# On the gradient flow of a one-homogeneous functional

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We consider the gradient flow of a one-homogeneous functional, whose dual involves the derivative of a constrained scalar function. We show in this case that the gradient flow is related to a weak, generalized formulation of a Hele-Shaw flow. The equivalence follows from a variational representation, which is a variant of well-known variational representations for the Hele-Shaw problem. As a consequence we get existence and uniqueness of a weak solution to the Hele-Shaw flow. We also obtain an explicit representation for the Total Variation flow in one dimension, and easily deduce basic qualitative properties, concerning in particular the "staircasing effect".

## 1. Introduction

This paper deals with the $L^2$-gradient flow of the functional

$$J_k(\omega) := \int_A |\mathrm{d}\omega|\,dx \qquad k \in \{0, \dots, N-1\}$$

defined on differential forms $\omega \in L^2(A, \Omega^k(\mathbb{R}^N))$, where $A \subseteq \mathbb{R}^N$ is an open set. We will focus on the particular case $k = N-1$: in that case, the dual variable is a scalar and this yields very particular properties of the functional $J_k$ and the associated flow.

Notice that, when $k = 0$, the functional $J_0$ reduces to the usual total variation. When $k = N-1$ we can identify by duality $\omega \in L^2(A, \Omega^{N-1}(\mathbb{R}^N))$ with a vector field $u \in L^2(A, \mathbb{R}^N)$, so that $J_{N-1}$ is equivalent to the functional

$$\mathcal{D}(u) := \int_A |\mathrm{div}\, u|\,dx \tag{1.1}$$

that is, the total mass of $\mathrm{div}\, u$ as a measure.

The gradient flow of $\mathcal{D}$ has interesting properties: we show in particular that it is equivalent to a constrained variational problem for a function $w$ such that $\Delta w = \mathrm{div}\, u$. Moreover, under some regularity assumption on the initial datum $u_0$, such a variational problem allows to define a weak formulation of the Hele-Shaw flow [9, 11] (see also [12] for a viscosity formulation). Therefore, it turns out that

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the flow of (1.1) provides a (unique) global weak solution to the Hele-Shaw flow, for a suitable initial datum $u_0$. But our formulation allows us to consider quite general initial data $u_0$, for which for instance $\operatorname{div} u_0$ may change sign, or be a measure.

The plan of the paper is the following: in Section 2 we introduce the general functional we are interested in, we write the Euler-Lagrange equation for its Moreau-Yosida approximation and, in Section 2.1, we express it in a dual form that will be the base of our analysis.

In Section 3 we focus on the case $k = 1$ which is analyzed in this paper. We show many interesting properties of the flow: comparison, equivalence with a weak Hele-Shaw flow if the initial datum is smooth enough, and qualitative behavior when the initial datum is not smooth. In Section 4.1 we observe that, in dimension 2, the case $k = N - 1$ also covers the flow of the $L^1$-norm of the rotation of a vector field, which appears as a particular limit of the Ginzburg-Landau model (see [16, 19] and references therein).

Another interesting consequence of our analysis is that it yields simple but original qualitative results on the solutions of the Total Variation flow in dimension one (see also [3, 5]). We show in Section 4.2 that the denoising of a noisy signal with this approach will, in general, almost surely produce a solution which is "flat" on a dense set. This undesirable artefact is the well-known "staircasing" effect of the Total Variation regularization and is the main drawback of this approach for signal or image reconstruction.

## 2. Gradient flow

Given an initial datum $\omega_0 \in L^2(A, \Omega^k(\mathbb{R}^N))$, the general theory of [6] guarantees the existence of a global weak solution $\omega \in L^2([0, +\infty), L^2(A, \Omega^k(\mathbb{R}^N)))$ of the gradient flow equation of $J_k$:

$$\omega_t \in -\partial J_k(\omega) \qquad t \in [0, +\infty)\,, \tag{2.1}$$

where $\partial J_k$ denotes the subgradient of the convex functional $J_k$. Given $\varepsilon > 0$ and $f \in L^2(A, \Omega^k(\mathbb{R}^N))$, we consider the minimum problem

$$\min_{\omega: A \to \mathbb{R}^N} J_k(\omega) \,+\, \int_A \frac{1}{2\varepsilon}|\omega - f|^2\, dx. \tag{2.2}$$

Notice that

$$\min_{\omega: A \to \mathbb{R}^N} J_k(u) + \int_A \frac{|\omega - \varepsilon f|^2}{2}\, dx = \varepsilon \min_{\omega: A \to \mathbb{R}^N} J_k(u) + \int_A \frac{1}{2\varepsilon}|\omega - f|^2\, dx.$$

The Euler-Lagrange equation corresponding to (2.2) is

$$\varepsilon(f - \omega) \in \partial J_k(\omega),$$

that is there exists a $(k+1)$-form $v$ with $|v| = 1$ such that $v = \mathrm{d}\omega/|\mathrm{d}\omega|$ if $\mathrm{d}\omega \neq 0$, and

$$\varepsilon(f - \omega) = \mathrm{d}^* v \text{ in } A \qquad \text{and} \qquad (*v)_T = 0 \text{ on } \partial A. \tag{2.3}$$

### 2.1. *Dual formulation*

Equation (2.3) is equivalent to

$$\omega \in \partial J_k^*(\varepsilon(f-\omega)),$$

where

$$J_k^*(\eta) := \sup_{w:A\to\mathbb{R}^N} \int_A \eta \cdot w\, dx - J_k(w) = \begin{cases} 0 & \text{if } \|\eta\|_* \le 1 \\ +\infty & \text{otherwise} \end{cases}$$

and

$$\|\eta\|_* = \sup\left\{ \int_A \eta \cdot w\, dx : \ J_k(w) \le 1 \right\}.$$

Note that

$$J_k(w) + J_k^*(\eta) \ge \int_A w \cdot \eta\, dx$$

for all $w, \eta$. The equality holds iff $\int_A \eta \cdot w\, dx = J_k(w)$, and in such case we have $\|\eta\|_* \le 1$.

Letting $u$ be a minimizer of (2.2) and $\eta = (f-u)/\varepsilon$ we then get

$$\int_A u \cdot \frac{f-u}{\varepsilon}\, dx = J_k(u),$$

which implies

$$J_k^*\left(\frac{f-u}{\varepsilon}\right) = 0 \qquad \text{that is} \qquad \varepsilon \ge \|f-u\|_* .$$

In particular, we showed the following (see also [15] for the same result in the case of the Total Variation).

**Proposition 2.1.** *The function $u = 0$ is a minimizer of (2.2) if and only if*

$$\varepsilon \ge \varepsilon_c := \|f\|_* . \tag{2.4}$$

Note that $\|\eta\|_* < \infty$ implies that

$$\int_A \eta w = 0$$

for all $w$ such that $\mathrm{d}w = 0$. By Hodge decomposition, this implies that $\eta = \mathrm{d}^* g$ for some 2-form $g$, with $g_N = 0$ on $\partial A$. It follows that

$$\|\eta\|_* = \sup_{\int_A |\mathrm{d}w|\le 1} \int_A \mathrm{d}^* g \cdot w\, dx = \sup_{\int_A |\mathrm{d}w|\le 1} \int_A g \cdot \mathrm{d}w\, dx + \int_{\partial A} w \wedge * g_N = \sup_{\int_A |\mathrm{d}w|\le 1} \int_A g \cdot \mathrm{d}w\, dx. \tag{2.5}$$

We then get

$$\|\eta\|_* = \inf_{\substack{\mathrm{d}^* g = \eta \\ g_N|\partial A = 0}} \|g\|_{L^\infty(A)}.$$

Indeed, it is immediate to show the $\leq$ inequality. On the other hand, by Hahn-Banach Theorem, there exists a form $g'$, with $\mathrm{d}^* g' = \mathrm{d}^* g = \eta$ such that

$$\|\eta\|_* = \sup_{\int_A |\mathrm{d}w| \leq 1} \int_A g \cdot \mathrm{d}w \, dx = \sup_{\int_A |\psi| \leq 1} \int_A g' \cdot \psi \, dx = \|g\|_{L^\infty(A)}.$$

Fix now $\phi_0$ such that $\mathrm{d}^*\phi_0 = \eta$. We can write $g = \phi_0 + \mathrm{d}^*\psi$, so that (2.5) becomes

$$\|\eta\|_* = \min_{\psi \colon (\phi_0 + \mathrm{d}^*\psi)\cdot\nu_A = 0} \|\phi_0 + \mathrm{d}^*\psi\|_{L^\infty(A)}. \tag{2.6}$$

The Euler-Lagrange equation of (2.6) is similar to the *infinity laplacian equation*

$$\mathrm{d}_\infty(\phi_0 + \mathrm{d}^*\psi) = 0.$$

By duality problem (2.6) becomes

$$\min_{\psi \in W_0^{1,\infty}(A)} \|\nabla\psi + \phi_0\|_{L^\infty(A)}, \tag{2.7}$$

and the corresponding Euler-Lagrange equation is

$$\langle (\nabla^2\psi + \nabla\phi_0)(\nabla\psi + \phi_0), (\nabla\psi + \phi_0)\rangle = 0. \tag{2.8}$$

## 3. The case $k = N - 1$

In this case, we recall that we are considering the gradient flow of the functional (1.1), which is defined, for any $u \in L^1_{\mathrm{loc}}(A;\mathbb{R}^N)$, as follows

$$\mathcal{D}(u) \; = \; \sup\left\{ \int_A -u\nabla v \, dx \; : \; v \in C_c^\infty(A)\,, |v(x)| \leq 1 \;\forall x \in A \right\}. \tag{3.1}$$

This is finite if and only if the distribution $\operatorname{div} u$ is a bounded Radon measure in $A$. We now see it as a (convex, l.s.c., with values in $[0,+\infty]$) functional over the Hilbert space $L^2(A;\mathbb{R}^N)$: it is then clear from (3.1) that it is the support function of

$$K \; = \; \left\{ -\nabla v \, : \, v \in H^1_0(A;[-1,1]) \right\}$$

and in particular $p \in \partial\mathcal{D}(u)$, the subgradient of $\mathcal{D}$ at $u$, if and only if $p \in K$ and $\int_A p \cdot u \, dx = \mathcal{D}(u) = \int_A |\operatorname{div} u|$:

$$\partial\mathcal{D}(u) \; = \; \left\{ -\nabla v \, : \, v \in H^1_0(A;[-1,1])\,, \; \int_A -\nabla v \cdot u \, dx \; = \; \int_A |\operatorname{div} u| \right\}.$$

We can define, for $u \in \operatorname{dom}\mathcal{D}$, the Radon-Nikodym density

$$\theta_{\operatorname{div} u}(x) \; = \; \frac{\operatorname{div} u}{|\operatorname{div} u|}(x) \; = \; \lim_{\rho\to 0} \frac{\int_{B(x,\rho)} \operatorname{div} u}{\int_{B(x,\rho)} |\operatorname{div} u|},$$

which exists $|\operatorname{div} u|$-a.e. (we consider that it is defined only when the limit exists and is in $\{-1,1\}$), and is such that $\operatorname{div} u = \theta_{\operatorname{div} u}|\operatorname{div} u|$. We can also introduce the Borel sets

$$\mathcal{E}^\pm_u \; = \; \{x \in A \, : \, \theta_{\operatorname{div} u}(x) = \pm 1\}\,.$$

Then, we have:

**Lemma 3.1.**

$$\partial\mathcal{D}(u) \;=\; \left\{-\nabla v\,:\, v\in H^1_0(A;[-1,1])\,, v=\pm 1 \;\; |\mathrm{div}\, u|-\text{ a.e. on } \mathcal{E}_u^\pm\right\}\,.$$

**Proof.** Consider $v\in H^1_0(A;[-1,1])$. Then we know [1] that it is the limit of smooth functions $v_n \in C^\infty_c(A;[-1,1])$ with compact support which converge to $v$ quasi-everywhere (that is, up to a set of $H^1$-capacity zero).

We recall that when $u\in L^2(A;\mathbb{R}^N)$, the measure $\mathrm{div}\, u\in H^{-1}(A)$ must vanish on sets of $H^1$-capacity 0 [1] : it follows that $v_n\to v$ $|\mathrm{div}\, u|$-a.e. in $A$. Hence, by Lebesgue's convergence theorem,

$$-\int_A \nabla v(x)u(x)\,dx \;=\; \lim_{n\to\infty}\int_A v_n(x)\theta_{\mathrm{div}\, u}(x)|\mathrm{div}\, u|(x) \;=\; \int_A v(x)\mathrm{div}\, u(x)\,.$$

It easily follows that if $v=\pm1$ $|\mathrm{div}\, u|$- a.e. on $\mathcal{E}_u^\pm$, $-\nabla v\in\partial\mathcal{D}(u)$ and conversely, that if $v\in\partial\mathcal{D}(u)$ then $v=\pm1$ $|\mathrm{div}\, u|$-a.e. on $\mathcal{E}_u^\pm$. □

We now define, provided $u\in\mathrm{dom}\,\partial\mathcal{D}$ (i.e., $\partial\mathcal{D}(u)\neq\emptyset$),

$$\partial^0\mathcal{D}(u) \;=\; \arg\min\left\{\int_A |p|^2\,dx\,:\, p\in\partial\mathcal{D}(u)\right\}\;:$$

it corresponds to the element $p=-\nabla v\in\partial\mathcal{D}(u)$ of minimal $L^2$-norm. Using Lemma 3.1, equivalently, $v$ is the function which minimizes $\int_A|\nabla v|^2\,dx$ among all $v\in H^1_0(A)$ with $v\geq\chi_{\mathcal{E}_u^+}$ and $v\leq-\chi_{\mathcal{E}_u^-}$, $|\mathrm{div}\, u|$-a.e.: in particular, we deduce that it is harmonic in $A\setminus\overline{\mathcal{E}_u^+\cup\mathcal{E}_u^-}$.

Let us now return to the flow (2.1). In this setting, it becomes

$$\begin{cases} u_t &= \nabla v\\ u(0) &= u_0\end{cases} \tag{3.2}$$

where $v$ satisfies $|v|\leq 1$ and

$$\mathcal{D}(u)+\int_A u\cdot\nabla v = 0.$$

It is well know, in fact, that the solution of (3.2) is unique and that $-\nabla v(t) = \partial^0\mathcal{D}(u(t))$ is the right-derivative of $u(t)$ at any $t\geq 0$ [6]. Given the solution $(u(t),v(t))$ of (3.2), we let

$$w(t) \;:=\; \int_0^t v(s)\,ds,$$

which takes its values in $[-t,t]$. We have

$$u(t) = u_0+\nabla w(t).$$

**Theorem 3.1.** *Assume $u_0\in L^2(A;\mathbb{R}^N)$. The function $w(t)$ solves the following obstacle problem*

$$\min\left\{\frac{1}{2}\int_A|u_0+\nabla w|^2\,dx\,:\, w\in H^1_0(A)\,, |w|\leq t \;\, a.e.\right\}. \tag{3.3}$$

Observe that in case we additionally have $\operatorname{div} u_0 \geq \alpha > 0$, this obstacle problem is well-known for being an equivalent formulation of the Hele-Shaw problem, see [9, 11].

**Proof.** Given $u_0 \in L^2(A;\mathbb{R}^N)$, we can recursively define $u_{n+1} \in L^2(A;\mathbb{R}^N)$ as the unique solution of the minimum problem

$$\min_{u \in L^2(A,\mathbb{R}^N)} \mathcal{D}_\varepsilon(u, u_n),$$

where

$$\mathcal{D}_\varepsilon(u, v) = \mathcal{D}(u) + \int_A \frac{1}{2\varepsilon}|u - v|^2\, dx.$$

Then, there exists $v_{n+1} \in \partial\mathcal{D}(u_{n+1})$ such that

$$u_{n+1} - u_n - \varepsilon \nabla v_{n+1} = 0. \tag{3.4}$$

It follows that $v_{n+1} \in H^1_0(A)$ minimizes the functional

$$\int_A |u_n + \varepsilon\nabla v|^2\, dx$$

under the constraint $|v| \leq 1$. Let now

$$w_n := \varepsilon \sum_{i=1}^{n} v_i.$$

The from (3.4) we get

$$u_n = u_0 + \nabla w_n, \tag{3.5}$$

and $w_n$ minimizes the functional

$$\int_A |u_0 + \nabla w|^2\, dx \tag{3.6}$$

under the constraint $|w - w_{n-1}| \leq \varepsilon$. Notice that $|w_n - w_{n-1}| \leq \varepsilon$ for all $n$ implies

$$|w_n| \leq n\varepsilon. \tag{3.7}$$

We now show that $w_n$ minimizes (3.6) also under the weaker constraint (3.7). Indeed, letting $\hat{w}_n$ be the minimizer of (3.6) under the constraint (3.7), we have

$$\hat{w}_n - \varepsilon \leq \hat{w}_{n+1} \leq \hat{w}_n + \varepsilon,$$

which follows by noticing that $\min\{\hat{w}_n, \hat{w}_{n+1} + \varepsilon\}$ and $\max\{\hat{w}_n, \hat{w}_{n+1} - \varepsilon\}$ minimize (3.6), hence they are both equal to $\hat{w}_n$. It then follows $w_n = \hat{w}_n$ for all $n$.

Passing to the limit in $n$ we get the corresponding result in the continuum case. □

**Remark 3.1.** The previous proof also shows that for any initial $u_0 \in L^2(A;\mathbb{R}^N)$, $u(t) = u_0 + \nabla w(t)$ is the unique minimizer of

$$\int_A |\operatorname{div} u| + \frac{1}{2t}|u - u_0|^2\, dx.$$

We recall that obviously, such property does not hold for general semigroups generated by the gradient flow of a convex function. It is shown in [2] to be the case for the Total Variation flow, in any dimension, when the initial function is the characteristic of a convex set.

### 3.1. *Some properties of the solution*

A first observation is that $t \mapsto w(t)$ is continuous (in $H^1_0(A)$, strong), as follows both from the study of the varying problems (3.3) and from the fact that the flow $u(t) = u_0 + \nabla w(t)$ is both continuous at zero and $L^2(A)$-Lipschitz continuous away from $t = 0$ (and up to $t = 0$ if $u_0 \in \operatorname{dom} \partial\mathcal{D}$).

In fact, one can check that $w$ is also $L^\infty$-Lipschitz continuous in time: indeed, it follows from the comparison principle that for any $s \le t$,

$$w(s) - t + s \;\le\; w(t) \;\le\; w(s) + t - s \tag{3.8}$$

a.e. in $A$, hence $\|w(t) - w(s)\|_{L^\infty(A)} \le |t - s|$. The comparison (3.8) is obtained by adding the energy in (3.3) of $\min\{w(t), w(s) + t - s\}$ (which is admissible at time $t$ and hence should have an energy larger than the energy of $w(t)$) to the energy of $\max\{w(t) - t + s, w(s)\}$ (which is admissible at time $s$), and checking that this sum is equal to the energy at time $t$ plus the energy at time $s$. This is quite standard, see [7, 12].

In particular, we can define for any $t$ the sets

$$E^+(t) \;=\; \{\tilde{w}(t) = t\} \;\text{ and }\; E^-(t) \;=\; \{\tilde{w}(t) = -t\}\,, \tag{3.9}$$

where $\tilde{w}(t)$ is the precise representative of $w(t) \in H^1(A)$, defined quasi-everywhere by

$$\tilde{w}(t,x) \;=\; \lim_{\rho\to 0} \frac{1}{\omega_N \rho^N} \int_{B(x,\rho)} w(t,y)dy \tag{3.10}$$

($\omega_N$ is the volume of the unit ball). It follows from (3.8) and (3.10) that if $\tilde{w}(t,x) = t$, then for any $s < t$, $x$ is also a point where $\tilde{w}(s,x)$ is well-defined, and its value is $s$; similarly if $\tilde{w}(t,x) = -t$ then $\tilde{w}(s,x) = -s$. Hence: the functions $t \mapsto E^+(t)$, $t \mapsto E^-(t)$ are nonincreasing.

Also, if $s < t$, one has from (3.8)

$$\frac{1}{\omega_N \rho^N} \int_{B(x,\rho)} w(s,y)dy - t + s \le \frac{1}{\omega_N \rho^N} \int_{B(x,\rho)} w(t,y)dy \le \frac{1}{\omega_N \rho^N} \int_{B(x,\rho)} w(s,y)dy - s + t$$

so that if $x \in E^+(s)$,

$$2s - t \;\le\; \liminf_{\rho\to 0} \frac{1}{\omega_N \rho^N} \int_{B(x,\rho)} w(t,y)dy \;\le\; \limsup_{\rho\to 0} \frac{1}{\omega_N \rho^N} \int_{B(x,\rho)} w(t,y)dy \;\le\; t$$

and sending $s$ to $t$, we find that if $x \in \bigcap_{s<t} E^+(s)$, $\tilde{w}(t,x) = t$ and $x \in E^+(t)$: hence these sets (as well as $E^-(\cdot)$) are left-continuous.

We define

$$E_r^+(t) \;=\; \bigcup_{s>t} E^+(s) \;\subseteq E^+(t) \;\text{ and }\; E_r^-(t) \;=\; \bigcup_{s>t} E^-(s) \;\subseteq E^-(t)\,, \qquad (3.11)$$

as well as $E(t) = E^+(t) \cup E^-(t)$, $E_r(t) = E_r^+(t) \cup E_r^-(t)$. Then, there holds the following lemma:

**Lemma 3.2.** *If $s \le t$, then*

$$E^-(t) \;\subseteq E^-(s) \text{ and } E^+(t) \;\subseteq E^+(s),$$

$$E_r^-(t) \;\subseteq E_r^-(s) \text{ and } E_r^+(t) \;\subseteq E_r^+(s).$$

*Moreover, for $t > 0$, $v(t) = \pm 1$ quasi-everywhere on $E_r^\pm(t)$ and $\mathcal{E}^\pm_{u(t)} \subseteq E_r^\pm(t)$, up to a set $|\mathrm{div}\, u(t)|$-negligible. In particular*

$$\mathrm{div}\, u(t) \llcorner (E_r^-(t))^c \ge 0, \quad \mathrm{div}\, u(t) \llcorner (E_r^+(t))^c \le 0, \quad \mathrm{div}\, u(t) \llcorner (E_r^+(t) \cup E_r^-(t))^c = 0.$$

Here, for a Radon measure $\mu$ and a Borel set $E$, $\mu \llcorner E$ denotes the measure defined by $\mu \llcorner E(B) := \mu(E \cap B)$.

**Proof.** The first two assertions, as already observed, follow from (3.8) and the definition of $E_r^\pm$. We know that the solution of equation (3.2) satisfies $\partial_t^+ u = -\partial^0 \mathcal{D}(u(t)) = \nabla v(t)$ for any $t > 0$, but the right-derivative of $u = u_0 + \nabla w(t)$ is nothing else as $\lim_{h\to 0} \nabla[w(t+h) - w(t)]/h$. We easily deduce that $v(t) = \lim_{h\to 0}[w(t+h) - w(t)]/h$ (which converges in $H_0^1$-strong). Since when $x \in E_r^+(t)$, $\tilde{w}(t,x) = t$ and $\tilde{w}(t+h,x) = t+h$ for $h$ small enough, we deduce that $v(x) = 1$ on that set, in the same way $v = 1$ on $E_r^-(t)$.

Observe that the Euler-Lagrange equation for (3.3) is the variational inequality

$$\int_A (u_0 + \nabla w(t)) \cdot (t\nabla v - \nabla w(t))\, dx \;\ge\; 0\,,$$

for any $v \in H_0^1(A; [-1,1])$. In other words since $u(t) = u_0 + \nabla w(t)$,

$$-\int_A u(t) \cdot \nabla \frac{w(t)}{t} \;\ge\; -\int_A u(t) \cdot \nabla v$$

for any $|v| \le 1$, and we recover that $-\nabla w(t)/t \in \partial\mathcal{D}(u(t))$.

Hence (using Lemma 3.1), $\mathcal{E}^\pm_{u(t)} \subseteq E^\pm(t)$. Now, if $\tilde{v} \in H_0^1(A;[-1,1])$ with $\tilde{v} = \pm 1$ on $E_r^\pm$, one deduces that for any $s > t$,

$$-\int_A \nabla\tilde{v} \cdot u(s)\, dx \;=\; \mathcal{D}(u(s))\,.$$

Sending $s \to t$, it follows

$$-\int_A \nabla\tilde{v} \cdot u(t)\, dx \;\ge\; \mathcal{D}(u(t))\,,$$

hence $\tilde{v} \in \partial\mathcal{D}(u(t))$. We deduce that $\mathcal{E}^\pm_{u(t)} \subseteq E_r^\pm(t)$, invoking Lemma 3.1. □

**Remark 3.2.** We might find situations where $|v(t)| = 1$ outside of the contact set. For instance, assume the problem is radial, $\operatorname{div} u_0$ is positive in a crown and negative in the center. Then one may have that $E^+$ is a crown ($w$ should be less than $t$ at the center) and $E^-$ is empty. In that case, $v$ should be equal to one also in the domain surrounded by the set $E^+$.

We show now another simple comparison lemma:

**Lemma 3.3.** *Let $u_0$ and $u_0'$ in $L^2(A;\mathbb{R}^N)$ such that*

$$\operatorname{div} u_0' \ \le\ \operatorname{div} u_0$$

*in $H^{-1}(A)$. Then for any $t \ge 0$, $w'(t) \le w(t)$, where $w'(t)$ and $w(t)$ are the solutions of the contact problem (3.3), the first with $u_0$ replaced with $u_0'$.*

**Proof.** Let $t > 0$, $\varepsilon > 0$, and $w^\varepsilon$ be the minimizer of

$$\min_{|w|\le t} \frac{1}{2}\int_A |\nabla w|^2\,dx \;-\; \int_A w(\operatorname{div} u_0' - \varepsilon)$$

which of course is unique. We now show that $w^\varepsilon \le w(t)$ a.e., and since $w^\varepsilon \to w'(t)$ as $\varepsilon \to 0$ the thesis will follow.

We have by minimality

$$\int_A \frac{|\nabla w(t)|^2}{2}\,dx - \int_A w(t)(\operatorname{div} u_0) \;\le\; \int_A \frac{|\nabla (w(t)\vee w^\varepsilon)|^2}{2}\,dx - \int_A (w(t)\vee w^\varepsilon)(\operatorname{div} u_0)\,,$$

$$\int_A \frac{|\nabla w^\varepsilon|^2}{2}\,dx - \int_A w^\varepsilon(\operatorname{div} u_0' - \varepsilon) \;\le\; \int_A \frac{|\nabla (w(t)\wedge w^\varepsilon)|^2}{2}\,dx - \int_A (w(t)\wedge w^\varepsilon)(\operatorname{div} u_0' - \varepsilon)\,,$$

where we denote $w(t)\vee w^\varepsilon := \max\{w(t), w^\varepsilon\}$ and $w(t)\wedge w^\varepsilon := \min\{w(t), w^\varepsilon\}$. Summing both inequalities we obtain

$$\int_A (w(t)\vee w^\varepsilon - w(t))\operatorname{div} u_0 \;\le\; \int_A (w^\varepsilon - w(t)\wedge w^\varepsilon)(\operatorname{div} u_0' - \varepsilon)\,,$$

from which it follows $\varepsilon \int_A (w^\varepsilon - w(t))^+\,dx \le 0$, which is our claim. □

**Corollary 3.1.** *Under the assumptions of Lemma 3.3,*

$$E^-(t) \ \subseteq E'^-(t) \ \text{and}\ E'^+(t) \ \subseteq\ E^+(t)\,, \tag{3.12}$$

*and it follows that $v'(t) \le v(t)$, for each $t > 0$.*

**Proof.** Eqn (3.12) follows at once from the inequality $w'(t) \le w(t)$ (Lemma 3.3). We deduce, of course, that also $E_r^-(t)\ \subseteq E'^-_r(t)$, and $E'^+_r(t)\ \subseteq\ E_r^+(t)$. Consider the function $v = v'(t)\wedge v(t) = \min\{v'(t), v(t)\}$. As it is $\pm 1$ on $E'^\pm_r(t)$, it follows from Lemmas 3.2 and 3.1 that $-\nabla v \in \partial\mathcal{D}(u'(t))$. In the same way, $v' = v'(t)\vee v(t) = \max\{v'(t), v(t)\}$ is such that $-\nabla v' \in \partial\mathcal{D}(u(t))$. Since

$$\int_A |\nabla v|^2\,dx + \int_A |\nabla v'|^2\,dx \;=\; \int_A |\nabla v(t)|^2\,dx + \int_A |\nabla v'(t)|^2\,dx\,,$$

either $\int_A |\nabla v|^2\,dx \le \int_A |\nabla v(t)|^2\,dx$ or $\int_A |\nabla v'|^2\,dx \le \int_A |\nabla v'(t)|^2\,dx$. By minimality (as $-\nabla v(t) = \partial^0\mathcal{D}(u(t))$) it follows that $v = v(t)$ and $v' = v'(t)$. $\square$

### 3.2. *The support of the measure* div $u$

Throughout this section we will assume that $\operatorname{div} u_0$ is a bounded Radon measure on $A$.

**Lemma 3.4.** *Let $u_0 \in L^2(A;\mathbb{R}^N) \cap \operatorname{dom}\mathcal{D}$, $\delta > 0$ and $u = (I + \delta\partial\mathcal{D})^{-1}(u_0)$. Then for a positive Radon measure $\mu \in H^{-1}(A)$, the Radon-Nikodym derivatives of* $\operatorname{div} u$ *and* $\operatorname{div} u_0$ *with respect to $\mu$ satisfy $(\operatorname{div} u/\mu)(x) \le (\operatorname{div} u_0/\mu)(x)$ for $\mu$-a.e. $x \in \mathcal{E}_u^+$, and $(\operatorname{div} u/\mu)(x) \ge (\operatorname{div} u_0/\mu)(x)$ for $\mu$-a.e. $x \in \mathcal{E}_u^-$. In particular, $\operatorname{div} u << \operatorname{div} u_0$ and $(\operatorname{div} u)^\pm \le (\operatorname{div} u_0)^\pm$.*

**Remark 3.3.** It follows from the Lemma that $\operatorname{div} u = \theta \operatorname{div} u_0 \llcorner (\mathcal{E}_u^+ \cup \mathcal{E}_u^-)$ for some weight $\theta(x) \in [0,1]$. We can build explicit examples where $\theta < 1$ at some point. Consider for instance, in $1D$, $A = (0,1)$ and the function $u_0(x) = 0$ if $x < 1/3$ and $x > 2/3$, and $2 - 3x$ if $1/3 < x < 2/3$. Then, one shows that $u(t)$ is given by

$$u(t,x) \;=\; \begin{cases} 3t & \text{if } x < \frac{1}{3} \\ 1 - 2\sqrt{3t} & \text{if } \frac{1}{3} < x < a(t) := \frac{1}{3} + \frac{2t}{\sqrt{3}} \\ 2 - 3x & \text{if } a(t) < x < b(t) := 1 - \frac{\sqrt{1+6t}}{3} \\ \sqrt{1+6t} - 1 & \text{if } x > b(t) \end{cases}$$

until $t = 1 - 2\sqrt{2}/3$. We have $\operatorname{div} u(t) = u(t)_x = (1 - 2\sqrt{3t} - 3t)\delta_{1/3} - 3\chi_{(a(t),b(t))}$ for such $t$: $E^+_{u(t)} = \{1/3\}$ stays constant for a while (and disappears suddenly right after $t = 1 - 2\sqrt{2}/3$), while the density of the measure $\operatorname{div} u(t)$ goes down monotonically until it reaches zero (notice that $v(t)$ will jump right after $1 - 2\sqrt{2}/3$), while $E^-_{u(t)} = (\alpha(t), \beta(t))$ shrinks in a continuous way, and carries the constant continuous part of the initial divergence $(-3)$.

**Proof.** We have $u = u_0 + \delta\nabla v$ with $-\nabla v \in \partial\mathcal{D}(u)$. Let $x \in \mathcal{E}_u^+$. Recall that the precise representative of $v$ is defined by

$$\tilde{v}(x) \;=\; \lim_{\rho\to 0} \frac{\int_{B(x,\rho)} v(y)\,dy}{\omega_N \rho^N},$$

where $\omega_N = |B(0,1)|$, and that this limit exists quasi-everywhere in $A$. We assume also that $\tilde{v}(x) = 1$.

Then, for a.e. $\rho > 0$, one may write

$$\begin{aligned}\int_{B(x,\rho)} \operatorname{div} u \;=\; &\int_{\partial B(x,\rho)} u\cdot\nu\, d\mathcal{H}^1\\ =\; &\int_{\partial B(x,\rho)} u_0\cdot\nu\, d\mathcal{H}^1 + \delta\int_{\partial B(x,\rho)} \nabla v\cdot\nu\, d\mathcal{H}^1\\ =\; &\int_{B(x,\rho)} \operatorname{div} u_0 \;+\;\delta\int_{\partial B(x,\rho)} \nabla v\cdot\nu\, d\mathcal{H}^1. \qquad (3.13)\end{aligned}$$

Now, let $f(\rho) = (1/\rho^{N-1})\int_{\partial B(x,\rho)} v\, d\mathcal{H}^1$ (which is well-defined for any $\rho$). Then, since $\tilde v(x) = 1$ and $v \le 1$ a.e.,

$$\limsup_{\rho\to 0} f(\rho) = N\omega_N\,.$$

One can also show that for a.e. $\rho > 0$, $f'(\rho) = (1/\rho^{N-1})\int_{\partial B(x,\rho)} \nabla v\cdot\nu\, d\mathcal{H}^1$, in fact $f$ is locally $H^1$ in some small interval $(0,\rho_0)$.

Since $v \le 1$ a.e., $f(\rho) \le N\omega_N$ a.e., so that

$$\begin{aligned}\liminf_{\varepsilon\to 0}\int_\varepsilon^\rho \frac{1}{r^{N-1}}\int_{\partial B(x,r)} \nabla v\cdot\nu\, d\mathcal{H}^1\, dr \;&=\; \liminf_{\varepsilon\to 0}\int_\varepsilon^\rho f'(r)\, dr\\ &=\; \liminf_{\varepsilon\to 0} f(\rho) - f(\varepsilon) \;\le\; 0\end{aligned}$$

for any $\rho$. If follows that for any $\rho$ small, the set $I^+_\rho = \{r\in[0,\rho] : \int_{\partial B(x,r)} \nabla v\cdot\nu\, d\mathcal{H}^1 \le 0\}$ has positive Lebesgue measure, and for any $r\in I^+_\rho$, we deduce from (3.13) that $\int_{B(x,r)} \operatorname{div} u \le \int_{B(x,r)} \operatorname{div} u_0$.

Now consider $\mu$ a positive Radon measure: $\mu$-a.e., we know that the limits

$$\frac{\operatorname{div} u}{\mu}(x) \;=\; \lim_{r\to 0}\frac{\int_{B(x,r)} \operatorname{div} u}{\mu(B(x,r))} \quad\text{and}\quad \frac{\operatorname{div} u_0}{\mu}(x) \;=\; \lim_{r\to 0}\frac{\int_{B(x,r)} \operatorname{div} u_0}{\mu(B(x,r))}$$

exist. If moreover, as before, $x\in\mathcal{E}^+_u$ and $\tilde v(x) = 1$ (which holds $\mu$-a.e., since $\mu\in H^{-1}(A)$), we can find a subsequence $r_n$ such that $\int_{B(x,r_n)} \operatorname{div} u \le \int_{B(x,r_n)} \operatorname{div} u_0$ for each $n$, and it follows $(\operatorname{div} u/\mu)(x) \le (\operatorname{div} u_0/\mu)(x)$. □

The following corollaries follows:

**Corollary 3.2.** *Let $t > s \ge 0$: then $(\operatorname{div} u(t))^\pm \le (\operatorname{div} u(s))^\pm$. In particular, $\mathcal{E}^\pm_{u(t)} \subseteq \mathcal{E}^\pm_{u(s)}$, $|\operatorname{div} u(s)|$-a.e. in $A$.*

**Proof.** Indeed: if $t > s$, then $u(t) = (I + (t-s)\partial\mathcal{D})^{-1}(u(s))$. We deduce that for quasi-every $x\in\mathcal{E}^+_{u(t)}$, $1 = \theta_{\operatorname{div} u(t)}(x) \le (\operatorname{div} u(s)/(\operatorname{div} u(t))^+)(x)$, and it follows $(\operatorname{div} u(t))^+ \le (\operatorname{div} u(s)/(\operatorname{div} u(t))^+)(\operatorname{div} u(t))^+ \le (\operatorname{div} u(s))^+$. □

**Corollary 3.3.** *We have that $(\operatorname{div} u(t))^\pm \overset{*}{\rightharpoonup} (\operatorname{div} u_0)^\pm$ as $t\to 0$, weakly-$*$ in the sense of measures. Moreover, $\mathcal{E}^\pm_{u_0} \subset E^\pm_r(0)$ (up to a $|\operatorname{div} u_0|$-negligible set), and $\operatorname{div} u_0 \llcorner (E^+_r(0)) \ge 0$, $\operatorname{div} u_0 \llcorner (E^-_r(0)) \le 0$.*

**Proof.** We know that as $t \to 0$, $u(t) \to u_0$ in $L^2(A;\mathbb{R}^N)$, and thanks to the boundedness of $\operatorname{div} u(t)$ it follows that $\operatorname{div} u(t) \xrightharpoonup{*} \operatorname{div} u_0$ in the sense of measures. Now consider a subsequence $(t_k)$ such that $(\operatorname{div} u(t_k))^+ \xrightharpoonup{*} \mu$, $(\operatorname{div} u(t_k))^- \xrightharpoonup{*} \nu$. Since $\mu - \nu = \operatorname{div} u_0$, it follows that $\mu \geq (\operatorname{div} u_0)^+$ and $\nu \geq (\operatorname{div} u_0)^-$. The reverse inequalities follow from Lemma 3.4 and the first part of the thesis follows.

From the previous results we obtain that for each $t$, one can write

$$(\operatorname{div} u(t))^+ \;=\; \theta_t(x)(\operatorname{div} u_0)^+$$

The function $\theta_t(x) = \liminf_{\rho\to 0}(\int_{B(x,\rho)} \operatorname{div} u(t)^+)/(\int_{B(x,\rho)}(\operatorname{div} u_0)^+)$ is well-defined on the set $\mathcal{E}^+_{u_0}$ which supports the measure $(\operatorname{div} u_0)^+$, and we find that $\theta_t(x) \leq 1$ is nonincreasing in $t$. Hence there exists for all $x \in \mathcal{E}^+_{u_0}$ the limit $\lim_{t\to 0}\theta_t(x) = \sup_{t>0}\theta_t(x)$, and this limit must be 1 $(\operatorname{div} u_0)^+$-a.e., otherwise this would contradict that $(\operatorname{div} u(t))^+ \xrightharpoonup{*} (\operatorname{div} u_0)^+$. It follows that up to a $(\operatorname{div} u_0)^+$-negligible set, $\mathcal{E}^+_{u_0} \subseteq \bigcup_{t>0}\{x \in \mathcal{E}^+_{u_0} : \theta_t(x) > 0\}$.

Now, if $x \in \mathcal{E}^+_{u_0}$ and $\theta_t(x) > 0$, then $x \in \mathcal{E}^+_{u(t)}$: indeed,

$$\frac{\int_{B(x,\rho)} \operatorname{div} u(t)}{\int_{B(x,\rho)} |\operatorname{div} u(t)|} \;=\; \frac{(\operatorname{div} u(t))^+(B(x,\rho)) - (\operatorname{div} u(t))^-(B(x,\rho))}{(\operatorname{div} u(t))^+(B(x,\rho)) + (\operatorname{div} u(t))^-(B(x,\rho))} \;\overset{\rho\to 0}{\longrightarrow}\; 1\,,$$

since

$$\begin{aligned}(\operatorname{div} u(t))^-(B(x,\rho)) \;&\leq\; (\operatorname{div} u_0^-)(B(x,\rho)) \\ &=\; o((\operatorname{div} u_0^+)(B(x,\rho))) \;\leq\; o((\operatorname{div} u(t))^+(B(x,\rho)))\end{aligned}$$

(the equality is because $x \in \mathcal{E}^+_{u_0}$, the last inequality because $\theta_t(x) > 0$). It follows that

$$\mathcal{E}^+_{u_0} \;\subseteq\; \bigcup_{t>0}\mathcal{E}^+_{u(t)}$$

and the conclusion follows from Lemma 3.2. □

### 3.3. *The regular case*

Let us now assume that $\operatorname{div} u_0 = g \in L^p(A)$, $p > 1$. The obstacle problem which is solved by $w(t)$ can be written

$$\min_{w\in H^1_0 : |w|\leq t} \frac{1}{2}\int_A |\nabla w(x)|^2\,dx \;-\; \int_A g(x)w(x)\,dx\,.$$

Standard results show that $w(t) \in W^{2,p}(A)$, (see Theorem 9.9 in [10]). In particular, we have that in the $L^p$ sense,

$$-\Delta w(t) \;=\; g\chi_{\{|w(t)|<t\}}$$

and, since $u(t) = u_0 + \nabla w(t)$, we deduce that in this case

$$\operatorname{div} u(t) \;=\; \operatorname{div} u_0 \chi_{E(t)} \tag{3.14}$$

for any $t > 0$. In particular, formally, we deduce from (3.2) that

$$\operatorname{div} u_0 \frac{\partial \chi_{E(t)}}{\partial t} \;=\; \Delta v(t)\,, \tag{3.15}$$

and since $\Delta v(t)$ is the jump of the normal derivative of $v(t)$ on $\partial E^{\pm}(t)$, we find that these sets shrink with a normal speed $|\nabla v(t)|/|\operatorname{div} u_0|$.

This can be written rigorously in the sense of distributions: $(E^+, E^-, v)$ are such that $v \in L^1([0,T); H^1_0(A;[-1,1]))$, $v = \pm 1$ on $E^{\pm}$ for a.e. $t$ and $x$, and for any $\phi \in C^\infty_c([0,T) \times A)$,

$$\int_A \operatorname{div} u_0(x)\phi(0,x)\,dx \;+\; \int_0^T \int_A \operatorname{div} u_0(x)\chi_{E(t)}(x)\frac{\partial \phi}{\partial t}(x,t)\,dx\,dt \\ -\; \int_0^T \int_A \nabla v(t,x)\cdot \nabla \phi(t,x)\,dx\,dt \;=\; 0. \tag{3.16}$$

We observe that the evolution equation (3.16) is reminiscent of the enthalpy formulation of the one-phase Stefan problem [18].

We expect that with either the additional information that $\operatorname{div} u_0$ is a.e. nonnegative on $E^+$ and nonpositive on $E^-$, or that the maps $E^{\pm}(t)$ are nonincreasing, then (3.16) characterizes the unique evolution (3.2). On the other hand, without this additional assumption, then a time-reversed evolution with will satisfy the same weak equation, with $u_0$ replaced with $-u_0$. With *both* assumptions we can actually show the following result:

**Proposition 3.1.** *Let $E^+, E^-$ be measurable subsets of $A \times [0,T]$, and $v \in L^1([0,T); H^1_0(A))$ with $|v| \le 1$ a.e., $v = \pm 1$ a.e. on $E^{\pm}$, and satisfying* (3.16)*. Assume in addition that $\pm\operatorname{div} u_0 \ge 0$ a.e. on $E^{\pm}$, and*

$$E^{\pm}(t) \;\subseteq\; E^{\pm}(s) \qquad \textit{for a.e. } t > s\,. \tag{3.17}$$

*Then $u(t,x) := u_0(x) + \nabla \int_0^t v(s,x)\,ds$ is the unique solution of* (3.2)*.*

**Proof.** Let $w(t) = \int_0^t v(s)\,ds$. Thanks to (3.17), we have that $|w(t,x)| \le t$ for a.e. $x \in A$, and $w(t,x) = \pm t$ for a.e. $x \in E^{\pm}(t)$, for all $t$. We can approach test functions of the form $\chi_{[0,t]}\phi(x)$, $\phi \in H^1_0(A)$, with smooth functions and pass to the limit to check that

$$\int_A \operatorname{div} u_0 \phi\,dx - \int_{E(t)} \operatorname{div} u_0 \phi\,dx \;=\; \int_A \nabla w(t)\cdot\nabla\phi\,dx\,,$$

for almost all $t$ (up to a negligible set, which we can actually choose independently of $\phi$, as $H^1_0(A)$ is separable).

If we choose $\phi - w(t, \cdot)$ as the test function in this equation, we find

$$\int_A \operatorname{div} u_0(x)\phi(x)\,dx - \int_A \operatorname{div} u_0(x)w(x,t)\,dx - \int_{E(t)} \operatorname{div} u_0(x)(\phi(x) - w(x,t))\,dx$$
$$= \int_A \nabla w(t,x)\cdot\nabla\phi(x)\,dx - \int_A |\nabla w(t,x)|^2\,dx\,,$$
$$= -\frac{1}{2}\int_A |\nabla w(t,x) - \nabla\phi(x)|^2\,dx + \frac{1}{2}\int_A |\nabla\phi(x)|^2\,dx - \frac{1}{2}\int_A |\nabla w(t,x)|^2\,dx\,.$$

If $|\phi| \le t$, we have that $-\operatorname{div} u_0(x)(\phi(x) - w(x,t)) \ge 0$ for a.e. $x \in E(t)$, so that $w(t)$ is the minimizer of (3.3) and the thesis follows. □

**Remark 3.4.** As mentioned above, it is a natural question whether assumption (3.17) is necessary to prove this result. For instance, in case $E^+$ and $E^-$ are closed sets in $[0,T)\times A$ with $E^+(t)\cap E^-(t) = \emptyset$ for any $t > 0$, and $\{\operatorname{div} u_0 = 0\}$ is a negligible set, then one can actually deduce (3.17) from (3.16). Indeed, using localized test functions $\phi(x)\chi_{[s,t]}$, one shows first that $v$ is harmonic in $A\setminus E(t)$ for a.e. $t$, and then that $\int_{E(s)} \operatorname{div} u_0\phi\,dx - \int_{E(t)} \operatorname{div} u_0\phi\,dx \ge 0$, and (3.17) follows.

**Remark 3.5.** When $p > N/2$, we can deduce some further properties of $w$ from the regularity theory for the obstacle problem [7]. Indeed, letting $\Psi \in H^1_0(A)\cap W^{2,p}(A)$ such that $-\Delta\Psi = g$, we have that $\tilde w = w - \Psi \in H^1_0(A)$ solves the obstacle problem

$$\min_{-t-\Psi\le\tilde w\le t-\Psi} \frac{1}{2}\int_A |\nabla\tilde w(x)|^2\,dx.$$

Since $p > N/2$, we have $w(t) \in C^\alpha(A)$, with $\alpha = 2 - N/p$, so that $E(t) = \{|w(t)| = t\}$ is a closed set. In this case, $v(t)$ can be defined as the harmonic function in $A\setminus E(t)$ with Dirichlet boundary condition $v(t) = 0$ on $\partial A$ and $v(t) = \pm 1$ on $E^\pm(t)$. Moreover, it is easy to check that $-\nabla v(t) \in \partial^0\mathcal{D}(u(t))$, and $v(t)$ is continuous out of the singular points of $\partial A\cup\partial E(t)$.

**Remark 3.6.** If $A = \mathbb{R}^N$ one can easily show easily by a translation argument that $u_0 \in H^1(A;\mathbb{R}^N) \Rightarrow u(t) \in H^1(A;\mathbb{R}^N)$ with same norm, so that the $H^1$-norm of $u(t)$ is nonincreasing. In this case, $\mathcal{E}^+_{u(t)}$ is a.e.-equivalent to the support of $(\operatorname{div} u)^+$ and since from the equation it follows $u = u_0$ a.e. on $E^\pm$ (since $v = \pm 1$ a.e. on $E^\pm$, so that $\nabla v = 0$ a.e., the problem being in general that this will not be true quasi-everywhere), we deduce that $\operatorname{div} u = \operatorname{div} u_0$ a.e. on $E^+\cup E^- = \operatorname{spt}(\operatorname{div} u)$.

## 4. Examples

### 4.1. *The antiplane case in dimension 2*

Let $N = 2$ and $k = 1$. We have

$$J(\psi) = |\operatorname{rot}\psi|(A) = \sup\left\{\int_A \nabla^\perp\cdot\psi \,:\, v \in C^\infty_c(A;[-1,1])\right\}$$

where $\operatorname{rot}\psi = \partial_1\psi_2 - \partial_2\psi_1$ and $\nabla^\perp = (\partial_2, -\partial_1)$. Then, we check easily that in $L^2(A;\mathbb{R}^2)$ the functional $J$ is the support function of the closed convex set

$$K = \left\{\nabla^\perp v \,:\, v \in H^1_0(A;[-1,1])\right\}.$$

As we mentioned in the Introduction, this functional appears as limit of the Ginzburg-Landau model in a suitable energy regime [19].

Letting $\psi^\perp = (\psi_2, -\psi_1)$, we get $J(\psi) = \int_A |\operatorname{div}\psi^\perp|$, so that the flow can be described as above.

**Proposition 4.1.** *Let $u_0 \in L^2(A;\mathbb{R}^2)$ with $\operatorname{rot} u_0 = g \in L^p(A)$, $p > 1$. Then for $t > 0$ there exist nonincreasing left-continuous closed (and disjoint) sets $E^\pm(t) \subset \{\pm g \geq 0\}$, such that $\operatorname{rot} u(t) = \operatorname{rot} u_0(\chi_{E^-(t)\cup E^+(t)})$. Moreover, letting $E^\pm = \overline{\cup_{t\geq 0}\{t\}\times E^\pm(t)}$, there exists a function $v(t,x)$ with $v = \pm 1$ a.e. on $E^\pm$ such that $(E^+, E^-, v)$ are the unique closed sets and function solution of the weak Hele-Shaw flow* (3.16).

### 4.2. *The one-dimensional Total Variation Flow*

Let now $N = 1$, $k = 0$: the previous analysis also provides interesting qualitative information on the behavior of the flow of the Total Variation, in dimension 1.

We consider $u_0 \in L^2((a,b))$, $a < b$, and the flow $u(t)$ of the total variation $J(u) := \sup\{\int_a^b uv'\,dt \,:\, v \in C^\infty_c(a,b;[-1,1])\}$. Notice that in this situation, the function $w$ which minimizes (3.3), being in $H^1_0(a,b)$, is also in $C^{1/2}([a,b])$ with $w(a) = w(b) = 0$. In particular, the sets $E^\pm(t)$ defined in (3.9) are closed, disjoint sets compactly contained in $(a,b)$.

We can state the following result.

**Proposition 4.2.** *The function $u(t)$ is the unique minimizer of*

$$\min_u J(u) + \frac{1}{2t}\int_a^b |u - u_0|^2\,dx.$$

*Moreover there exist nonincreasing, disjoint closed sets $E^\pm(t) \subset (a,b)$ such that $u(t) = u_0$ a.e. on $E^\pm(t)$, $u_0$ is nondecreasing on any interval contained in $E^+(t)$, nonincreasing on any interval contained in $E^-(t)$, and $u(t)$ is constant on each connected component of $(a,b)\setminus(E^+(t)\cup E^-(t))$.*

If $u_0$ is smooth enough, one can also characterize the speed of the boundary points of $E^\pm(t)$ in term of $u_0$ and the size of the intervals of $(a,b)\setminus(E^+(t)\cup E^-(t))$.

**Proof.** The first part of the thesis is a consequence of Remark 3.1. Then, if $u_0 \in BV(a,b)$, the thesis is a consequence of Lemma 3.2. Indeed, for a.e. $x$ on $E^\pm(t)$, we have $\partial_x w(t,x) = 0$ and $u(t,x) = u_0(x) + \partial_x w(t,x) = u_0(x)$. If $I \subset E^+(t)$ is an interval, since the measure $Du(t) \llcorner I$ must be nonnegative, $u(t)$ is nondecreasing on $I$, but as $u(t) = u_0$ a.e. on $I$ it follows that $u_0$ is nondecreasing on $I$.

If $u_0 \notin BV(a,b)$, we use the fact that for all $\varepsilon > 0$, $u(\varepsilon) \in BV(a,b)$. Then the Proposition holds for $t > \varepsilon$, and we have $u(t) = u(\varepsilon)$ a.e. on $E^{\pm}(t)$, $u(\varepsilon)$ is nondecreasing on any interval contained in $E^+(t)$, nonincreasing on any interval contained in $E^-(t)$, and $u(t)$ is constant on each connected component of $(a,b) \setminus (E^+(t) \cup E^-(t))$. The sets do not depend on $\varepsilon$, as they are defined as the contact sets in (3.3). Sending then $\varepsilon \to 0$ we deduce the result. $\square$

We can deduce the following, quite interesting result (see also [17, 5, 14] for other results on the one-dimensional Total Variation flow).

**Corollary 4.1.** *Let $u_0 = \bar{u}_0 + n$ where $\bar{u}_0 \in BV(a,b)$ and $n$ is a stochastic process $(a,b)$ with $n \in L^2(a,b)$ a.s. and such that $|Dn|(I) = +\infty$ for any interval $I \subset (a,b)$, almost surely. Let $u(t)$ be the total variation flow starting from $u_0$. Then almost surely, at $t > 0$, there is "staircaising" everywhere in the interval $(a,b)$: $u(t)$ is constant on each connected component of an open set $A(t)$ which is dense in $(a,b)$.*

**Remark 4.1.** The property that $|Dn|(I) = +\infty$ for any interval $I$, almost surely, is satisfied for instance by the Wiener process (as its quadratic variation is positive a.s.). For a Gaussian stationary process, it will depend on the behaviour of the autocorrelation function and can be characterized by conditions on the power spectrum of the process, see for instance [4] for (non sharp) conditions.

**Proof.** We let $A(t) = (a,b) \setminus (E^+(t) \cup E^-(t))$, and from the previous result we know that $u(t)$ is constant on each connected component of $A(t)$ while $u = u_0$ on $(a,b) \setminus A(t)$. Now assume there is an interval $I$ with $I \cap A(t) = \emptyset$: without loss of generality we may assume that $I \subset E^+(t)$. Then $u_0$ must be nondecreasing on $I$, in particular there exists $I' \subset I$ with $|Du_0|(I') < +\infty$. But this yields that $|Dn|(I') < +\infty$, which is a.s. impossible. $\square$

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