$ for any λ such that $m = m^\lambda$. On the other hand, if v' is another minimizer of (3.1), for $m = m^\lambda$, then since $\Phi(v') + \lambda \int_\Omega v' dx = \Phi(v^\lambda) + \lambda \int_\Omega v^\lambda dx$, v' also minimizes (3.2) and by uniqueness $v' = v^\lambda$.

In addition, given m, m' and corresponding λ, λ', we have

$$f(m') + \lambda m' = \Phi(v^{\lambda'}) + \lambda \int_\Omega v^{\lambda'} - u^0 dx \geq \Phi(v^\lambda) + \lambda \int_\Omega v^\lambda - u^0 dx = f(m) + \lambda m,$$

showing that

$$f \text{ is convex and } -\lambda \in \partial f(m). \tag{3.3}$$

Observe that, in case F is not superlinear or not strictly convex, one can still build by approximation an increasing family of minimizers with increasing masses, minimizing the obstacle problem for some non-increasing multipliers, but one might lose uniqueness.

Proposition 3.4. *Let F be strictly convex and superlinear, and let u^0 be a subsolution, then $u^n = v^{\lambda_n}$ for any $n \geq 1$, where $\lambda_n := \|u^n - u^{n-1}\|_1/\tau$.*

Proof. By the above analysis u^n is the unique solution of

$$\min_{v \geq u^{n-1}} \Phi(v) + \lambda_n \int_\Omega v dx$$

with $\lambda_n = \|u^n - u^{n-1}\|_1/\tau$. We show by induction that this is also v^{λ_n}, knowing that it is true for $n = 1$. Assume it holds for u^{n-1}, then by comparison principle and the fact λ_n is non-increasing (see Theorem 2.7), one has $v^{\lambda_n} \geq u^{n-1}$. Hence we get

$$\Phi(u^n) + \lambda_n \int_\Omega u^n dx \leq \Phi(v^{\lambda_n}) + \lambda_n \int_\Omega v^{\lambda_n} dx.$$

But since $u^n \geq u^0$, the reverse inequality is also true, hence u^n and v^{λ_n} are both minimizers of the obstacle problem for λ_n. By uniqueness, we deduce that they coincide. □

[2] If domain of F is not the entire space, the maximal reachable mass $\bar{m}$ can be finite.

Observe that λ_n can be also built as follows: given λ_{n-1}, when λ decreases from λ_{n-1} to 0, then $\|v^\lambda - u^{n-1}\|_1/\tau$ increases from 0 to $\|v^0 - v^{\lambda_{n-1}}\|_1/\tau > 0$, and there is a value in $(0, \lambda_{n-1})$ for which they coincide. Moreover, letting $m_n = \int_\Omega u^n - u^0 dx$, by Proposition 3.4 and (3.3) we have

$$\frac{m_n - m_{n-1}}{\tau} = \lambda_n \in -\partial f(m_n), \tag{3.4}$$

for any $n \geq 1$, so that the sequence $(m_n)_n$ solves the discrete implicit Euler scheme for the gradient flow of the convex function f.

Theorem 3.5. *Let F be strictly convex and superlinear, and let u^0 be a subsolution with $\Phi(u^0) < +\infty$. Then there exists a unique limit solution u given by Theorem 2.3 with initial datum u^0. Moreover, the function u is non-decreasing in t, and $u(t) = v^{\lambda(t)}$ for a.e. $t > 0$, where $\lambda(t) \in L^2((0, +\infty))$ is positive and non-increasing.*

Proof. The monotonicity of u in t follows directly from Proposition 3.3.

Letting $\lambda_\tau(t) = \lambda_{\lfloor t/\tau \rfloor + 1}$ and $m_\tau(t) = m_{\lfloor t/\tau \rfloor + 1}$ for $t \geq 0$, by Helly's Theorem we may assume that, up to a subsequence, λ_τ and m_τ converge pointwise to functions $\lambda(t)$ and $m(t)$ which are respectively non-increasing and non-decreasing. By Proposition 3.4 we then get that $u_\tau(t) \to u(t) = v^{\lambda(t)}$ and $\int_\Omega u_\tau(t) - u^0 dx \to m(t) = m^{\lambda(t)}$ as $\tau \to 0$, for a.e. $t > 0$.

Recalling (3.4) we also have that m is the unique solution of the gradient flow

$$\dot{m} + \partial f(m) \ni 0$$

with initial value $m(0) = 0$, see for instance [5]. It follows that $u(t)$ is the solution of (3.1) for $m = m(t)$, and since the latter is unique, we deduce that also the limit flow $u(t)$ is unique, and that $u_\tau \to u$ as $\tau \to 0$, without passing to a subsequence. The fact that $\lambda \in L^2((0, +\infty))$, by the dissipation estimate 2.8. □

Remark 3.6. Observe that $\|u^{n\tau} - u^{m\tau}\|_1 = \tau \sum_{l=m+1}^n \lambda_l = \int_{m\tau}^{n\tau} \lambda_\tau(s) ds$, hence

$$\|u(t_2) - u(t_1)\|_1 = \int_{t_1}^{t_2} \lambda(s) ds \qquad \text{for all } 0 \leq t_1 < t_2,$$

which is equivalent to

$$m(t) = \int_\Omega v^{\lambda(t)}(x) - u^0(x) dx = \int_0^t \lambda(s) ds \qquad \text{for all } t \geq 0. \tag{3.5}$$

If F is of class C^1, recalling that the functions v^λ satisfy

$$-\operatorname{div} \nabla F(Dv^\lambda) + \lambda = 0 \qquad \text{a.e. in } \{v^\lambda > u^0\},$$

equation (3.5) implies (2.15).

Remark 3.7. If F is of class C^1 one can check that two different values of λ yield different functions (when $v^\lambda > u^0$, since in that case one has $-\operatorname{div} \nabla F(Dv^\lambda) + \lambda = 0$ a.e. in $\{v^\lambda > u^0\}$). Then, using that $t \mapsto u(t) = v^{\lambda(t)}$ is Hölder continuous in $L^1(\Omega)$ by Theorem 2.3, it follows that $\lambda(t)$ is continuous.

4. The Dirichlet energy

In this section, we consider the simplest case $F(\xi) = |\xi|^2/2$, so that

$$\Phi(u) = \frac{1}{2}\int_\Omega |Du|^2$$

is the Dirichlet energy of u.

In what follows, we shall consider either the case of Dirichlet boundary conditions ($\mathrm{dom}(\Phi) = u^0 + H^1_0(\Omega)$), or the case of homogeneous Neumann boundary conditions ($\mathrm{dom}(\Phi) = H^1(\Omega)$).

4.1. **Uniqueness.** Assuming that $\Phi(u^0) < +\infty$, the limit solution u provided by Theorem 2.3 satisfies

$$\int_\Omega |Du(t,x)|^2 dx + \frac{1}{2}\int_0^t \|\dot u(s)\|_1^2 ds + \frac{1}{2}\int_0^t \|\Delta u(s)\|_\infty^2 ds \le \int_\Omega |Du^0|^2 dx, \tag{4.1}$$

which is (2.8), and we take into account (*cf* Theorem 2.11) that $\dot u \in L^\infty([t,+\infty]; H^1(\Omega))$ for any $t > 0$, and $\partial\Phi(u(t)) = \{-\Delta u(t)\}$ for a.e. t. As usual, this can be rewritten:

$$\int_0^t \left(\int_\Omega Du(s,x)\cdot D\dot u(s,x)dx + \frac{1}{2}\|\dot u(s)\|_1^2 + \frac{1}{2}\|\Delta u(s)\|_\infty^2\right) ds \le 0,$$

which is possible only if $\Delta u(s) \in \|\dot u\|_1 \mathrm{sign}(\dot u)$ a.e. in Ω, for a.e. $s \in [0,T]$, and we obtain the equations

$$\begin{cases} |\Delta u| \le \|\dot u\|_1 & \text{a.e. in } \Omega \\ \dot u \Delta u = |\dot u|\|\dot u\|_1 & \text{a.e. in } \Omega \end{cases} \quad \text{a.e. in } [0,T], \tag{4.2}$$

cf (2.15).

It turns out that this defines a *unique* evolution starting from $u^0 \in H^1(\Omega)$. Indeed, for different solutions u, v of (4.2) we have

$$\frac{d}{dt}\int_\Omega |Du - Dv|^2 dt = 2\int_\Omega (Du - Dv)\cdot D(\dot u - \dot v)dx = -2\int_\Omega (\Delta u - \Delta v)\cdot(\dot u - \dot v)dx \le 0. \tag{4.3}$$

Theorem 4.1. *For any $u^0 \in H^1(\Omega)$, there is a unique flow $u \in C^0([0,+\infty); H^1(\Omega))$ which solves (4.2). In addition, the minimizing movements $\hat u_\tau$ converge to u in $C^0((0,+\infty); H^1(\Omega))$ (ie., locally uniformly in time), as $\tau \to 0$. The semi-norm of the speed $\|D\dot u\|_2$ is non-increasing in time.*

Proof. The uniqueness follows from (4.3) in the case of Dirichlet boundary conditions. In the case of Neumann conditions, assume we have two different solutions $u(t)$ and $u(t) + c(t)$, $c(t) \in \mathbb{R}$ (with $c \in H^1(\mathbb{R}_+)$ thanks to (4.1)). The equations state, then, that for a.e. t and a.e. $x \in \Omega$, both $\dot u\Delta u = |\dot u|\|\dot u\|_1$ and $(\dot u + \dot c)\Delta u = |\dot u + \dot c|\|\dot u + \dot c\|_1$. In case $\{\Delta u = 0\}$ has positive measure, we deduce that either $\|\dot u\|_1 = 0$ but then $0 = |\dot c|^2|\Omega|$ on a set of positive measure, meaning $\dot c = 0$, or $|\dot u| = 0$ on a set of positive measure and on the same set, $|\dot c|\|\dot u + \dot c\|_1 = 0$. Hence again, $\dot c = 0$ (or we would have $\dot u \equiv -\dot c \ne 0$ a.e., a contradiction).

Hence, we assume $|\{\Delta u = 0\}| = 0$. Since in addition, $\int_\Omega \Delta u = 0$ (in the case of Neumann conditions), Ω is split into two sets $\Omega^\pm$ of positive measure, with $\Delta u > 0$ a.e. in Ω^+, hence $\dot u \ge 0$, and $\Delta u < 0$ a.e. in Ω^-, hence $\dot u + \dot c \le 0$. This contradicts $\dot u \in H^1(\Omega)$ if $\dot c > 0$, since one would have $|\{-\dot c \le \dot u \le 0\}| = 0$. Symmetrically, one cannot have $\dot c < 0$, hence $\dot c = 0$. We deduce that $c(t) = 0$ (since c must be continuous with $c(0) = 0$), and it proves uniqueness in the Neumann case.

The convergence of $\hat{u}_\tau$ to the unique possible limit u is guaranteed by Theorem 2.3 and Proposition 2.5, at least in $C^0([0,T]; L^p(\Omega))$ for any $T > 0$ and $p < d/(d-1)$. In addition, it follows from (2.14) by standard arguments that $\|D\hat{u}_\tau(t) - D\hat{u}_\tau(s)\| \le 2/\min\{t,s\} \int_\Omega |Du^0|^2 dx$, from which we also deduce the uniform convergence on any interval $[t,T]$, $T > t > 0$.

Eventually, the fact that the speed is non-increasing in H^1 follows from the fact that, using (4.3), $\|Du(t+\varepsilon) - Du(t)\|_2 \le \|Du(s+\varepsilon) - Du(s)\|_2$ for any $t > s > 0$ and any $\varepsilon > 0$. □

We observe that the contraction property in the continuous setting also has a counterpart for the discrete flow:

Lemma 4.2. *Let $v, v' \in L^1(\Omega)$, $v - v' \in H^1(\Omega)$ (resp. $H^1_0(\Omega)$ in the case of Dirichlet boundary conditions) and assume u is a minimizer of*

$$\frac{1}{2\tau}\|u - v\|_1^2 + \Phi(u)$$

and u' a minimizer of the same problem with v replaced with v'. Then:

$$\frac{1}{2}\int_\Omega |D(u-u')|^2 dx \le \frac{1}{2}\int_\Omega |D(v-v')|^2 dx - \frac{1}{2}\int_\Omega |D(u-u'-v+v')|^2 dx. \tag{4.4}$$

and in particular

$$\|Du - Du'\|_2 \le \|D(v-v')\|_2.$$

Proof. Subtracting the Euler-Lagrange equations for u and u', multiplying by $(u-v)-(u'-v)$ and integrating by parts, we get:

$$\int_\Omega (Du - Du')\cdot(D(u - v - u' + v')dx \le 0$$

thanks to the monotonicity of the subgradient of $\|\cdot\|_1^2/2$. It follows

$$\int_\Omega |D(u-u')|^2 dx \le \int_\Omega D(u-u')\cdot D(v-v')dx$$

from which we deduce (4.4). □

Specializing (4.4) to the case $v = u^n$, $v' = u^{n-1}$, for $n \ge 1$, we find:

$$\int_\Omega |D(u^{n+1} - u^n)|^2 dx \le \int_\Omega |D(u^n - u^{n-1})|^2 dx - \int_\Omega |D(u^{n+1} - 2u^n + u^{n-1})|^2 dx. \tag{4.5}$$

which shows that also for the discrete flow one has that $\|D\dot{\hat{u}}_\tau\|_2$ is non-increasing in time.

4.2. **Energy decay estimate.**

In the case of homogeneous Neumann or Dirichlet boundary conditions, we expect that $\lim_{t\to\infty}\int_\Omega |Du(t)|^2 dx = 0$. Actually, we have the rate:

Proposition 4.3. *Let u solve (4.2), with $u^0 \in H^1(\Omega)$ (with Neumann boundary conditions) or $u^0 \in H^1_0(\Omega)$ (with Dirichlet boundary conditions). Then, for any $t > 0$ we have*

$$\int_\Omega |Du(t)|^2 dx \le e^{-\frac{t}{C_\Omega}}\int_\Omega |Du^0|^2 dx,$$

where C_Ω is the constant in the Poincaré-Wirtinger (Neumann) or Poincaré (Dirichlet) inequality.

Proof. We consider the minimizing movement scheme, and, given $\tau > 0$, $n \geq 1$, we compare the energy of u^n with the energy of $u_a := u^{n-1} + a\tau(u^{n-1} - m^{n-1})$, $a \in \mathbb{R}$, where $m^{n-1} = \int_\Omega u^{n-1} dx/|\Omega|$ is the average of u^{n-1} for Neumann boundary conditions, and 0 for homogeneous Dirichlet boundary conditions. One has in particular, thanks to the Poincaré(-Wirtinger) inequality:

$$\|u^{n-1} - u_a\|_1^2 = a^2\tau^2\|u^{n-1} - m^{n-1}\|_1^2 \leq a^2\tau^2 C_\Omega \int_\Omega |Du^{n-1}|^2.$$

Hence:

$$\frac{1}{2}\int_\Omega |Du^n|^2 dx + \frac{1}{2\tau}\|u^n - u^{n-1}\|_1^2$$
$$\leq (1 + 2a\tau + \tau^2 a^2)\frac{1}{2}\int_\Omega |Du^{n-1}|^2 dx + \frac{\tau}{2}a^2 C_\Omega \int_\Omega |Du^{n-1}|^2.$$

Choosing $a = -1/C_\Omega$, we find:

$$\frac{1}{2}\int_\Omega |Du^n|^2 dx \leq \frac{1}{2}\int_\Omega |Du^{n-1}|^2 \left(1 - \tfrac{\tau}{C_\Omega} + \tfrac{\tau^2}{C_\Omega^2}\right).$$

Hence,

$$\frac{1}{2}\int_\Omega |Du^n|^2 dx \leq \frac{1}{2}\int_\Omega |Du^0|^2 \left(1 - \tfrac{\tau}{C_\Omega} + \tfrac{\tau^2}{C_\Omega^2}\right)^n.$$

We conclude using that

$$\lim_{n\to\infty, n\tau\to t} \left(1 - \tfrac{\tau}{C_\Omega} + \tfrac{\tau^2}{C_\Omega^2}\right)^n = e^{-\frac{t}{C_\Omega}}.$$

□

5. Gradient flow of anisotropic perimeters

Given a norm φ on $\mathbb{R}^d$, we consider the anisotropic perimeter

$$E \mapsto P_\varphi(E) := \int_{\partial E} \varphi(\nu)\, d\mathcal{H}^{d-1}.$$

Letting φ^o be the dual norm of φ, we recall that the convex set

$$W_\varphi := \{x \in \mathbb{R}^d :\ \varphi^o(x) \leq 1\},$$

usually called *Wulff Shape*, is the unique volume-constrained minimizer of P_φ, up to translations and dilations. We say that φ is smooth (resp. elliptic) if the function $\varphi^2/2$ is smooth (resp. strongly convex).

We now introduce the geometric L^1-minimizing movement scheme. Given $\tau > 0$ and $E \subset \mathbb{R}^d$, we consider the minimum problem

$$\min_F P_\varphi(F) + \frac{1}{2\tau}|E \triangle F|^2, \tag{5.1}$$

and we let $T_\tau E$ be a (possiblynon-unique) minimizer of (5.1).

Given an initial set $E^0 \subset \mathbb{R}^d$, for all $n \in \mathbb{N}$ we let $E^n := T_\tau^n E_0$. For $(x,t) \in \mathbb{R}^d \times (0, +\infty)$, we also let

$$E_\tau(t) := E^{\lfloor t/\tau \rfloor} \qquad u_\tau(x,t) := \chi_{E_\tau(t)}(x).$$

The function $t \mapsto E_\tau(t)$ is the discrete L^1-gradient flow of P_φ, with initial datum E^0.

We point out that the analogous concept for the L^2-gradient flow of P_φ, where (5.1) is replaced by the problem

$$\min_F P_\varphi(F) + \frac{1}{2\tau}\int_{E\triangle F} \operatorname{dist}(x,\partial E)\,dx, \tag{5.2}$$

was originally introduced in [1, 16] as a discrete approximation of the mean curvature flow.

From (5.1) it follows that

$$P_\varphi(E^n) \le P_\varphi(E^{n-1}) \qquad \text{and} \qquad \frac{1}{2\tau}|E^n\triangle E^{n-1}|^2 \le P_\varphi(E^{n-1}) - P_\varphi(E^n), \tag{5.3}$$

for all $n\in\mathbb{N}$, so that

$$\begin{aligned} |E^n\triangle E^m|^2 &\le \left(\sum_{k=m+1}^{n} |E^k\triangle E^{k-1}|\right)^2 \\ &\le 2\tau(n-m)\sum_{k=m+1}^{n}\Big(P_\varphi(E^{k-1}) - P_\varphi(E^k)\Big) \\ &\le 2\tau(n-m)P_\varphi(E^0), \end{aligned} \tag{5.4}$$

for all $0\le m<n$.

Reasoning as in the proof of Theorem 2.3, from (5.3) and (5.4) we get the following result.

Theorem 5.1. *Assume that $E^0\subset\mathbb{R}^d$ is a set of finite perimeter. Then there exist a sequence $\tau_k\to\infty$ and a function $u(x,t)\in L^\infty((0,+\infty),BV(\mathbb{R}^d))\cap C^{1/2}((0,+\infty),L^1(\mathbb{R}^d))$, with $u(x,t)=\chi_{E(t)}(x)$ for some family of sets $E(t)$, such that*

$$\lim_{k\to+\infty}\sup_{t\in[0,T]}\|u(\cdot,t)-u_{\tau_k}(\cdot,t)\|_{L^1(\mathbb{R}^d)} = 0 \qquad \forall T>0.$$

Following [16, Lemma 1.3, Remark 1.4], we show a density estimate for minimizers of (5.1).

Lemma 5.2. *There exists $c>0$ depending only on φ and the dimension d such that the following holds: let $E\subset\mathbb{R}^d$ and F a minimizer of (5.1), then*

(1) for a.e. $x\in F\setminus E$ and all $r>0$ such that $|B(x,r)\cap E|=0$, $|B(x,r)\cap F|\ge cr^d$;
(2) for a.e. $x\notin E$, and $r>0$ such that $|B(x,r)\cap E|=0$, if $|B(x,r)\cap F|\le cr^d/2$, then $B(x,r/2)\cap F=\emptyset$.

Proof. Following [16] we compare the energy of F and $F\setminus B(x,r)$ in (5.1) and, introducing $b>a>0$ s.t. $a|x|\le\varphi(x)\le b|x|$, we observe that for a.e. $r>0$, if $|B(x,r)\cap E|=0$:

$$P_\varphi(F) + \frac{1}{2\tau}|E\triangle F|^2 \le P_\varphi(F\setminus B(x,r)) + \frac{1}{2\tau}|(E\triangle F)\setminus B(x,r)|^2$$

implies $a\mathcal{H}^{d-1}(\partial F\cap B(x,r))\le b\mathcal{H}^{d-1}(\partial B(x,r)\cap F)$. Introducing $f(r)=|F\cap B(x,r)|$, this is rewritten $(a+b)\mathcal{H}^{d-1}(\partial(F\cap B(x,r)))\le f'(r)$, and using the isoperimetric inequality we deduce that for some constant $\gamma>0$ depending only on d,a,b, $\gamma f(r)^{1-1/d}\le f'(r)$. The thesis follows from a version of Gronwall's Lemma. □

Remark 5.3. A symmetric statement holds for points $x\in E$, with $B(x,r)\cap F$ replaced with $B(x,r)\setminus F$.

Remark 5.4. A similar proof (see [16] again) shows that there exists $r(\tau) > 0$ such that for $r < r(\tau)$, for a.e. $x \in F$, $|B(x,r) \cap F| \geq cr^d$, and for a.e. $x \notin F$, $|B(x,r) \cap F^c| \geq cr^d$. In particular, the points of Lebesgue density 1 (resp. 0) of F form an open set, the reduced boundary of F is $\mathcal{H}^{d-1}$-essentially closed, and there is no abuse of notation in denoting it ∂F.

5.1. **Outward minimizing case.**

Definition 5.5. *Let $\Omega \subset \mathbb{R}^d$ be open. We say that a set $E \subset \Omega$ is outward minimizing if*

$$P_\varphi(E) \leq P_\varphi(F) \qquad \forall F \supset E, F \subset \Omega.$$

Notice that if φ is smooth and E is an outer minimizer with boundary of class C^2 then E is φ-mean convex, that is, $H_\varphi(x) \geq 0$ for any $x \in \partial E$, where $H_\varphi(x)$ is the φ-mean curvature of ∂E at x (see for instance [9] for a precise definition). Conversely, if $H_\varphi(x) \geq \delta > 0$ for any $x \in \partial E$ one can build $\Omega \supset\supset E$ such that E is outward minimizing in Ω. Notice also that a convex set is always outward minimizing.

We recall the following result proved in [12, Lemma 2.5] (see also [9, Section 2.1]).

Lemma 5.6. *E is outward minimizing if and only if*

$$P_\varphi(E \cap F) \leq P_\varphi(F) \qquad \forall F \subset \Omega.$$

From Lemmas 5.2 and 5.6 we obtain the following result.

Proposition 5.7. *Assume that $E \subset\subset \Omega$ is outward minimizing in Ω. Then, for τ small enough (depending only on φ, $\operatorname{dist}(E, \partial\Omega)$, and the dimension) we have that $T_\tau E \subseteq E$ and $T_\tau E$ is outward minimizing in Ω.*

In particular, the limit flow obtained in Theorem 5.1 is non-increasing and outward minimizing in Ω.

Proof. The first assertion follows from the minimality of $T_\tau E$ and from the fact that

$$P_\varphi(T_\tau E \cap E) + \frac{1}{2\tau}|(T_\tau E \cap E) \triangle E|^2 \leq P_\varphi(T_\tau E) + \frac{1}{2\tau}|T_\tau E \triangle E|^2,$$

with equality iff $T_\tau E \subseteq E$. We use here Lemma 5.6 which holds if we can prove first that $T_\tau E \subset \Omega$. Let $r := \operatorname{dist}(E, \partial\Omega)/2$. Then for τ small enough, we have (comparing the energy of $T_\tau E$ and E in (5.1)) that $|T_\tau E \setminus E| \leq \sqrt{2\tau P_\varphi(E)} \leq cr^d/2$ where c is the constant in Lemma 5.2. Using point (2) in Lemma 5.2, it follows that $\{x : r/2 < \operatorname{dist}(x, E) < 3r/2\} \cap T_\tau E = \emptyset$ and we deduce that $T_\tau E \subset \{x : \operatorname{dist}(x, E) \leq r/2\} \subset\subset \Omega$.

In order to prove the second assertion, we fix F such that $T_\tau E \subset F \subset \Omega$, and we notice that

$$P_\varphi(T_\tau E) \leq P_\varphi(F \cap E) + \frac{1}{2\tau}|(F \cap E) \triangle E|^2 - \frac{1}{2\tau}|T_\tau E \triangle E|^2 \leq P_\varphi(F \cap E) \leq P_\varphi(F),$$

where the last inequality follows from the outward minimality of E.

□

Remark 5.8. From Proposition 5.7 and (5.1) it follows that the set $T_\tau E$ solves the minimum problem

$$\min_{F \subset E} P_\varphi(F) - \frac{1}{\tau}|E|\,|F| + \frac{1}{2\tau}|F|^2, \tag{5.5}$$

hence $T_\tau E$ is also a solution of the volume-constrained isoperimetric problem (see also (5.12) later on)

$$\min_{F \subset E, |F| = |T_\tau E|} P_\varphi(F). \tag{5.6}$$

If φ is smooth and elliptic (that is, $\varphi^2/2$ is smooth and strongly convex), from (5.5) it follows that $T_\tau E \cap \mathrm{int}(E)$ is smooth and satisfies the Euler-Lagrange equation

$$H_\varphi(x) = \frac{|E \setminus T_\tau E|}{\tau} \qquad \text{for } x \in \partial T_\tau E \cap \mathrm{int}(E). \tag{5.7}$$

If in addition ∂E is of class $C^{1,1}$, by classical regularity results for the obstacle problem [6,8] $\partial T_\tau E$ is also of class $C^{1,1}$ outside a closed singular set of Hausdorff dimension $d-2$, and satisfies the Euler-Lagrange inequality

$$0 \le H_\varphi(x) \le \frac{|E \setminus T_\tau E|}{\tau} \qquad \text{for a.e. } x \in \partial T_\tau E. \tag{5.8}$$

Passing to the limit in (5.7) and (5.8) as $\tau \to 0$, and reasoning as in Theorem 2.13, we may expect that the limit flow $E(t)$ satisfies the equations

$$\begin{cases} 0 \le H_\varphi \le -\dfrac{d}{dt}|E(t)| & \text{a.e. on } \partial E(t) \\[2ex] H_\varphi = -\dfrac{d}{dt}|E(t)| & \text{a.e. on } \partial E(t) \cap \mathrm{int}(E^0), \end{cases} \tag{5.9}$$

for a.e. $t > 0$.

Remark 5.9. We cannot expect that there always exists a constant $\lambda > 0$ such that $T_\tau E$ is a solution of

$$\min_{F \subset E} P_\varphi(F) - \lambda |F|, \tag{5.10}$$

as it happens in the case of functions (see Section 3). In the sequel we shall see that, in the case E is convex, this is true only if E does not coincide with its Cheeger set, and τ is small enough so that λ is greater than the Cheeger constant of E.

5.2. **Convex case.** We now consider the special case of a convex initial set.

Proposition 5.10. *Let $d = 2$ and assume that φ is smooth and elliptic. Assume also that E^0 is a bounded convex set. Then the limit flow $E(t)$ obtained in Theorem 5.1 is given by a decreasing family of convex subsets of E^0.*

Proof. As in [17, Section 4.1] one can easily show that $T_\tau E$ is a convex subset of E with boundary of class $C^{1,1}$, satisfying (5.7) and (5.8). As in [4, Section 9] (see also [15, Theorem 2.3]), it follows that each connected component of $\partial T_\tau E \cap \mathrm{int}(E)$ is a graph and it is contained in $r\partial W_\varphi$, with $r = \tau/|E \setminus T_\tau E|$. As a consequence, if r is greater than the inradius of E, then

$$T_\tau E = E_r^- := \bigcup_{x + rW_\varphi :\, (x + rW_\varphi) \subset E} (x + rW_\varphi),$$

otherwise $T_\tau E = rW_\varphi + s$ for some segment $s \subset E$, and s is a point if r is smaller than the inradius of E.

By iterating the previous argument, and taking the limit as $\tau \to 0$, get the thesis.

□

Remark 5.11. By the argument above we get that

$$E(t) = E^-_{r(t)} \qquad t \in [0, T],$$

where $r(t)$ is continuous, increasing, and $T \ge 0$ is the first time such that $r(T)$ equals the inradius of E. In particular, the limit flow is unique on $[0, T]$.

Remark 5.12. By approximating a general norm φ with a sequence of smooth and elliptic norms, following the proof of Proposition 5.10, we obtain that there exists $r > 0$ such that $T_\tau E = E_r^-$ or $T_\tau E = rW_\varphi + s$ for some segment $s \subset E$.

As a consequence, also in the case of a general norm, there exists at least one limit flow $E(t)$ given by a decreasing family of convex subsets of E^0. We point out that, in the general case, we do not prove uniqueness of the limit flow.

In [7] it has been proved that, in any dimension $d \geq 2$, a volume-constrained minimizer of P_φ inside a convex set E is unique and convex if its volume is greater or equal than the volume of the Cheeger set of E, which is the minimizer F^* of the variational problem

$$\min_{F \subset E} \frac{P_\varphi(F)}{|F|} =: \lambda^*$$

(λ^* is called the Cheeger constant of E), and is unique when E is convex [2, 8, 14]. The Cheeger set is also characterized as the largest minimizer ($\emptyset$ being the smallest one) of the problem $\min_{F \subset E} P_\varphi(F) - \lambda^*|F|$, which has value 0.

For $\lambda > \lambda^*$, there is a unique minimizer F^λ to (5.10), which is convex, and coincides with the above volume-constrained minimizer (and is continuous with respect to λ, see [7]). Moreover, if φ is smooth and elliptic, λ coincides with the mean curvature H_φ of $\partial F^\lambda \cap \mathrm{int} E$ (otherwise it can be thought of as a variational mean curvature).

It follows that, as long as $|E^n| \geq |F^*|$, where F^* is the Cheeger set of E^0, we can define a non-increasing sequence $\lambda_n \geq \lambda^*$ such that $E^n = F^{\lambda_n}$ and which satisfies, for $n \geq 1$,

$$\frac{|E^{n-1}| - |E^n|}{\tau} = \lambda_n,$$

or equivalently for all $n \geq 1$,

$$|E^0| - |E^n| = \tau \sum_{k=0}^{n} \lambda_k.$$

In the limit $\tau \to 0$, similarly to Section 3, up to a subsequence the non-increasing function $\lambda_{\lfloor t/\tau \rfloor + 1}$ converges pointwise to a non-increasing function $\lambda(t)$, while $E_\tau(t)$ converges to $F^{\lambda(t)}$. In particular, in the limit we find that:

$$|E^0| - |E(t)| = \int_0^t \lambda(s) ds \tag{5.11}$$

for all $0 \leq t \leq T^*$, where $\lambda(T^*) = \lambda^*$.

If φ is smooth and elliptic, since the sets F^λ are all different, the function $t \mapsto \lambda(t)$ is continuous on $[0, T^*]$, so that the function $t \mapsto |E(t)|$ is of class C^1 by (5.11). We deduce that (5.9) holds for all $t \in (0, T^*)$.

We then obtain a partial extension of Proposition 5.10 to arbitrary dimensions and for a general norm φ.

Proposition 5.13. *Assume that E^0 is a bounded convex set not coinciding with its Cheeger set. Then there exists $T^* > 0$ such that limit flow $E(t)$ is given by a decreasing family of convex subsets of E^0 for $t \in [0, T^*]$. Moreover, each set $E(t)$ is a volume-constrained minimizer of P_φ inside E^0, and $E(T^*) = F^*$ is the Cheeger set of E^0. In particular, for $t \in (0, T^*]$ $E(t)$ is the unique minimizer $F^{\lambda(t)}$ of (5.10) for some $\lambda(t) > \lambda^*$ which solves (5.11).*

For $m \in [0, |E^0|]$ we let

$$f(m) := \min \left\{ P_\varphi(F) : \, F \subset E^0, |E^0 \setminus F| = m \right\}. \tag{5.12}$$

Reasoning as in Section 3 we have that, for any $m \in [0, |E^0 \setminus F^*|]$ there exists a unique $\lambda^m \geq \lambda^*$ such that $|E^0 \setminus F^{\lambda^m}| = m$ and $P_\varphi(F^{\lambda^m}) = f(m)$. Moreover the function $m \to \lambda^m$ is non-increasing in this interval, and

$$-\lambda^m \in \partial f(m), \tag{5.13}$$

which implies that f is convex on $[0, |E^0 \setminus F^*|]$. With almost the same proof as Theorem 3.5, we can show the following uniqueness result for the limit flow $E(t)$.

Proposition 5.14. *Assume that E^0 is a bounded convex set not coinciding with its Cheeger set. Then the flow $E(t)$ given by Proposition 5.13 is unique and satisfies*

$$\frac{d|E(t)|}{dt} = -\lambda(t) \tag{5.14}$$

for all $t \in (0, T^*)$, where $\lambda(t)$ coincides with the mean curvature of $\partial E(t)$ inside E^0 and $E(T^*)$ is the Cheeger set of E^0.

Remark 5.15. If φ is smooth and elliptic, from (5.14) it follows that $E(t)$ satisfies (5.9).

Recalling the proof of Proposition 5.10, when $d = 2$ the minimizer E^m in (5.12) is uniquely characterized and coincides with the set $E^-_{r^m}$ as long as $m \geq |E^-_{r^0}|$, where r^0 is the inradius of E^0 and $r^m \geq r^0$ is such that $|E^-_{r^m}| = m$. When $m < |E^-_{r^0}|$ the minimizer E^m is only unique up to translations.

If in addition $\varphi(x) = |x|$, it has been proved in [15] that the function f is convex on $[0, m^0]$, where $m^0 = |E^0| - |B_{r^0}|$ and E^m is a solution of (5.10) with $\lambda = 1/r_m$, among sets of volume greater of equal to $|B_{r_m}|$. Observing also that $f(m) = 2\sqrt{\pi(|E^0| - m)}$ for $m \in [m_0, |E^0|)$, reasoning as above we get that f satisfies (5.13) for all $m \in (0, |E^0|)$, so that we can partly extend the result in Proposition 5.14.

Proposition 5.16. *Let $d = 2$, $\varphi(x) = |x|$, and assume that E^0 is a bounded convex set. Then the flow $E(t)$ is defined on a maximal time interval $[0, T_{\max})$, with*

$$\lim_{t \to T_{\max}} |E(t)| = 0,$$

it is unique up to translations, and satisfies (5.14) for all $t \in (0, T_{\max})$, where $\lambda(t)$ coincides with the curvature of $\partial E(t)$ inside E^0. Moreover, $E(t)$ is unique as long as $|E(t)| \geq |E^-_{r^0}|$.

We show with two simple examples that uniqueness of the flow cannot be expected for $t > T^*$. In the following we fix $d = 2$ and $\varphi(x) = |x|$.

Example 1. Let $E^0 = B_R(x_0)$ for some $R > 0$ and $x_0 \in \mathbb{R}^2$. Then, by the isoperimetric inequality, $T_\tau E^0$ is a ball contained in E^0 of radius r minimizing the function

$$r \mapsto 2\pi r + \frac{\pi^2}{2\tau}(R^2 - r^2)^2, \tag{5.15}$$

that is, $r = R - \tau/(2\pi R^2) + o(\tau)$ as $\tau \to 0$. By iteration, it follows that the discrete evolutions $E_\tau(t)$ converge, up to a subsequence as $\tau \to 0$, to $E(t) = B_{R(t)}(x(t))$, with

$$R(t) := \left(R^3 - \frac{3t}{2\pi} \right)^{\frac{1}{3}} \qquad \text{for } t \in \left[0, \frac{2}{3}\pi R^3 \right)$$

and $x(t)$ is a Lipschitz function such that $|\dot{x}(t)| \leq |\dot{R}(t)|$ for a.e. $t \in [0, \frac{2}{3}\pi R^3)$.

Notice that in this case the limit evolution is non-unique.

Example 2. Let $E^0 = B_{R_1}(x_1) \cup B_{R_2}(x_2)$ for some $R_1 \geq R_2 > 0$ and $x_1, x_2 \in \mathbb{R}^2$ such that $|x_1 - x_2| > R_1 + R_2$. As in the previous example, we have that $T_\tau E^0 = B_{r_1}(\tilde{x}_1) \cup B_{r_2}(\tilde{x}_2)$, with $B_{r_i}(\tilde{x}_i) \subseteq B_{R_i}(x_i)$ for $i \in \{1, 2\}$, and the radii r_i minimize the function

$$(r_1, r_2) \mapsto 2\pi(r_1 + r_2) + \frac{\pi^2}{2\tau}\left(R_1^2 - r_1^2 + R_2^2 - r_2^2\right)^2. \tag{5.16}$$

If $R_1 > R_2$, by an easy computation it follows that $r_1 = R_1$ and r_2 minimize the function in (5.15) with R replaced by R_2, that is, $r_2 = R_2 - \tau/(2\pi R_2^2) + o(\tau)$ as $\tau \to 0$. In particular, in the limit as $\tau \to 0$, we obtain the evolution $E(t) = B_{R_1}(x_1) \cup B_{R_2(t)}(x_2(t))$, with $R_2(t)$ and $x_2(t)$ as in the previous case of a single ball.

On the other hand, If $R_1 = R_2 = R$ then either $r_1 = R$ and r_2 minimize the function in (5.15), or viceversa $r_2 = R$ and r_1 minimize the function in (5.15). This implies that, in the limit as $\tau \to 0$, only one of the two balls start shrinking, whereas the other does not move until the first ball disappears. As above the limit evolution is non-unique.

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