LIMITING MODELS IN CONDENSED MATTER PHYSICS AND GRADIENT FLOWS OF 1-HOMOGENEOUS FUNCTIONALS

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Abstract. We survey some recent results on variational and evolution problems concerning a certain class of convex 1-homogeneous functionals for vector-valued maps related to models in phase transitions (Hele-Shaw), superconductivity (Ginzburg-Landau) and superfluidity (Gross-Pitaevskii). Minimizers and gradient flows of such functionals may be characterized as solutions of suitable non-local vectorial generalizations of the classical obstacle problem.

Contents

1. Introduction

In this note we survey some results obtained in [8, 6, 2] on variational and evolution problems concerning a certain class of convex 1-homogeneous functionals for vector-valued maps related to models in phase transitions (Hele-Shaw), superconductivity (Ginzburg-Landau) and superfluidity (Gross-Pitaevskii). It has been recognized that minimizers and gradient flows of such functionals may be characterized as solutions of suitable vectorial generalizations of obstacle-type problems, where the constraints are non-local in nature.

This motivated us to investigate the general structure and qualitative properties of such solutions in analogy with classical obstacle problems, trying in particular to characterize situations where the non-local constraint is saturated, as well as qualitative properties of the corresponding coincidence sets, and interpreting their physical meaning according to the considered model.

The functionals we are interested in have the general form

$$J(u) = \int_\Omega |A(x){\cdot}Du| = \sup\left\{\int_\Omega u\cdot \mathrm{div}(A(x)^t\cdot\xi)\,dx : \xi\in C^1_c(\Omega;\mathbb{R}^{kq}),\ |\xi|\le 1\right\}, \quad (1.1)$$

where $\Omega\subset\mathbb{R}^n$, $u\in L^1(\Omega;\mathbb{R}^k)$ and $A\in C^1(\Omega,\mathbb{R}^{qn})$. The functional J is convex and 1-homogeneous, and in case $u\in BV(\Omega;\mathbb{R}^k)$, i.e. $Du=\nu|Du|$ is a $\mathbb{R}^{nk}$-valued measure with total variation $|Du|$ and $|\nu|=1$ $|Du|$-a.e., the functional $J(u)$ represents the total variation of the measure $A(x)\cdot\nu|Du|\llcorner\Omega$.

1.1. **Some examples.** A first important example is given by the weighted Total Variation functional (see e.g. [4, 10, 11, 12, 15])

$$J_0(u) = \int_\Omega \rho(x)|Du| = \sup\left\{\int_\Omega u\cdot\mathrm{div}(\rho(x)\xi)\,dx : \xi\in C^1_c(\Omega;\mathbb{R}^n),\ |\xi|\le 1\right\}, \qquad (1.2)$$

defined for $u\in L^1(\Omega)$ and a regular nonnegative weight $\rho(x)\ge 0$. Further examples are given, for $u\in L^1(\Omega;\mathbb{R}^n)$ a vector field, by

$$J(u)=\int_\Omega |\nabla\cdot u|,$$

i.e. the total variation of the divergence of u, and, in case $n=3$, by

$$J(u)=\int_\Omega |\nabla\times u|,$$

i.e. the total variation of the curl of u. More generally, if $u(x)\in\Lambda^k\mathbb{R}^n$ for $x\in\Omega$, i.e. u is a k-differential form, we may consider

$$J_k(u) = \int_\Omega \rho(x)|du| = \sup\left\{\int_\Omega u\cdot d^*(\rho(x)\xi)\,dx : \xi\in C^1_c(\Omega;\Lambda^{k+1}\mathbb{R}^n),\ |\xi|\le 1\right\}, \quad (1.3)$$

so that J_{n-1} corresponds to the (weighted) total variation of the divergence, and J_1 to the (weighted) total variation of the curl in case $n=3$, and J_0 yields again the weighted TV functional.

1.2. **Gradient flows.** Given an initial datum $u_0\in L^2(\Omega;\mathbb{R}^k)$, the L^2-gradient flow for $J(u)$ is defined by the differential inclusion

$$u_t\in -\partial J(u) \qquad t\in[0,+\infty)\,, \qquad (1.4)$$

where ∂J denotes the subgradient of the convex functional J. The general theory of [9] guarantees the existence of a global weak solution $u\in H^1([0,+\infty),L^2(\Omega;\mathbb{R}^k))$ of (1.4). The (weighted) Moreau-Yosida regularization of the convex functional J, which yields a discrete approximation scheme for (1.4) is given by

$$I(u) = J(u)+\lambda\int_\Omega |u-u_0|^2 g(x)\,dx\,, \qquad (1.5)$$

with $u_0:\Omega\to\mathbb{R}^k$ square integrable with respect to the measure $g(x)\,dx$, where $g(x)\ge 0$ is a nonnegative regular weight function and $\lambda>0$ a parameter. For $n=2$, $k=1$ and $f(x)=g(x)\equiv 1$, (1.5) corresponds to the Rudin-Osher-Fatemi Total Variation based denoising model [21]. Anisotropic versions of TV functionals and applications to active contour and edge detection have been studied in [12]. Related homogeneous functionals for the description of landsliding have been also studied in [15, 10].

1.3. **Formulation for differential forms.** For u a k-differential form on Ω, define

$$I_k(u) = J_k(u) + \lambda \int_\Omega |u - u_0|^2 g(x)\, dx\,,$$

with u_0 a k-form in $L^2(g(x)\,dx)$. Functionals like I_k enter in the description of some phenomena in condensed matter Physics such as superconductivity and superfluidity (e.g. Bose-Einstein condensation). For instance, in case $n = 3$, $\lambda = 1$, $u_0(x) = \omega(xdy - ydx)$, $\Omega = \{\rho(x) \geq 0\}$ and $g(x) = \rho(x)$, the functional I_1 arises as a reduced model describing, in suitably scaled units, the vortex density distribution in a trapped Bose-Einstein condensate rotating around the z-axis with an angular velocity $\omega > 0$. Here u corresponds to a superfluid current density, so that its exterior differential (corresponding to the curl) describes the vorticity of the superfluid. The derivation of this model as a limiting description of the Gross-Pitaevsky energy functional in certain asymptotic regimes has been rigorously proved in [5, 6] through a Γ-convergence analysis valid for general Ginzburg-Landau type models. In axisymmetric domains (see [2]) the superfluid current can be expressed, in cylindrical coordinates, by $u = v(r, z)d\theta$, so that the situation is actually described by a weighted TV regularization functional $I_0(v)$, defined on $\tilde{\Omega} = \Omega \cap \{\theta = \text{cost.}\}$ with $g(r, z) = r^{-1}\rho(r, z)$.

Finally, the functional I_{n-1}, corresponding to the discretization of the L^2-gradient flow $u_t = -\partial J_{n-1}(u)$, has been studied in [8], where some rigorous connection with a (weak formulation of the) Hele-Shaw model in phase transitions has been established.

In the following sections we will analyze some properties of J_k and I_k focusing mainly on the cases $k = 1$ and $k = n - 1$.

2. Gradient flow of J_k

In order to analyze the gradient flow (1.4) for the functional J_k one is led to consider first the approximating scheme (1.5) given, fixing $\epsilon > 0$, by the minimum problem (we consider for simplicity uniform densities $\rho = g = 1$)

$$\min I_k(u) = J_k(u) \, + \int_\Omega \frac{1}{2\epsilon}|u - f|^2\, dx, \tag{2.1}$$

for a given k-form $f \in L^2(\Omega, \Lambda^k(\mathbb{R}^n))$. The Euler-Lagrange equation corresponding to (2.1) is

$$\frac{f - u}{\epsilon} \in \partial J_k(u)\,.$$

Notice that since J_k is convex positively 1-homogeneous we have $\eta \in \partial J_k(u)$ if and only if

$$J_k(u) = \langle \eta, u\rangle \text{ and } \langle \eta, w\rangle \leq J_k(w) \quad \forall\, \eta \in \partial J_k(u),\ \forall\, w \in\, L^2(\Omega; \Lambda^k\mathbb{R}^n)\,. \tag{2.2}$$

In particular, the convex conjugate function J_k^* corresponds to the indicatrix function $J_k^*(\eta) = 0$ if $\sup\langle \eta, w\rangle \leq J(w)$, and $J_k^*(\eta) = +\infty$ otherwise. An element $\eta \in \partial J_k(u)$

may be represented as $\eta = d^*v$ (in the distributional sense), for a $(k+1)$-form v such that $|v| \le 1$ a.e. in Ω and $(*v)_T = *(v_N) = 0$ on $\partial\Omega$, where d^* correspond to the adjoint operator to d with respect to the Hodge star duality, and ω_T (resp. ω_N) denotes the tangential (resp. normal) component of a form ω on $\partial\Omega$: in particular we have $(*\omega)_T = *(\omega_N)$, and the boundary condition $(*v)_T = 0$ corresponds to the vanishing of the normal component of v on $\partial\Omega$. Moreover, the condition $J_k(u) = \langle \eta, u\rangle = \int_\Omega v \cdot du$ implies that $v = du/|du|$ (and in particular $|v| = 1$) whenever $du \ne 0$. We may hence write the Euler-Lagrange equation (2.1) as

$$\frac{f-u}{\epsilon} = d^*v \text{ in } \Omega \qquad \text{and} \qquad (*v)_T = 0 \text{ on } \partial\Omega. \tag{2.3}$$

Accordingly, the gradient flow (1.4) corresponds to the differential system

$$u_t = d^*v \text{ in } \Omega \qquad \text{and} \qquad (*v)_T = 0 \text{ on } \partial\Omega \tag{2.4}$$

under the constraints $|v| \le 1$ and $J_k(u) = \int_\Omega v \cdot du$ valid for any time $t > 0$.

2.1. **Dual formulation.** Equation (2.3) is equivalent to

$$u \in \partial J_k^*\left(\frac{f-u}{\epsilon}\right), \tag{2.5}$$

where

$$J_k^*(\eta) := \sup_{w\in L^2(\Omega,\Lambda^k\mathbb{R}^n)} \int_\Omega \eta\cdot w\,dx - J_k(w) = \begin{cases} 0 & \text{if } \|\eta\|_* \le 1 \\ +\infty & \text{otherwise}\end{cases}$$

and

$$\|\eta\|_* = \sup\left\{\int_\Omega \eta\cdot w\,dx :\ J_k(w) \le 1\right\}.$$

Note that

$$J_k(w) + J_k^*(\eta) \ge \int_\Omega w\cdot\eta\,dx$$

for all w, η. The equality holds iff $\int_\Omega \eta\cdot w\,dx = J_k(w)$, and in such case we have $\|\eta\|_* \le 1$.

Letting u be a minimizer of (2.1) and $\eta = (f-u)/\epsilon$ we get from (2.5)

$$\eta - \frac{f}{\epsilon} + \frac{1}{\epsilon}\partial J_k^*(\eta) \ \ni\ 0 \tag{2.6}$$

which shows that η is the unique minimizer of

$$\min_\eta J_k^*(\eta) + \frac{\epsilon}{2}\int_\Omega \left|\eta - \frac{f}{\epsilon}\right|^2 dx \ = \ \frac{\epsilon}{2}\min_{\|\eta\|_*\le 1}\int_\Omega \left|\eta - \frac{f}{\epsilon}\right|^2 dx. \tag{2.7}$$

This corresponds to the dual problem of (2.1), which can be interpreted as the L^2-projection of f on the convex set $\{\|\eta\|_* \le \epsilon\}$. In particular, we deduce the existence of a critical threshold ϵ_c below which minimizers are necessarily trivial (see also [18] for the same result in the case of the Total Variation model corresponding to J_0). This can be summarized in the following

Proposition 2.1. *The k-form $u \equiv 0$ is a minimizer of* (2.1) *if and only if*

$$\epsilon \ge \epsilon_c := \|f\|_*\,. \tag{2.8}$$

2.2. **Non local obstacle-type problems.** Note that $\|\eta\|_* < \infty$ implies that

$$\int_\Omega \eta w = 0$$

for all w such that $dw = 0$. By Hodge decomposition, this implies that $\eta = d^* g$ for some $(k+1)$-form g, with $g_N = 0$ on $\partial\Omega$. It follows that

$$\begin{aligned} \|\eta\|_* &= \sup_{\int_\Omega |dw| \le 1} \int_\Omega d^* g \cdot w \, dx \\ &= \sup_{\int_\Omega |dw| \le 1} \int_\Omega g \cdot dw \, dx + \int_{\partial\Omega} w \wedge * g_N \;=\; \sup_{\int_\Omega |dw| \le 1} \int_\Omega g \cdot dw \, dx. \end{aligned} \tag{2.9}$$

We then get the following characterization of the norm $\|\cdot\|_*$:

$$\|\eta\|_* = \inf_{\substack{d^* g = \eta \\ g_N|_{\partial\Omega} = 0}} \|g\|_{L^\infty(\Omega, \Lambda^{k+1}(\mathbb{R}^n))}. \tag{2.10}$$

We may hence view the norm $\|\cdot\|_*$ related to the dual convex function J_k^* as a non-local L^∞ norm and, accordingly, the dual problem (2.7) can be interpreted as a non-local vector-valued version of the classical obstacle problem, which reduces to the classical one in the case $k = n-1$, since in (2.9) we may choose n-forms dw such that the functions $*dw$ approximate a Dirac mass located where the essential supremum of $|g|$ is attained.

Let us briefly deduce (2.10). Indeed, it is immediate to show the $\le$ inequality. On the other hand, by the Hahn-Banach Theorem, there exists a differential form $g' \in L^\infty(\Omega; \Lambda^{k+1}(\mathbb{R}^n))$, with $d^* g' = d^* g = \eta$ (in the distributional sense) such that

$$\|\eta\|_* = \sup_{\int_\Omega |dw| \le 1} \int_\Omega g \cdot dw \, dx = \sup_{\int_\Omega |\psi| \le 1} \int_\Omega g' \cdot \psi \, dx = \|g'\|_{L^\infty(\Omega; \Lambda^{k+1}(\mathbb{R}^n))}.$$

Fix now φ_0 such that $d^* \varphi_0 = \eta$. We can write $g = \varphi_0 + d^* \psi$, so that (2.10) becomes

$$\|\eta\|_* = \min_{\psi\colon (\varphi_0 + d^*\psi)\cdot \nu_A = 0} \|\varphi_0 + d^* \psi\|_{L^\infty(A)}. \tag{2.11}$$

The Euler-Lagrange equation of (2.11) is a kind of generalization of the *infinity Laplacian equation* (see e.g. [7, 20])

$$\mathrm{d}_\infty(\varphi_0 + d^* \psi) = 0.$$

Indeed when $k = n-2$, by duality, problem (2.11) becomes

$$\min_{\psi \in W_0^{1,\infty}(\Omega)} \|\nabla\psi + \varphi_0\|_{L^\infty(\Omega)}, \tag{2.12}$$

whose corresponding Euler-Lagrange equation is

$$\langle (\nabla^2 \psi + \nabla\varphi_0)(\nabla\psi + \varphi_0), (\nabla\psi + \varphi_0) \rangle = 0, \tag{2.13}$$

which is a non-homogeneous ∞-Laplacian equation reminiscent of the Aronsson problem [1]. For such class of equations it is not known if there are conditions on φ_0 guaranteeing uniqueness of solutions.

2.3. **Some properties of the gradient flow of J_{n-1}.** When $k = n-1$ we can identify by duality a square integrable $(n-1)$-form in Ω with a vector field $u \in L^2(\Omega, \mathbb{R}^N)$, so that J_{n-1} is equivalent to the functional

$$J_{n-1}(u) := \int_\Omega |\nabla \cdot u| \, dx \tag{2.14}$$

that is, the total mass of $\nabla \cdot u$ as a measure.

The gradient flow of J_{n-1} is actualy equivalent to a constrained variational problem for a function w such that $\Delta w = \nabla \cdot u$. Consider indeed the formulation

$$\begin{cases} u_t & = \nabla v \\ u(0) & = u_0 \end{cases} \tag{2.15}$$

where v satisfies $|v| \leq 1$ and

$$J_{N-1}(u) + \int_\Omega u \cdot \nabla v = 0.$$

It is well-known that the solution of (2.15) is unique and that $-\nabla v(t) = \partial^0 J_{N-1}(u(t))$ is the right-derivative of $u(t)$ at any $t \geq 0$ [9]. Given the solution $(u(t), v(t))$ of (2.15), we let $w(t) := \int_0^t v(s)\, ds$, which takes its values in $[-t, t]$. It holds $u(t) = u_0 + \nabla w(t)$, and the function $w(t)$ solves the following obstacle problem (see [8]):

$$\min \left\{ \frac{1}{2} \int_\Omega |u_0 + \nabla w|^2 \, dx \, : \, w \in H^1_0(\Omega)\, , |w| \leq t \text{ a.e.} \right\}. \tag{2.16}$$

Observe that in case we additionally have $\nabla \cdot u_0 \geq \alpha > 0$, this obstacle problem is known to be an equivalent formulation of the Hele-Shaw flow [14, 16] (see also [17] for a viscosity formulation). Therefore, it turns out that the flow of J_{n-1} provides a (unique) global weak solution to the Hele-Shaw problem, under suitable regularity assumptions on the initial datum u_0. Moreover this formulation allows to consider quite general initial data u_0, for which for instance $\nabla \cdot u_0$ may change sign, or be a measure. Further regularity properties of the function $w(t)$ and the evolution law of the contact set are deduced via comparison principles (see [8] for details).

3. The functional I_1

Let us turn to analyze the functional I_1 which expresses a Moreau-Yosida regularization of the functional $J_1(u) = \int_\Omega \rho |du|$, i.e. the total (weighted) mass of the exterior differential of a square integrable 1-form u. In the case $N = 3$ (and $\rho = 1$) it corresponds to the total mass of the curl of a vector field $u \in L^2(\Omega; \mathbb{R}^N)$ as a measure. This type of functional arises as a reduced description of vortex density in superfluids and respectively superconductors corresponding to asymptotic regimes of the three dimensional Gross-Pitaevskii model of Bose-Einstein condensates (resp. the Ginzburg-Landau model for 3-d superconductivity)

3.1. **Asymptotics for the Gross-Pitaevskii model.** Consider a Bose-Einstein condensate with mass m confined in a domain $\Omega \subset \mathbb{R}^3$ by a smooth trapping nonnegative potential $0 \leq a \in C^\infty(\mathbb{R}^3)$, $a(x) \to +\infty$ as $|x| \to +\infty$, and subjected to a forcing Φ_ϵ that in general depends on a scaling parameter ϵ. In the model case corresponding to a rotation around the z-axis, one has $\Phi_\epsilon := \frac{1}{2}c_\epsilon(x_1 dx^2 - x_2 dx^1)$, and $a(x)$ grows at least quadratically in $|x|$. The Gross-Pitaevskii functional in the ϵ-scaling regime reads

$$G_\epsilon(u) := \int_{\mathbb{R}^3} \frac{1}{2}|\nabla u|^2 - \Phi_\epsilon \cdot ju + \frac{1}{\epsilon^2}(\frac{|u|^4}{4} + a(x)\frac{|u|^2}{2})$$

where u is a complex-valued wave function whose modulus describes the superfluid density, $\Phi_\epsilon = |\log\epsilon|\Phi$ for some fixed Φ and the $j(u) = \frac{i}{2}(ud\bar{u} - \bar{u}du)$ measures the superfluid current. A stable condensate may be described by a (local or global) minimizer of G_ϵ in the function space

$$H^1_a(\mathbb{R}^3;\mathbb{C}) := H^1_a := \text{ completion of } C^\infty_c(\mathbb{R}^3;\mathbb{C}) \text{ with respect to } \|\cdot\|_a, \tag{3.1}$$

where the norm $\|\cdot\|_a$ is defined by $\|u\|_a^2 := \int_{\mathbb{R}^3}|du|^2 + (1+a)|u|^2$. Define also

$$H^1_{a,m}(\mathbb{R}^3;\mathbb{C}) := H^1_{a,m} := \{u \in H^1_a \ : \ \int |u|^2 = m\}.$$

In order to study the behavior of minimizers of G_ϵ in $H^1_{a,m}$ it is convenient to rewrite the energy as follows: define

$$\rho(x) := (\lambda - a(x))^+, \quad w(x) := (\lambda - a(x))^-, \quad \text{ for } \lambda \text{ such that } \int_{\mathbb{R}^3} \rho \, dx = m. \tag{3.2}$$

The last condition clearly determines λ uniquely. The function ρ is called the *Thomas-Fermi density* in the physics literature, and gives to the leading-order the condensate density, in the limit $\epsilon \to 0$. Since $\int \lambda|u|^2 = \lambda m$ for all $u \in H^1_{a,m}$, it follows that u minimizes G_ϵ in $H^1_{a,m}$ if and only if u minimizes

$$\mathcal{G}_\epsilon(u) \ := \ \int_{\mathbb{R}^3} \frac{1}{2}|\nabla u|^2 - \Phi_\epsilon \cdot ju + \frac{1}{4\epsilon^2}(\rho - |u|^2)^2 + \frac{w}{2\epsilon^2}|u|^2 \tag{3.3}$$

in $H^1_{a,m}$. We will henceforth write the Gross-Pitaevskii functional in the more convenient form (3.3), and define $\Omega = \{x \in \mathbb{R}^3 : \rho(x) > 0\}$. The following convergence result is a consequence of the analysis in [5, 6].

Proposition 3.1. *Assume that $\Phi_\epsilon = |\log\epsilon|\Phi$, with $\Phi \in L^4_{loc}(\Lambda^1\mathbb{R}^3)$ and that $|\Phi(x)|^2 \le Ca(x)$ outside some compact set K. Assume that u_ϵ minimizes $\mathcal{G}_\epsilon$ in $H^1_{a,m}$. Then*

$$|u_\epsilon| \to \rho \quad \text{ in } L^4(\mathbb{R}^3)$$

for ρ defined in (3.2)*, and there exists $j_0 \in L^{4/3}(\Lambda^1\Omega)$ such that*

$$|\log\epsilon|^{-1} ju_\epsilon \rightharpoonup j_0 \text{ weakly in } L^{4/3}(\mathbb{R}^3).$$

Moreover, $j_0 = \rho v_0$, where v_0 is the unique minimizer of

$$\mathcal{G}(v) := \int_\Omega \rho\left(\frac{|v|^2}{2} - v\cdot\Phi + \frac{1}{2}|dv|\right). \tag{3.4}$$

in the space

$$L^2_\rho(\Lambda^1\Omega) := \left\{ v \in L^1_{loc}(\Lambda^1\Omega) \ : \ \int_\Omega \rho|v|^2 \, dx < \infty \right\}. \tag{3.5}$$

(We set $\mathcal{G}(v) = +\infty$ if dv is not a Radon measure or if ρ is not $|dv|$-integrable.)

This convergence results for the Gross-Pitaevskii functional parallel those obtained in [5] for the Ginzburg-Landau functional. The functional $\mathcal{G}$ of the limiting variational problem corresponds to I_1, and the 1-form v measures the asymptotic superfluid current and hence its curl, corresponding to dv, gives a measure of the leading-order vortex density of the superfluid. From the analysis of $\mathcal{G}$ (compare with the analysis of I_1 leading to Proposition 2.1) one may thus characterize when minimizers of the limiting problem are vortex-free to the leading order, and the magnitude of the critical threshold of the forcing (resp. the first critical magnetic field in the superconductivity case) above which there is vortex nucleation, by obtaining a description of minimizers of $\mathcal{G}$ as solutions of a nonlocal vector-valued obstacle problem of the type of (2.7).

Let us now state more precisely a necessary and sufficient condition on Φ and ρ for minimizers of the limiting functional $\mathcal{G}$ to be vortex-free, by which we mean that $dv_0 = 0$ in Ω. Denoting by

$$(v, w)_\rho := \int_\Omega \rho\, v \cdot w \, dx, \qquad \|v\|_\rho := (v, v)_\rho^{1/2},$$

respectively the inner product and norm on the Hilbert space $L^2_\rho(\Lambda^1\Omega)$, let P_ρ denote the orthogonal projection with respect to the L^2_ρ inner product onto $(\ker d)_\rho$, where

$$(\ker d)_\rho := L^2_\rho\text{-closure of } \{\varphi \in C^\infty(\Lambda^1\Omega) : d\varphi = 0,\ \|\varphi\|_\rho < \infty\}. \tag{3.6}$$

We will also write $P_\rho^\perp$ for the complementary orthogonal projection. Note that if $w \in \mathrm{Image}(P_\rho^\perp) = (\ker d)_\rho^\perp$, then $\int (\rho w)\cdot\varphi = 0$ for all $\varphi \in (\ker d)_\rho \supset \ker d$. Thus $\rho w \in (\ker d)^\perp$, and so it follows from the standard (unweighted) Hodge decomposition that

$$\forall\ w \in (\ker d)_\rho^\perp,\ \exists \beta \in H^1_N(\Lambda^2\Omega) \text{ such that } w = \frac{d^*\beta}{\rho} \text{ and } \int_\Omega \frac{|d^*\beta|^2}{\rho} \ = \ \|w\|^2_\rho. \tag{3.7}$$

Thus if $\Phi \in L^2_\rho$, there exists $\beta_\Phi \in H^1_N$ such that $d^*\beta_\Phi \in L^2_\rho$ and

$$\Phi = P_\rho\Phi + \frac{d^*\beta_\Phi}{\rho}. \tag{3.8}$$

We have the followig result (see [6]):

Theorem 3.2. *Suppose that Ω is a bounded, open subset of $\mathbb{R}^3$ and that $0 \le \rho \in C^1(\Omega)$ and $\Phi \in L^4_{loc}(\Lambda^1\mathbb{R}^3) \cap L^2_\rho(\Lambda^1\Omega)$ are given. Let $\beta_\Phi \in H^1_N(\Lambda^2\Omega)$ be such that $P_\rho^\perp\Phi = \frac{d^*\beta_\Phi}{\rho}$, and let β_0 minimize the functional*

$$\beta \mapsto \frac{1}{2}\int_\Omega \frac{|d^*\beta|^2}{\rho} \tag{3.9}$$

in the space

$$\left\{ \beta \in H^1_N(\Lambda^2\Omega) \ : \ \frac{d^*\beta}{\rho} \in L^2_\rho(\Lambda^1\Omega), \quad \|\beta - \beta_\Phi\|_{\rho*} \le \frac{1}{2} \right\}, \tag{3.10}$$

where

$$\|\beta\|_{\rho*} := \sup\left\{ \int_\Omega \beta \cdot dw : \ w \in C^\infty(\Lambda^1\bar\Omega), \int_\Omega \rho|dw| \le 1 \right\}. \tag{3.11}$$

Then $v_0 = P_\rho\Phi + \frac{d^*\beta_0}{\rho}$ *is the unique minimizer of* $\mathcal{G}(\cdot)$ *in* $L^2_\rho(\Lambda^1\Omega)$*. Moreover,*

$$\int_\Omega (\beta_\Phi - \beta_0)\cdot dv_0 = \frac{1}{2}\int_\Omega \rho|dv_0|\,. \tag{3.12}$$

Finally, $dv_0 = 0$ *if and only if* $\|\beta_\Phi\|_{\rho*} \le \frac{1}{2}$*.*

Note that (3.12) states that the action of the vorticity distribution dv_0 on the potential $\beta_0 - \beta_\Phi$ is the largest possible given the constraint (3.10).

Observe that the form of the constraint in the limiting dual variational problem depends on the dimension. In particular, in 2d, as in 3d, it is the case that if v_0 minimizes $\mathcal{G}$, then $dv_0 = d(\frac{d^*\beta_0}{\rho})$, where the potential β_0 minimizes the functional (3.9) subject to the constraint (3.10). The difference is that in 2d, the potentials β are 2-forms on $\mathbb{R}^2$, and so they can be identified with functions. Since it is not hard to check that $\{d\omega : \int_\Omega \rho|d\omega| \le 1\}$ is weakly dense in the set of signed measures μ such that $\int_\Omega \rho d|\mu| \le 1\}$, the 2d constrained problem reduces to minimizing (3.9) in the set

$$\left\{\beta \in H^1(\Lambda^2\Omega) : \|\frac{1}{\rho}(\beta - \beta_\Phi)\|_{L^\infty} \le \frac{1}{2}\right\}. \tag{3.13}$$

This is a classical (weighted) 2-sided obstacle problem, and for many Φ's, using the maximum principle, it in fact reduces to a one-sided obstacle problem. Thus we can view the problem in Theorem 3.2 as a nonlocal, vector-valued analog of the classical obstacle problem.

3.2. **Rotational symmetry and weighted TV minimization.** In the presence of rotational symmetry, the functional $\mathcal{G}$ reduces to a simpler 2-dimensional model corresponding to the weighted Total Variation minimization functional I_0. More precisely, assume that there exist $\tilde\Omega \subset [0,\infty)\times\mathbb{R}$, $\tilde\rho : \tilde\Omega \to (0,\infty)$ and $\varphi : \tilde\Omega \to \mathbb{R}$ such that $\Omega = \{(r\cos\alpha, r\sin\alpha, z) : (r,z) \in \tilde\Omega, \alpha \in \mathbb{R}\}$, $\rho(r\cos\alpha, r\sin\alpha, z) = \tilde\rho(r,z)$ and $\Phi(r\cos\alpha, r\sin\alpha, z) = \varphi(r,z)d\theta$.

Then it is easy to see that the unique minimizer v_0 of $\mathcal{G}$ is given in cylindrical coordinates by $v_0 = w_0(r,z)d\theta$, where w_0 minimizes the functional (of the type I_0)

$$\mathcal{G}^{red}(w) := \frac{1}{2}\int_{\tilde\Omega} \tilde\rho\left(|\nabla w| + \frac{(w-\varphi)^2}{r}\right)dr\,dz \tag{3.14}$$

in the space of functions $w : \tilde\Omega \to \mathbb{R}$ such that $\int_{\tilde\Omega} \frac{\tilde\rho}{r} w^2\,dr\,dz < \infty$.

One can use duality to rewrite the problem of minimizing $\mathcal{G}^{red}$ as a constrained variational problem. For instance, one can verify that v_0 minimizes $\mathcal{G}^{red}$ if and only if it minimizes the functional

$$w \mapsto \int_{\tilde\Omega} \frac{\tilde\rho}{r} w^2\; dr\,dz \tag{3.15}$$

subject to the constraint

$$\int_{\tilde\Omega} \frac{\tilde\rho}{r}(\varphi - w)\zeta\; dr\; dz \le \frac{1}{2}\int_{\tilde\Omega} \tilde\rho|\nabla\zeta| \qquad \text{for all } \zeta \in C^\infty(\tilde\Omega). \tag{3.16}$$

For the velocity field represented by the 1-form $v = w(r,z)d\theta$, the associated vorticity 2-form is $dv = \partial_r w\, dr \wedge d\theta + \partial_z w\, dz \wedge d\theta$. The vorticity vector field, that is, the vector field dual to dv, is then $\frac{1}{r}(\partial_r w\, \hat{e}_z - \partial_z w\, \hat{e}_r)$, where $\hat{e}_z$ and $\hat{e}_r$ denote unit vectors in the (upward) vertical and (outward) radial directions respectively. It is natural to interpret integral curves of this vector field as "vortex curves". Since the vorticity vector field has no $\hat{e}_\theta$ component and is always tangent to the level surfaces of w, we conclude that vortex curves have the form "$\theta = \text{constant}, w = \text{constant}$", at least for regular values of w. Thus in the reduced 2d model, we interpret level sets of a minimizer w_0, or more precisely sets of the form $\partial\{(r,z) : w_0(r,z) > t\}$, as representing vortex curves. For similar reasons, one should think to the "vorticity measure " as being given by $\nabla^\perp w_0$, rather than ∇w_0.

3.3. **Contact curves and vortex curves.** It is interesting to ask whether one can define a useful analog of the "contact set" (as normally defined for classical obstacle problems) for the variational problems with nonlocal constraints formulated in Theorem 3.2. We address this question first for Bose-Einstein condensates in the presence of rotational symmetry, as discussed above. Thus, we assume that $w_0 : \tilde{\Omega} \to \mathbb{R}$ minimizes the functional (3.15) subject to the constraint (3.16). Starting from (3.16), an approximation argument shows that if E is a set of locally finite perimeter in $\tilde{\Omega}$, then

$$\int \frac{\tilde{\rho}}{r}(\varphi - w_0)\chi_E \; dr \; dz \le \frac{1}{2}\int \tilde{\rho}|\nabla \chi_E|, \tag{3.17}$$

where χ_E denotes the characteristic function of E. We say that ∂E is a *contact curve* if equality holds in (3.17) (where ∂E should be understood as the 1-dimensional set that carries $|\nabla \chi_E|$).

Lemma 3.3. *For a.e. t, $\partial\{w_0 > t\}$ is a contact curve.*

It is natural to interpret $\partial\{w_0 > t\}$ as a "vortex curve", so the Lemma states, heuristically, that every vortex curve for w_0 is also a contact curve.

Proof. By using rotational symmetry to rewrite (3.12) in the (r,z) variables, or by using the fact that $0 = \frac{d}{dt}\mathcal{G}^{red}(e^t w_0)\big|_{t=0}$, we find that

$$\frac{1}{2}\int \tilde{\rho}|\nabla w_0| + \int \frac{\tilde{\rho}}{r}(w_0 - \varphi)w_0 \; dr \; dz = 0.$$

Using the coarea formula, we then get

$$\int_{-\infty}^{\infty} \left(\frac{1}{2}\int \tilde{\rho}|\nabla \chi_{\{w_0>t\}}| \; + \int \frac{\tilde{\rho}}{r}(w_0 - \varphi)\chi_{\{w_0>t\}} \, dr\, dz \right) dt = 0. \tag{3.18}$$

It follows from (3.17) that

$$\frac{1}{2}\int \tilde{\rho}|\nabla \chi_{\{w_0>t\}}| \; + \int \frac{\tilde{\rho}}{r}(w_0 - \varphi)\chi_{\{w_0>t\}} \, dr\, dz \ge 0$$

for every t, and then (3.18) implies that the equality holds for a.e. t. □

It is probably not true that every contact curve for the minimizer w_0 is also a vortex curve, in the generality that we consider here, due to the possibility of degenerate (nonlocal) obstacles, as in the classical obstacle problem. One might hope, however, that the vortex curves and contact curves coincide under reasonable physical assumptions (for example $\Phi = r^2 d\theta$, corresponding to a rotation of a condensate around the z axis).

In the work [2] we have investigated (also numerically) further properties of the contact set: in particular we prove that vortex curves are smooth, of finite length, and meet orthogonally the boundary of $\tilde{\Omega}$ (see [19], and compare also with [13] in the superconductivity case). Moreover, the level set corresponding to $\sup w_0$ is necessarily flat, hence the union of vortex curves forms a proper subset of $\tilde{\Omega}$. In the peculiar case of $\tilde{\Omega}$ being an ellipsoid with suitable eccentricity, following [3] one can also prove that the level set corresponding to $\inf w_0$ is also flat, whence one deduces the existence of a vortex-free zone around the rotation axis.

The situation is more complicated for Bose-Einstein condensates in a general domain $\Omega \subset \mathbb{R}^3$ without rotational symmetry, since in this case the analogs of vortex curves and contact curves may not in fact be curves and do not in general admit a very easy concrete characterization. Abstractly, they may be described as follows: if we write $\mathcal{Z}$ to denote the closure (in the sense of distributions) of

$$\{d\alpha : \alpha \in L^2(\Lambda^1\Omega), \int_\Omega \rho|d\alpha| \leq 1\},$$

then one can think of the set $\mathrm{extr}\mathcal{Z}$ of extreme points of (the convex set) $\mathcal{Z}$ as analogous to the objects — distributional boundaries of sets of finite weighted perimeter — used above to describe vortex and contact curves. Indeed, by the arguments in Remark 3 of [22] and general convexity considerations, one can show that $\mathrm{extr}\mathcal{Z}$ is a nonempty Borel subset of a suitable metric space, and for any T in the vector space generated by $\mathcal{Z}$ (that is, the space $\cup_{\lambda>0}\lambda\mathcal{Z}$), there is a measure μ_T on $\mathrm{extr}\mathcal{Z}$ such that

$$T = \int_{\mathrm{extr}\mathcal{Z}} \omega \; d\mu_T(\omega) \tag{3.19}$$

and

$$\int_\Omega \rho \, d|T| \;=\; \int_{\mathrm{extr}\mathcal{Z}} \left(\int_\Omega \rho \, d|\omega| \right) \; d\mu_T(\omega). \tag{3.20}$$

We remark that in the closely related situation of divergence-free vector fields on $\mathbb{R}^n$, a concrete characterization of elements of the analog of $\mathrm{extr}\mathcal{Z}$ as "elementary solenoids" is established in [22].

With this notation, an analog of Lemma 3.3 is

Lemma 3.4. *Let β_0 be the minimizer of the constrained variational problem* (3.9), (3.10)*), so that $v_0 = P_\rho\Phi + \frac{d^*\beta_0}{\rho}$ is the minimizer of $\mathcal{G}(\cdot)$. Then*

$$\int_\Omega (\beta_\Phi - \beta_0) \cdot d\omega \;\leq\; \frac{1}{2} \int_\Omega \rho d|\omega|. \tag{3.21}$$

for every $\omega \in \mathcal{Z}$. We say that $\omega \in \mathrm{extr}\mathcal{Z}$ is a "generalized contact curve" if the above condition holds with equality.

Furthermore, let μ_{dv_0} denote a measure on extr$\mathcal{Z}$ satisfying (3.19), (3.20) *(with T replaced by dv_0). Then μ_{dv_0} a.e. ω is a generalized contact curve.*

The proof is exactly as in Lemma 3.3, except for the fact that (3.19), (3.20) are used instead of the coarea formula. Then (3.21) follows immediately from the fact that β_0 satisfies (3.10), and the last assertion is a consequence of (3.12).

It would presumably be possible to adapt the results of [22] to the closely related situations considered here, in order to obtain concrete descriptions of $\mathrm{extr}\mathcal{Z}$, although we are not sure that this would add much insight. It would also be interesting to know whether, if we consider the model case of uniform rotation about the z axis (for Bose-Einstein) or a constant applied magnetic field (for Ginzburg-Landau), the complexities sketched above do not in fact occur, and the vortex curves and contact curves for minimizers can in fact be identified with curves of finite length; this seems likely to be the case.

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