ON THE REGULARITY OF TIMELIKE EXTREMAL SURFACES

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ABSTRACT. We study a class of timelike weakly extremal surfaces in flat Minkowski space $\mathbb{R}^{1+n}$, characterized by the fact that they admit a C^1 parametrization (in general not an immersion) of a specific form. We prove that if the distinguished parametrization is of class C^k, then the surface is regularly immersed away from a closed singular set of euclidean Hausdorff dimension at most $1 + 1/k$, and that this bound is sharp. We also show that, generically with respect to a natural topology, the singular set of a timelike weakly extremal cylinder in $\mathbb{R}^{1+n}$ is 1-dimensional if $n = 2$, and it is empty if $n \geq 4$. For $n = 3$, timelike weakly extremal surfaces exhibit an intermediate behavior.

1. INTRODUCTION

In this paper we study timelike extremal surfaces in $(1+n)$-dimensional flat Minkowski space. In particular, we focus on extremal immersions of a cylinder $\mathbb{R} \times \mathbb{S}^1$ into $\mathbb{R}^{1+n}$, which arise in models of closed cosmic strings and have been extensively studied in the physics community (see [7, 19, 2] and references therein), as well as in the more recent mathematical literature [17, 15, 16, 14, 13, 4, 11, 18], and have recently been proved [5, 11] to describe the dynamics of topological defects in various relativistic field theories in certain scaling limits.

As for many geometric problems, timelike extremal surfaces present various kinds of singularities. For instance, it has been shown in [4] that a closed convex string in $\mathbb{R}^2$ with zero initial velocity shrinks to a point in finite time, while its shape approaches that of a circle. An analogous phenomenon can be found in other geometric evolutions such as the planar curvature flow [10] and the hyperbolic curvature flow of convex curves [12]. However more complicated singularities can occur during the evolution (typically the formation of cusps), and a partial classification has been provided in [8], where the authors study self-similar singularity formation. A theory of generalized extremal surfaces in the varifolds sense (see [1]) has been recently proposed in [5, 6].

In [18] it has been shown that any (immersed) timelike extremal cylinder in $\mathbb{R}^{1+2}$ necessarily develops singularities in finite time. In the same paper, the authors conjecture that this does not hold in $\mathbb{R}^{1+n}$ for $n \geq 3$, where existence of smooth timelike extremal cylinders is expected. On the other hand, there exist globally smooth timelike extremal surfaces with noncompact slices in $\mathbb{R}^{1+2}$, which are small perturbations of timelike planes [15, 14].

The arguments of [18] rely heavily on a particular representation of extremal immersed cylinders, which we call the *orthogonal gauge*, known for a long time in the physics literature and first proved to be valid, as far as we know, in [4]. This representation also yields global weak solutions in the sense of [5, 6]. The main goal of this paper is to estimate the dimension of the singular set of these weak solutions, which have the good

property that they are images of C^1 maps (in general *not immersions*) of a specific form, see (2.7), (2.8) below. In particular we prove that, if the map is of class C^k, then the dimension of the singular set is bounded above by $1 + \frac{1}{k}$, and the bound is sharp. The upper bound on the dimension turns out to follow immediately from a classical refinement of Sard's Theorem, due to Federer [9], so the construction of examples of extremal surfaces attaining this bound is the harder part of this result.

We also show that the singular set is generically empty when $n > 3$, confirming the conjecture of [18] in such dimensions. More precisely we show that, generically for $n > 3$, given a closed curve Γ immersed in $\mathbb{R}^n$ and a velocity field $v : \Gamma \to \mathbb{R}^n$, with $|v| < 1$ and orthogonal to Γ, there exists a smooth globally immersed timelike extremal surface containing Γ and tangent to $(1, v)$. For $n = 3$, roughly speaking, both globally smooth immersed solutions and solutions that develop singularities occur for large sets of initial data (that is, sets with nonempty interior.)

We start in Section 2 by quickly recalling some properties of the orthogonal gauge, including existence and (restricted) uniqueness of solutions of a Cauchy problem for timelike extremal surfaces. We also present some examples in Section 4 showing that uniqueness may fail without the restrictions imposed in Section 2.

2. Timelike extremal surfaces in the orthogonal gauge

Given an open interval $I \subset \mathbb{R}$, and an immersion $\psi : I \times \mathbb{R} \to \mathbb{R}^{1+n}$, possibly periodic with respect to the second variable, for an open set $U \subset I \times \mathbb{R}$ we define the Minkoswkian area of $\psi(U)$ to be

$$\int_U \sqrt{|g|}\,, \qquad g := \det(g_{ij}), \qquad g_{ij} := (\partial_i \psi, \partial_j \psi)_m$$

where $(\cdot, \cdot)_m$ denotes the Minkowski inner product. This functional is also sometimes called the Nambu-Goto action. The surface parametrized by ψ is said to be *timelike* if $g < 0$ everywhere, and a timelike surface is *extremal* if ψ is a critical point of the Minkowskian area functional with respect to compactly supported variations.

It is noted in [18] that any timelike immersion of a surface into $\mathbb{R}^{1+n}$ can be reparametrized locally to have the form

$$\psi(t, x) = (t, \gamma(t, x)). \tag{2.1}$$

Here we will consider the initial value problem for timelike extremal surfaces with initial data of the form

$$\gamma(0, x) = \gamma_0, \quad \gamma_t(0, x) = v_0, \tag{2.2}$$

where $\gamma_0 \in C^1(\mathbb{R}; \mathbb{R}^n)$ is an immersion and $v_0 \in C^0(\mathbb{R}; \mathbb{R}^n)$ satisfies $v_0 \cdot \gamma_0' = 0$ and $|v_0| < 1$ everywhere. We call such a pair an *admissible couple*, and we say that an admissible couple is *periodic* if γ_0 and v_0 are periodic with the same period, which implies in particular that γ_0 parametrizes a closed curve. We remark that if γ_0 is an embedding, or more generally if $v_0 \circ \gamma_0^{-1}$ is single-valued on $\mathrm{Image}(\gamma_0)$, then the initial condition (2.2) can be restated in the form

γ_0 parametrizes $\{x \in \mathbb{R}^n : (0, x) \in M\}$, and $(1, v_0(x)) \in T_{\psi(0,x)}M$ for every $x \in \mathbb{R}$.
$$\tag{2.3}$$

Two admissible couples (γ_0, v_0), $(\hat{\gamma}_0, \hat{v}_0)$ are considered to be equivalent if there is a C^1 diffeomorphism $\lambda : \mathbb{R} \to \mathbb{R}$ such that $(\gamma_0, v_0) = (\hat{\gamma}_0, \hat{v}_0) \circ \lambda$. Equivalent couples encode exactly the same geometric data, and to any timelike surface M, whose $t = 0$ slice is an immersed curve, one can assign an (equivalence class of) admissible couples, indeed possibly multiple equivalence classes if the curve is not embedded.

Our approach is based on the observation, classical in the physics literature and straightforward to verify (see [19, 4]), that if $\gamma \in C^k(I \times \mathbb{R}; \mathbb{R}^n)$, $k \geq 1$ satisfies

$$|\gamma_x|^2 - |\gamma_t|^2 = 1 \tag{2.4}$$
$$\gamma_x \cdot \gamma_t = 0 \tag{2.5}$$
$$\gamma_{tt} - \gamma_{xx} = 0 \tag{2.6}$$

for all $(t, x) \in I \times \mathbb{R}$, then $\psi(x, t) = (t, \gamma(t, x))$ is a solution of the Euler-Lagrange equations associated to the Minkowski area functional wherever $g \neq 0$, and hence is an extremal immersion near such points. This holds in the distributional sense if $k = 1$ and classically if $k \geq 2$. In view of (2.4)-(2.5), we will call such a parametrization the *orthogonal gauge*.

The general solution γ of (2.4) - (2.6) has the form

$$\gamma(t, x) = \frac{a(x + t) + b(x - t)}{2} \tag{2.7}$$

where $a, b \in C^1(\mathbb{R}; \mathbb{R}^n)$ are maps satisfying

$$|a'| = |b'| = 1 \quad \text{in } \mathbb{R}. \tag{2.8}$$

Indeed, (2.7) is just d'Alembert's formula, and once γ is known to have the form (2.7), then the constraints (2.4), (2.5) are easily seen to be equivalent to (2.8).

Given a function $\gamma(t, x) = \frac{1}{2}(a(x + t) + b(x - t))$, with a, b satisfying (2.8), we shall write in the sequel $\psi(t, x) := (t, \gamma(t, x))$ and $M := \text{Image}(\psi)$. We also define the singular set of M as

$$Sing := \{\psi(t, x) : \text{rank}(\nabla \psi)(t, x) < 2\} = \{\psi(t, x) : \gamma_x(t, x) = 0\}.$$

We have that M is timelike and regularly immersed in an open neighborhood of every point of $M \setminus Sing$, while, at every point of $Sing$, the orthogonal coordinate system degenerates and, as we will prove in Theorem 3.1 below, M fails to be timelike. A stricter notion of singular set is

$$Sing^* := \{p \in Sing : \lim_{q \in M, q \to p} \tau(q) \text{ does not exist}\},$$

where $\tau(\cdot)$ is the (spatial) tangent

$$\tau(p) = \frac{\gamma_x}{|\gamma_x|} \circ \psi^{-1}(p)$$

defined wherever it makes sense, which is at points $p \in M \setminus Sing$ where the set $\{\frac{\gamma_x}{|\gamma_x|}(t, x) : \psi(t, x) = p\}$ consists of exactly one element.

We note that the definitions of $Sing$ and $Sing^*$ both have the drawback that they depend on the parametrization of M.

We collect some known results in the following

Proposition 2.1. *Given an admissible couple* $(\hat{\gamma}_0, \hat{v}_0) \in C^k \times C^{k-1}$, *there exists an equivalent admissible couple* (γ_0, v_0) *and a map* $\gamma \in C^k(\mathbb{R} \times \mathbb{R}; \mathbb{R}^n)$ *of the form* (2.7), (2.8), *such that the initial condition* (2.2) *holds. In addition,*

1. *$\psi(t, x) = (t, \gamma(t, x))$ is timelike and an immersion in a neighborhood of every point where $\gamma_x \neq 0$, and it is neither timelike nor an immersion at points where γ_x vanishes.*

2. *ψ is an extremal immersion wherever it is a immersion, and in particular this holds for (t, x) in a neighborhood of $\{0\} \times \mathbb{R}$.*

3. *If $\hat{\psi}$ is any extremal immersion of the form $\hat{\psi}(t, x) = (t, \hat{\gamma}(t, x))$ for $(t, x) \in I \times \mathbb{R}$ for some interval $I \subset \mathbb{R}$ containing 0, and if $(\hat{\gamma}(0, \cdot), \hat{\gamma}_t(0, \cdot))$ is equivalent to (γ_0, v_0), then ψ is a reparametrization of $\hat{\psi}$, and thus $\psi(I \times \mathbb{R}) = \hat{\psi}(I \times \mathbb{R})$.*

4. *$M = Image(\psi)$ can be identified with a global weak solution of the extremal surface equation, in the sense of* [5, 6].

This proposition implies in particular the local existence of a smooth timelike extremal surface M satisfying the initial condition (2.3) for an admissible couple (γ_0, v_0) such that $v_0 \circ \gamma_0^{-1}$ is single-valued, as well as the global existence of a weak solution.

We show in Proposition 4.1 below that the restriction of the uniqueness assertion **3** to the class of surfaces parametrized by maps to the form (2.1) is in fact necessary; without this condition, uniqueness can fail.

Proof. Given any admissible couple $(\hat{\gamma}_0, \hat{v}_0)$, we can always find an equivalent couple (γ_0, v_0) such that

$$|\gamma_0'|^2 + |v_0|^2 = 1. \tag{2.9}$$

Letting γ denote the solution of the wave equation (2.6) with initial data (2.2), it is easy to check (see for example (2.12) and (2.13) below) that γ satisfies (2.7), (2.8), thus proving the existence of an extremal immersion for the admissible couple (γ_0, v_0).

The proof of **1** is given in the proof of Theorem 3.1 below. The only subtle part is checking that M is not timelike at $\psi(t, x)$, if $\gamma_x(t, x) = 0$; everything else follows easily from the definitions and (2.7).

Concerning **2**, we have already noted that a straightforward computation shows that ψ is an extremal immersion wherever it is an immersion, and it follows from **1** that ψ is an immersion in a neighborhood of $\{0\} \times \mathbb{R}^n$.

Finally, conclusions **3** and **4** are established in [4] and [5, 6] respectively. They are proved for γ which is periodic in the x variable, but both facts are essentially local (due to finite propagation speed) and so the proofs work without change in the general case. $\square$

Remark 2.2. In [15, Theorem 4.1], global existence of C^2 solutions is proved for an equation that, like (2-4)-(2-6), is equivalent to the equation for timelike extremal surfaces as long as the surfaces associated to the solutions remain immersed. In this result, the orthogonal gauge is not imposed, and the equations considered are thus nonlinear.

We record some standard formulas. Differentiating (2.7) we obtain

$$\gamma_x(t, x) = \frac{a'(x + t) + b'(x - t)}{2} \tag{2.10}$$

$$\gamma_t(t, x) = \frac{a'(x + t) - b'(x - t)}{2}. \tag{2.11}$$

Letting $t = 0$ in (2.10) - (2.11) and recalling (2.2), we deduce that

$$a'(x) = \gamma_0'(x) + v_0(x) \tag{2.12}$$
$$b'(x) = \gamma_0'(x) - v_0(x) \tag{2.13}$$

We define a *cylinder* to be a set $M \subset I \times \mathbb{R}^n$ that can be written *globally* as the image of a map ψ of the form (2.1), where $\gamma(t, \cdot)$ is periodic with fixed period E for every $t \in I$.

It is straightforward to check that if one starts with a representative $(\hat{\gamma}_0, \hat{v}_0)$ of a periodic admissible couple such that (2.9) does not hold, with $(\hat{\gamma}_0, \hat{v}_0)$ periodic of period L, then an equivalent couple (γ_0, v_0) that satisfies (2.9) is periodic with period

$$E_0 := \int_0^L \frac{|\hat{\gamma}_0'(x)|}{\sqrt{1 - |\hat{v}_0(x))|^2}} \, dx.$$

Then $a + b$ is periodic, and we see from (2.12), (2.13) that a', b' are periodic as well, all with period E_0. Hence, if $(\hat{\gamma}_0, \hat{v}_0)$ is a periodic admissible couple, then the surface associated to $(\hat{\gamma}_0, \hat{v}_0)$ by Proposition 2.1 is a cylinder.

Notice that, given a solution γ, the corresponding couple (a, b) is uniquely determined up to additive constants. In particular, the othogonal gauge provides a one-to-one correspondence between the set of all equivalence classes of admissible couples and the set

$$X := \left\{ (a, b) \in C^1(\mathbb{R}; \mathbb{R}^n) \times C^1(\mathbb{R}; \mathbb{R}^n) : a' + b' \text{ never vanishes, } |a'| = |b'| = 1 \right\} / \sim$$

where $(a, b) \sim (c, d)$ iff there exist $x_0 \in \mathbb{R}$, $z_0 \in \mathbb{R}^n$ and $\sigma_0 \in \{\pm 1\}$ such that

$$c(x) = a(\sigma_0 x + x_0) + z_0 \qquad d(x) = b(\sigma_0 x + x_0) - z_0 \qquad \text{for all } x \in \mathbb{R}.$$

Similarly, equivalence classes of periodic admissible couples are parametrized by

$$X_{\mathrm{per}} = \left\{ [(a, b)] \in X : a', b', a + b \text{ periodic with the same period} \right\}$$

where $[\cdot]$ denotes an equivalence class. When $(a, b) \in X_{\mathrm{per}}$, we shall denote by E_0 the common period of a, b.

We shall consider the topology induced by $C^1(\mathbb{R}; \mathbb{R}^n) \times C^1(\mathbb{R}; \mathbb{R}^n)$ on X (or equivalently on the set of admissible couples) and we refer to it as the X-topology. We say that a property holds *generically* if it holds for all admissible couples out of a closed set with empty interior with respect to this topology.

3. GENERIC REGULARITY

In this section we study the regularity properties of extremal surfaces, which hold generically with respect to the X-topology. We start with a general regularity result which follows directly from the orthogonal gauge parametrization.

Theorem 3.1. *Given an admissible couple (γ_0, v_0), there exists a global timelike extremal surface M of the form* (2.1), *containing $\Gamma_0 = \mathrm{Image}(\gamma_0)$ and tangent to $(1, v_0)$, if and only if*

$$a'(s) \neq -b'(\sigma) \qquad \text{for all } s, \sigma \in \mathbb{R}. \tag{3.1}$$

If $(a, b) \in X_{\mathrm{per}}$ then M is an extremal cylinder.

We have only defined *timelike* for immersed surfaces. A surface M given as the image of a map ψ may be smooth even where ψ is not an immersion. In this case, we say that M is timelike at a point $p \in M$ if $T_p M$ exists and is timelike, and in addition the spatial unit tangent τ is continuous at p.

Proof. Assume (3.1). Then it is clear from the form (2.7) of γ that γ_x never vanishes, and from the form (2.1) of ψ, it follows that that $Sing = \emptyset$ and hence that ψ is a global immersion. It follows from (2.4) that $|\gamma_t| < 1$ whenever $\gamma_x \neq 0$, and from this it is easy to check that ψ is a timelike immersion everywhere.

If (3.1) fails, then $\gamma_x(t,x) = 0$ for some $(t,x) \in \mathbb{R} \times \mathbb{R}$, and by (2.4) we have $|\gamma_t(t,x)| = 1$. We will show that M is not timelike at $\psi(t,x)$. This is clearly the case if $p \in Sing^*$, so we assume that $p \notin Sing^*$. Then we can define a spatial tangent $\tau(p)$, and $T_p M$ is spanned by $(0, \tau(p))$ and $(1, \gamma_t(t,x))$. Thus it suffices to show that

$$\tau(p) \cdot \gamma_t(t,x) = 0, \tag{3.2}$$

since then it is easy to check that $T_p M$ contains no timelike vectors.

To prove (3.2), fix a sequence (t_k, x_k) in $M \setminus Sing$ such that $p_k := \gamma(t_k, x_k) \to \gamma(t,x)$. (We prove in Theorem 5.1 below that $\mathcal{H}^2(Sing) = 0$, so such a sequence exists.) Then since τ is continuous at p,

$$\tau(p) := \lim_k \frac{\gamma_x(t_k, x_k)}{|\gamma_x(t_k, x_k)|}, \qquad \gamma_t = \lim_k \gamma_t(t_k, x_k). \tag{3.3}$$

We write $\gamma(t,x) = \frac{1}{2}(a(x+t) + b(x-t))$ as usual, and we use the notation

$$m_k := a'(x_k + t_k), \qquad n_k := -b'(x_k - t_k).$$

If we define $n_0 = a'(x+t)$ then, using the (3.3) and the fact that $\gamma_x(t,x) = 0$, we find that

$$m_k \text{ and } n_k \to n_0, \qquad \text{as } k \to \infty, \qquad \text{and } n_0 = \gamma_t(t,x). \tag{3.4}$$

Then $\gamma_x(t_k, x_k) = m_k - n_k$ and $n_0 = \gamma_t(t,x)$, so (3.2) reduces to showing that if (3.4) holds and $|n_k| = |m_k| = 1$ for all k, then

$$|(m_k - n_k) \cdot n_0| = o(|n_k - m_k|) \qquad \text{as } k \to \infty.$$

Writing $\theta_k := \cos^{-1}(m_k \cdot n_0)$ and $\varphi_k := \cos^{-1}(n_k \cdot n_0)$, it is not hard to see that $|n_k - m_k| \geq |\sin \theta_k - \sin \varphi_k| \geq \frac{1}{2}|\theta_k - \varphi_k|$ for k sufficiently large, and then it suffices to check that

$$|\cos \theta_k - \cos \varphi_k| = o(|\theta_k - \varphi_k|)$$

for $\theta_k, \varphi_k \to 0$, which is clear.

$\square$

Notice that condition (3.1) is equivalent to say that the two curves $a', -b' : \mathbb{R} \to \mathbb{S}^{n-1}$ do not intersect.

The following result has been proved in [18].

Corollary 3.2. *Let $n = 2$ and let (γ_0, v_0) be a periodic admissible couple. Then the curve $\Gamma_0 = \text{Image}(\gamma_0)$ cannot be immersed in a global timelike extremal cylinder tangent to $(1, v_0)$.*

Remark 3.3. We emphasize that the corollary applies only to extremal *cylinders*. The proof does not rule out the possibility of smooth timelike extremal surfaces in $\mathbb{R}^{1+2}$ that are locally (but not globally) cylindrical, see Proposition 4.1 below.

Proof. By Theorem 3.1 it is enough to show that there exist $s, \sigma \in [0, E_0]$ such that

$$a'(s) + b'(\sigma) = 0.$$

As $|a'| = |b'| = 1$ and

$$\int_0^{E_0} a'(s)\, ds = \int_0^{E_0} -b'(\sigma)\, d\sigma\,, \tag{3.5}$$

the supports of the curves a' and $-b'$ are two connected arcs of $\mathbb{S}^1$, which necessarily intersect. The thesis then follows from Theorem 3.1. □

Corollary 3.4. *Let $n = 3$ and let (γ_0, v_0) be a periodic admissible couple. If $\Gamma_0 = Image(\gamma_0)$ can be immersed in a global timelike extremal cylinder tangent to $(1, v_0)$, then the same holds for any periodic admissible couple $(\hat{\gamma}_0, \hat{v}_0)$, sufficiently close to (γ_0, v_0) in the X-topology.*

Conversely, if Γ_0 cannot be immersed in a global timelike extremal cylinder tangent to $(1, v_0)$, then generically the same holds for any couple $(\hat{\gamma}_0, \hat{v}_0)$ sufficiently close to (γ_0, v_0).

Proof. The first assertion follows immediately from the fact that the set of couples (a, b) satisfying (3.1) is open in X_{per}.

The second assertion follows by noticing that the set of couples (a, b) such that the support of a' intersects at least two connected components of the complement in $\mathbb{S}^2$ of the support of $-b'$ is an open set in X_{per}, while the set of couples (a, b) such that the support of a' intersects only one connected component of the complementary of the support of $-b'$ is a closed set with empty interior. □

Remark 3.5. If we consider data (γ_0, v_0) parametrized by $X^2 := X_{\mathrm{per}} \cap \left(C^2(\mathbb{R}) \times C^2(\mathbb{R}) \right)$, endowed with the stronger topology induced by $C^2(\mathbb{R}) \times C^2(\mathbb{R})$, then Γ_0 can generically be immersed in a global E_0-periodic surface tangent to $(1, v_0)$, which is a timelike extremal surface away from a discrete set of singular points, parametrized by the finite set $Sing$. Moreover, the cardinality of the singular set $Sing$ is invariant for small perturbations of (γ_0, v_0) in the X^2-topology. Indeed, we observe that the couples (a, b) such that the curves a' and $-b'$ have a finite number of transversal intersections is a dense open set in X_{per} with respect to the X^2-topology, and the number of intersections is locally constant. Hence the curve γ given by (2.7) parametrizes a E_0-periodic timelike extremal cylinder tangent to $(1, v_0)$, away from a singular set which is finite in $[0, E_0] \times \mathbb{R}^3$, and the number of singularities is invariant for small perturbations of (γ_0, v_0).

An example of admissible couple in $\mathbb{R}^3$ which is immersed in a global timelike extremal cylinder has been given in [18]. More generally, we prove in Lemma 3.6 below that any curve in $\mathbb{S}^2$ whose convex hull contains a neighborhood of the origin can be realized as the set of tangent vectors of a closed curve a such that $|a'| = 1$. Hence one can easily find pairs $a, b : \mathbb{R} \to \mathbb{R}^n, n \geq 3$ of periodic curves with the same period, such that a' and $-b'$ trace out disjoint curves in $\mathbb{S}^{n-1}$. By Theorem 3.1, each such pair yields an example of a globally smooth timelike extremal cylinder.

Lemma 3.6. *Assume that $c : \mathbb{S}^1 \to \mathbb{S}^{n-1}$ is a smooth closed curve such that 0 belongs to $co(Image(c))$, where $co(\cdot)$ denotes the convex hull. Then there exists a closed curve $a : \mathbb{S}^1 \to \mathbb{R}^n$, of the same smoothness as c, such that $Image(a') = Image(c)$.*

Proof. We write c as a 2π-periodic function from $\mathbb{R}$ to $\mathbb{S}^{n-1}$. By assumption there exist points $0 < x_0 < \ldots < x_n \leq 2\pi$ such that

$$0 \in \text{int} \left(\text{co}\{c(x_0), \ldots, c(x_n)\} \right) \tag{3.6}$$

Let $p : \mathbb{R} \to \mathbb{R}$ be a smooth increasing function such that $p(x + 2\pi) = p(x) + 2\pi$,

$$p(x_i) = x_i, \qquad \text{and} \qquad \frac{d^k p}{dx^k}(x_i) = 0 \text{ for every } k \in N \text{ and } i = 0, \ldots, n.$$

Then, given positive numbers $\ell_0, \ldots, \ell_n$, let $L_i := \sum_{j=0}^{i} \ell_j$ and define

$$\tilde{c}(x) := \begin{cases} c(p(x)) & \text{for } 0 \leq x \leq x_0 \\ c(x_0) & \text{for } x_0 \leq x \leq x_0 + L_0 \\ c(p(x - L_0)) & \text{for } x_0 + L_0 \leq x \leq x_1 + L_0 \\ \vdots & \vdots \\ c(x_n) & \text{for } x_n + L_{n-1} \leq x \leq x_n + L_n \\ c(p(x - L_n)) & \text{for } x_n + L_n \leq x \leq 2\pi + L_n. \end{cases}$$

We claim that one can choose positive (ℓ_i) so that $\int_0^{2\pi + L_n} \tilde{c}(x) \, dx = 0$. Indeed, since

$$\int_0^{2\pi + L_n} \tilde{c}(x) \, dx = 0 \ = \ \int_0^{2\pi} c(p(x)) dx + \sum_{i=0}^{n} \ell_i c(x_i),$$

the claim follows from (3.6).

We now fix (ℓ_i) as above and define $\hat{c}(x) := \tilde{c}\left(\frac{(2\pi + L_n)x}{2\pi} \right)$. Then $a(x) := \int_0^x \hat{c}(y) dy$, for $0 \leq x \leq 2\pi$, defines a closed curve with the the the required properties. $\square$

Corollary 3.7. *Let $n > 3$ and let (γ_0, v_0) be a periodic admissible couple. Then Γ_0 can be generically immersed in a global timelike extremal cylinder tangent to $(1, v_0)$.*

Proof. The assertion follows as before from the fact that the set of couples (a, b) satisfying (3.1) is open in X_{per}, while the set of couples (a, b) such that the curves $a', -b' : [0, E_0] \to \mathbb{S}^2$ intersect is a closed set with empty interior. $\square$

Remark 3.8. A related question is what happens if we assume γ_0 to be an embedded curve in $\mathbb{R}^n$ and ask if it is contained in a global *embedded* timelike extremal surface in $\mathbb{R}^{1+n}$. It is easy to check that Corollary 3.2 still holds in this case, and we expect that Corollaries 3.4 and 3.7 also hold, with similar proofs.

Remark 3.9. As the set of periodic admissible couples which can be immersed in a global timelike extremal cylinder is parametrized by an open subset $\mathcal{O} \subset X_{\text{per}}$, it is natural to speculate on the number of connected components of $\mathcal{O}$. While it is clear from Corollary 3.2 that $\mathcal{O} = \emptyset$ if $n = 2$, it is not difficult to show that $\mathcal{O}$ has infinitely many connected components if $n = 3, 4$, while $\mathcal{O}$ is connected if $n > 4$.

Indeed, if $n = 3$ and $a', -b'$ are two disjoint closed curves in $\mathbb{S}^2$, then the winding number of a' around the image of $-b'$ is constant on connected components of $\mathcal{O}$, and one can easily find admissible couples with any prescribed winding number. If $n = 4$ the linking number in $\mathbb{S}^3$ of the curves $a', -b'$ is constant on connected components of $\mathcal{O}$, and one can find admissible couples with any prescribed linking number. If $n > 4$ the assertion follows from the fact the every knot is trivial in $\mathbb{S}^n$.

4. Nonuniqueness of smooth extremal surfaces

In the following statement, we say that a surface $M \subset \mathbb{R}^{1+n}$ is *locally cylindrical* if, for every $t_0 \in \mathbb{R}$, there exists an open interval $I \subset \mathbb{R}$ such that $t_0 \in I$ and $M \cap (I \times \mathbb{R}^2)$ is a cylinder in $I \times \mathbb{R}^n$, i.e. it can be written in the form (2.1).

Proposition 4.1. *If $n \geq 3$ there exist two distinct globally C^∞ timelike extremal surfaces M^1, M^2 in $\mathbb{R}^{1+n}$, both locally cylindrical, such that M^1 and M^2 coincide when $t \in [0, \delta]$ for some $\delta > 0$, in the sense that*

$$\left\{ (t, x) \in M^1 : t \in [0, \delta] \right\} = \left\{ (t, x) \in M^2 :\in [0, \delta] \right\}. \tag{4.1}$$

The surface M^2 that we construct below has the property that it is locally cylindrical but not globally cylindrical.

In general, in geometric evolution problems, self-intersections can give rise to nonuniqueness. The proposition shows that, even if we require smoothness and impose the "locally cylindrical" topological constraint, one can still take advantage of self-intersections to generate examples of nonuniqueness.

Proof. For $i = 1, \ldots, 3$ let a_i, b_i be distinct C^∞ maps $\mathbb{R} \to \mathbb{R}^n$, periodic with period 1, such that $|a_i'| = |b_i'| = 1$ and such that

$$a_i(x) = b_j(x) = (x, 0, \ldots, 0) \qquad \text{for all } i, j \text{ and all } x \in [-\delta, \delta] \text{ for some } \delta < \frac{1}{2}. \tag{4.2}$$

Assume in addition that

$$\{ (s, \sigma) \in \mathbb{R} \times \mathbb{R} : a_i'(s) + b_j'(\sigma) = 0 \text{ for some } i, j \} = \emptyset. \tag{4.3}$$

This says that no b_j' ever passes through any point that is antipodal to any point on any a_i'. It follows easily from Lemma 3.6 that this can be accomplished.

Now for any permutation $\pi : \{1, 2, 3\} \to \{1, 2, 3\}$, let (a^π, b^π) and be periodic curves $\mathbb{R} \to \mathbb{R}^n$ with period 3, defined by

$$(a^\pi, b^\pi)(x) = (a_{\pi(i)}(x), b_{\pi(i)}(x)) \qquad \text{for } x \in [i-1, i] \mod 3.$$

Next, define $\gamma^\pi(t, x) := \frac{1}{2}(a^\pi(x+t) + b^\pi(x-t))$,

Letting *id* denote the identity permutation, we claim that for every π,

$$\gamma^\pi(t, \cdot) \text{ and } \gamma^{id}(t, \cdot) \text{ parametrize the same curve for } 0 \leq t \leq \delta. \tag{4.4}$$

Indeed, if $t \geq 0$, we have

$$\gamma^\pi(t, x) = \begin{cases} \frac{1}{2}(a_{\pi(i)}(x+t) + b_{\pi(i)}(x-t)) & \text{if } i-1 \leq x-t \leq x+t \leq i \mod 3 \\ \frac{1}{2}(a_{\pi(i+1)}(x+t) + b_{\pi(i)}(x-t)) & \text{if } i-1 \leq x-t \leq i \leq x+t \mod 3, \end{cases} \tag{4.5}$$

where addition of indices is understood mod 3. If $0 \leq t \leq \delta$, it follows from this and (4.2) that

$$\gamma^\pi(t, x) = (x - i, 0 \ldots, 0) \quad \text{if } i - 1 \leq x - t \leq i \leq x + t,$$

for *every* permutation π. Then one can see by inspection of (4.5) that (4.4) holds (in fact it also holds for $t \in [-\delta, 0]$, by essentially the same argument). Next, note that (4.5) implies that

$$\gamma^\pi(\tfrac{1}{2}, x) = \tfrac{1}{2}(a_{\pi(i+1)}(x + \tfrac{1}{2}) + b_{\pi(i)}(x - \tfrac{1}{2})) \quad \text{if } i - \tfrac{1}{2} \leq x \leq i + \tfrac{1}{2} \mod 3$$

and from this one can see that in general $\gamma^{id}(\tfrac{1}{2}, \cdot) \neq \gamma^\pi(\tfrac{1}{2}, \cdot)$ if for example π is an odd permutation.

Finally, define $\psi^\pi(t, x) = (t, \gamma^\pi(t, x))$. Let M^1 be the surface parametrized by ψ^{id}, and let M^2 be the surface that agrees with M^1 when $t \leq \delta$, and for $t \geq \delta/2$ is parametrized by $\psi^\pi(t, x)$ for some odd permutation π. This definition makes sense in view of (4.4). These surfaces have all the stated properties. In particular, it follows from (4.3) and Theorem 4.1 that M^1, M^2 are both smoothly immersed and locally cylindrical.

(Indeed, note that since γ^π and γ^{id} as constructed above are both periodic with period 3 in the t variable, we are free to switch back and forth at will between γ^π and γ^{id} every 3 units of t.) □

Remark 4.2. When $n = 2$, a similar argument yields two functions γ^{id}, γ^π of the form (2.7), (2.8) that parametrize the same curve for $|t| \leq \delta$, but not for all t. These functions γ^{id}, γ^π fail to be global timelike immersions, see Theorem 5.1 below, but it is presumably possible to arrange that the breakdown of uniqueness (for the image manifolds) occurs before the breakdown of regularity.

Remark 4.3. In the proof of Proposition 4.1, if $a_1 = a_2 = a_3$ and b_1, b_2, b_3 are distinct, then one can see from (4.5) that $\gamma^{id}(t, \cdot)$ and $\gamma^\pi(t, \cdot)$ parametrize the same curve for every t. This shows that the different admissible pairs can generate the same extremal surface.

Remark 4.4. We now provide an example of nonuniqueness of (weakly) extremal surfaces, due to the appearance of singularities in the evolution. Let $\gamma_1(t, x) = \tfrac{1}{2}(a_1(x + t) + a_1(x - t))$ and $\gamma_2(t, x) = \tfrac{1}{2}(a_2(x + t) + a_2(x - t))$ be orthogonal parametrization of two different global extremal cylinders M_1 and M_2, with a_1, a_2 arclength parametrizations of the boundaries of two distinct uniformly convex, centrally symmetric planar sets, both periodic with period E_0. Symmetry implies that $a_i(x + E_0/2) = -a_i(x)$ for all x and $i = 1, 2$, and thus

$$\gamma_i(E_0/4, x) = \frac{1}{2}\Big(a_i(x + E_0/4) + a_i(x - E_0/4)\Big) = 0 \qquad \text{for } x \in \mathbb{R}, i \in \{1, 2\}.$$

In other words, γ_1, γ_2 both have an extinction singularity at the origin at time $\bar{t} := E_0/4$.

Note that the time derivatives at time $\bar{t}$ of γ_1 and γ_2 are respectively given by $a_1'(x + \bar{t})$ and $a_2'(x + \bar{t})$.

Define now $\gamma(t, x) = \gamma_1(t, s(x))$ for $t < \bar{t}$ and $\gamma(t, x) = \gamma_2(t, x)$ for $t \geq \bar{t}$, where $s(x)$ is a reparametrization of $[0, E_0]$ such that

$$a_1'(s(x) + \bar{t}) = a_2'(x + \bar{t})$$

for all $x \in [0, E_0]$, i.e.
$$s(x) = -\bar{t} + (a_1')^{-1} \circ a_2'(x + \bar{t}).$$
It follows that the derivatives of γ are continuous at $(x, \bar{t})$ for any $x \in \mathbb{R}$, and hence γ may be suitably extended to a C^1 *nonorthogonal* parametrization of a global (weakly) extremal cylinder M that agrees respectively with M_1 and M_2 on disjoint time intervals.

5. DIMENSION OF THE SINGULAR SET

Given $\gamma(t, x) = (a(x+t) + b(x-t))/2$, with a, b satisfying (2.8), and $\psi(t, x) = (t, \gamma(t, x))$, we now prove some upper bounds on the size of the singular sets $Sing$ and $Sing^*$ associated to the cylinder $M = \text{Image}(\psi)$.

In the following theorem, which is one of the main results of this paper, "dim" always means (Euclidean) Hausdorff dimension[1].

Theorem 5.1. *Assume that $a, b \in C^k(\mathbb{R}, \mathbb{R}^n)$, with $k \in \mathbb{N}$, with $(a, b) \in X$. Then,*

   1. *$\mathcal{H}^{1 + \frac{1}{k}}(Sing) = 0$ and $dim(Sing^*) \leq dim(Sing) \leq 1 + \frac{1}{k}$.*

   2. *It can happen that $dim(Sing^*) = 1 + \frac{1}{k}$.*

   3. *When $n = 2$ and $(a, b) \in X_{\text{per}}$, (at least) one of the following properties holds:*

     - there exists t_0 such that $\gamma_x(t_0, x) = 0$ for all x,

     - $Sing^*$ is at least one-dimensional, and the set

$$\{t \in \mathbb{R} : \exists x \in \mathbb{R} \text{ such that } \psi(t, x) \in Sing^*\}$$

     contains an open interval.

Remark 5.2. In fact it is clear from the proof that conclusion **1** holds for any surface that is given locally as the image of a C^k map, including any surface that can be written locally in the form (2.7) with $a, b \in C^k$. In particular, this applies to noncompact surfaces, as well as local cylinders of the type appearing in Proposition 4.1.

Also, we prove conclusion **2** for global cylinders, the most restrictive (topological) class of functions considered in this paper, so it follows that it holds for other classes of surfaces — noncompact, locally cylindrical — as well.

Note also that Remark 3.3 applies to conclusion **3**.

It is natural to wonder whether the results we prove here for the weak solutions given by the explicit formula (2.7) still hold in a larger class of weak solutions, and also whether any analogous results hold for higher-dimensional extremal surfaces.

Conclusion **3** is a refinement of a result from [18]. Our proof gives more details than [18] concerning the situation described in (5.4), since we found this point not completely straightforward. The proof of **3** shows that, if for instance a is a nonconvex curve in $\mathbb{R}^2$ and $b(x) = -a(x + E_0/2)$, then *both* the alternatives of conclusion **3** hold.

The rest of this section is devoted to the proof of Theorem 5.1.

*Proof of **1***. The estimate $\mathcal{H}^{1 + \frac{1}{k}}(Sing) = 0$ follows directly from a refined version of Sard's Theorem, see Federer [9, 3.4.3]. $\qquad\square$

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[1]It is arguably slightly unnatural to characterize a singular set in Minkowski space by the Euclidean Hausdorff dimension, but note that this quantity is invariant with respect to Lorentz transformations.

Remark 5.3. In [3] one can find a version of Sard's Theorem more refined than the one cited above, which gives a necessary and sufficient condition for a set $A \subset \mathbb{R}$ to be the set of critical values of some function in $C^{k,\alpha}(\mathbb{R}, \mathbb{R}^n)$. If $a, b \in C^{k,\alpha}$ for $\alpha \in (0,1)$ and k is a positive integer, these result implies that

$$\mathcal{H}^{\frac{1}{k+\alpha}}\left(Sing \cap \left(\{s\} \times \mathbb{R}^n\right)\right) = 0 \qquad \text{for every } s \in \mathbb{R}.$$

It is reasonable to conjecture, and may be even easy to prove, that under these hypotheses one has $\mathcal{H}^{1+\frac{1}{k+\alpha}}(Sing) = 0$, but this does not immediately follow from [3]. However, straightforward modifications of the proof of **2** below show that it can happen that $\dim(Sing^*) = 1 + \frac{1}{k+\alpha}$, when $a, b \in C^{k,\alpha}$.

Proof of **2**. It is enough to construct an example when $n = 2$. In order to do it, we will need the following result:

Lemma 5.4. *For every positive integer k, there exists $f \in C^k([0,1]; \mathbb{R})$ such that*

$$dim(f(\Sigma)) = \frac{1}{k}, \qquad \text{where } \Sigma := \left\{x \in [0,1] : f' \text{ changes sign near } x\right\}. \qquad (5.1)$$

The proof, which we defer to the end of this section, is a small modification of a classical argument used by Federer to prove the sharpness of his refined version of Sard's Theorem which we cited in the proof of **1** above.

We may assume that the function f from Lemma 5.4 satisfies $|f'| \leq \frac{1}{2}$, since multiplying a function by a constant does not change the dimension of the associated set Σ.

We define $g \in C^k([0,1])$ such that $g' = (1 - f'^2)^{1/2}$ for $x \in [0,1]$, and we fix two periodic maps $a, b \in C^k(\mathbb{R}; \mathbb{R}^2)$, with the same period $E_0 > 3$, parametrized by arclength and such that

$$a(x) = (f(x), g(x)) \text{ for } x \in [0,1], \qquad b(x) = (0, -x) \text{ for } x \in [-1, 2].$$

Then

$$\gamma_x(t,x) = \frac{1}{2}(f'(x+t), g'(x+t) - 1) \quad \text{if } x + t \in [0,1], \ |t| \leq \frac{1}{2},$$

and since $|g' - 1| \leq C f'^2$, it follows that $\frac{\gamma_x}{|\gamma_x|}$ is discontinuous at all points (t, x) such that $|t| \leq \frac{1}{2}$ and $x + t \in \Sigma$. We then deduce that

$$Sing^* \supset \left\{(t, \frac{f(x+t)}{2}, \frac{g(x+t) - (x-t)}{2}) : \ x + t \in \Sigma, \ |t| \leq \frac{1}{2}\right\}$$

$$= \left\{(0, \frac{f(s)}{2}, \frac{g(s) - s}{2}) + (t, 0, t) : \ s \in \Sigma, \ |t| \leq \frac{1}{2}\right\}.$$

If we let

$$A_0 := \left\{\frac{1}{2}(f(x), g(x) + x) : x \in \Sigma\right\},$$

then $\dim(Sing^*) \geq \dim(A_0) + 1$, since $Sing^* \subset \mathbb{R}^{1+2}$ contains a copy of $A_0 \subset \mathbb{R}^2$ translated along a line segment. Moreover, since $\frac{1}{2}f(\Sigma)$ is the projection of A_0 on the x-axis, we conclude from (5.1) that

$$\dim(Sing^*) \geq \ 1 + \dim(A_0) \geq \ 1 + \dim\left(\frac{1}{2}f(\Sigma)\right) = 1 + \frac{1}{k}.$$

We remark that, although the map γ constructed above is singular for $t = 0$, one can easily modify the construction to arrange that γ is regularly immersed at $t = 0$ and develops singularities as described above at a later time. Indeed, γ is a regular immersion at $t = 0$ if $a' = e^{i\alpha}, b' = -e^{i\beta}$, and $\alpha < \beta < \alpha + 2\pi$, and this condition can be achieved, while essentially preserving the above construction, by choosing E_0 large enough, taking a certain amount of care in how α is defined in $[1, E_0]$ and β in $[2, E_0 - 1]$, and then replacing $a(\cdot)$ by $a(\cdot - E_0/2)$. $\qquad\square$

Proof of **3**. Extend a', b' to E_0-periodic maps from $\mathbb{R}$ to $\mathbb{S}^1$, and let $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ be two continuous functions such that

$$a'(x) = e^{i\alpha(x)}, \qquad -b' = e^{i\beta(x)}. \tag{5.2}$$

As in the proof of Corollary 3.2, a' and b' satisfy (3.5), which implies that the images of a' and $-b'$ are closed arcs with intersection of positive length. In particular, by adding $2\pi k$ to α for an appropriate integer k, we can assume that the set $\text{Image}(\alpha) \cap \text{Image}(\beta)$ contains an interval of positive length. It then follows that the function

$$F(t, x) := \alpha(x + t) - \beta(x - t)$$

takes both positive and negative values. For example, to find a point where $F > 0$, choose $s, \sigma \in \mathbb{R}$ such that $\alpha(s) > \beta(\sigma)$, and let $(t, x) = (\frac{1}{2}(s - \sigma), \frac{1}{2}(s + \sigma))$.

We shall consider 2 cases:

Case 1. For every t_0, the function $x \mapsto F(t_0, x)$ does not change sign.

Then, since F assumes both positive and negative values, there must be some t_0 such that $F(t_0, x) = 0$ for all x. It follows that $\gamma_x(t_0, x) = 0$ for all x.

Case 2. There exists some t_0 such that $x \mapsto F(t_0, x)$ changes sign.

Then by continuity $x \mapsto F(t, x)$ changes sign for all t in a neighborhood of t_0.

Fix such a $\bar{t}$, and let S be a connected component of the set $\{x : F(\bar{t}, x) = 0\}$ such that F assumes both positive and negative values in every neighborhood of S. As observed in [18], it follows from (5.2) that the unit tangent $\tau = \frac{\gamma_x}{|\gamma_x|}$ is given by

$$\frac{\gamma_x}{|\gamma_x|}(\bar{t}, x) = \text{sign}\left(\sin\left(\frac{1}{2}F(\bar{t}, x)\right)\right) i\, e^{\frac{i}{2} G(\bar{t},x)}, \qquad G(\bar{t}, x) := \alpha(x + \bar{t}) + \beta(x - \bar{t}) \tag{5.3}$$

wherever $\gamma_x \neq 0$ (for simplicity of notation we identify $\mathbb{R}^2$ with $\mathbb{C}$). Therefore, if S consists of a single point $(\bar{t}, x)$, then $\lim_{y \to x} \frac{\gamma_x}{|\gamma_x|}(\bar{t}, y)$ does not exist, which implies that $\psi(\bar{t}, x) \in Sing^*$.

Suppose now that S is an interval, say $S = [s_0, s_1]$. Then $\gamma_x(\bar{t}, x) = 0$ for all $x \in S$, so that $x \mapsto \gamma(\bar{t}, x)$ is constant for $x \in S$. It follows that $\psi(\bar{t}, s_0) \in Sing^*$ unless

$$\tau(\bar{t}, s_0^-) = \tau(\bar{t}, s_1^+), \tag{5.4}$$

where $\tau(\bar{t}, s_0^-) := \lim_{x \nearrow s_0} \tau(\bar{t}, x)$ and $\tau(\bar{t}, s_1^+) := \lim_{x \searrow s_1} \tau(\bar{t}, x)$. (Condition (5.4) includes the assertion that both limits exist.) Recalling that F changes sign near S, we deduce from (5.3) that (5.4) can only occur if $\text{sign}(F(\bar{t}, s_0^-)) = -\text{sign}(F(\bar{t}, s_1^+))$ and

$$\frac{1}{2}(G(\bar{t}, s_1) - G(\bar{t}, s_0)) = \pi \mod 2\pi. \tag{5.5}$$

Assume this holds and let

$$x_0 := \min\left\{x \in S : |\alpha(x + \bar{t}) - \alpha(s_0 + \bar{t})| = \frac{\pi}{3}\right\} > s_0.$$

Since $0 = F(\bar{t}, x) = \alpha(x + \bar{t}) - \beta(x - \bar{t})$ for all $x \in S$, we get

$$\begin{aligned}
F(\bar{t} + \varepsilon, x_0 - |\varepsilon|) &= \alpha(x_0 + \bar{t} + \varepsilon - |\varepsilon|) - \beta(x_0 - \bar{t} - \varepsilon - |\varepsilon|) \\
&= \alpha(x_0 + \bar{t} + \varepsilon - |\varepsilon|) - \alpha(x_0 + \bar{t} - \varepsilon - |\varepsilon|) \neq 0
\end{aligned}$$

for all ε such that $x_0 - 2|\varepsilon| > s_0$. The last inequality follows from the fact that $|\alpha(x_0 + \bar{t}) - \alpha(s_0 + \bar{t})| = \pi/3$, while $|\alpha(x_0 + \bar{t} - 2|\varepsilon|) - \alpha(s_0 + \bar{t})| < \pi/3$, by the choice of x_0.

From the equality $F(\bar{t}, x) = 0$, for these ε it also follows that

$$F(\bar{t} - \varepsilon, x_0 - |\varepsilon|) = -F(\bar{t} + \varepsilon, x_0 - |\varepsilon|).$$

In particular, the function $\varepsilon \mapsto F(\bar{t} + \varepsilon, x_0 - |\varepsilon|)$ changes sign at $\varepsilon = 0$.

Fix now $y_0 < s_0$ such that $F(\bar{t}, y_0) \neq 0$ and

$$|\alpha(y + \bar{t}) - \alpha(s_0 + \bar{t})| < \frac{\pi}{3} \quad \text{for all } y \in [y_0, s_0].$$

For all ε sufficiently small, $F(t + \varepsilon, y_0 + |\varepsilon|)$ has the same sign as $F(\bar{t}, y_0)$. Thus, for all ε in an interval of the form $(-\delta, 0)$ or $(0, \delta)$, the function $x \mapsto F(\bar{t} + \varepsilon, x)$ must change sign between $y_0 + |\varepsilon|$ and $x_0 - |\varepsilon|$. For such ε, the interval $(y_0 + |\varepsilon|, x_0 - |\varepsilon|)$ must contain a connected component $\hat{S} = [\hat{s}_0, \hat{s}_1]$ of $\{x : F(\bar{t} + \varepsilon, x) = 0\}$ such that F assumes both positive and negative values in every neighborhood of $\hat{S}$. Moreover, our choice of y_0 and x_0 guarantees that, possibly reducing δ, we have

$$\begin{aligned}
\pi &> |\alpha(\hat{s}_1 + \bar{t} + \varepsilon) - \alpha(s_0 + \bar{t})| + |\alpha(s_0 + \bar{t}) - \alpha(\hat{s}_0 + \bar{t} + \varepsilon)| \\
&> |\alpha(\hat{s}_1 + \bar{t} + \varepsilon) - \alpha(\hat{s}_0 + \bar{t} + \varepsilon)| \\
&= \frac{1}{2}|G(\bar{t} + \varepsilon, \hat{s}_1) - G(\bar{t} + \varepsilon, \hat{s}_0)|
\end{aligned}$$

where the last equality follows from the fact that $F(\bar{t} + \varepsilon, s) = 0$ in $[\hat{s}_0, \hat{s}_1]$. Thus (5.5) cannot hold, and hence for all ε in the interval that we have found, $\psi(\bar{t} + \varepsilon, \hat{s}_0) \in Sing^*$, thus completing the proof of **3**. $\square$

Finally we give a proof of Lemma 5.4.

Proof of Lemma 5.4. We divide the proof into three steps.
Step 1. We first recall Federer's proof that for any $k \in \mathbb{N}$ and $\mu \in (0, \frac{1}{k})$, there exists $g \in C^k([0,1])$ such that

$$\mathcal{H}^\mu\Big(\{g(x) : g'(x) = 0\}\Big) > 0. \tag{5.6}$$

For $\sigma \in (0, 1)$ we will write C_σ to denote the "middle σ" Cantor-type set, so that

$$C_\sigma = \cap_{\ell=1}^\infty \cup_{i \in \{0,1\}^\ell} C_\sigma(i)$$

where, for every ℓ and every $i \in \{0,1\}^\ell$, $C_\sigma(i)$ is a closed interval of length $(\frac{1-\sigma}{2})^\ell$, and $C_\sigma(i_1, \ldots, i_\ell, 0), C_\sigma(i_1, \ldots, i_\ell, 1)$ are obtained by removing from $C_\sigma(i_1, \ldots, i_\ell)$ a centered open interval of length $\sigma(\frac{1-\sigma}{2})^\ell$. As usual we start with $C_\sigma(0) = [0, \frac{1}{2}(1 - \sigma)]$ and $C_\sigma(1) = [\frac{1}{2}(1 + \sigma)2, 1]$, and we label the intervals so that $C_\sigma(i_1, \ldots, i_\ell, 0)$ lies to the left of $C_\sigma(i_1, \ldots, i_\ell, 1)$.

Fix now ν and $\delta > 0$ such that $(k + \delta)\mu = \nu < 1$, and let $\alpha, \beta \in (0, 1)$ satisfy

$$\left(\frac{1-\alpha}{2}\right)^\mu = \left(\frac{1-\beta}{2}\right)^\nu = \frac{1}{2}.$$

These numbers are chosen so that C_α and C_β have dimension μ and ν respectively, and $\mathcal{H}^\mu(C_\alpha), \mathcal{H}^\nu(C_\beta) > 0$.

Notice that there is a natural map $g_0 : C_\beta \to C_\alpha$, characterized by

$$g_0(C_\beta \cap C_\beta(i)) = C_\alpha \cap C_\alpha(i) \qquad \text{for every } \ell \in \mathbb{N} \text{ and } i \in \{0, 1\}^\ell.$$

As Federer noted in [9, 3.4.4], g_0 extends to a C^k map $g : [0, 1] \to [0, 1]$ by a routine application of the Whitney extension Theorem. The point is that, given $x, y \in C_\beta$, we can fix $\ell \in \mathbb{N}$ and $i \in \{0, 1\}^\ell$ such that $x, y \in C_\beta(i)$, but x and y belong to different subintervals of $C_\beta(i)$. Then $g_0(x)$ and $g_0(y)$ both belong to $C_\alpha(i)$, and from this information one can easily check that

$$|x - y| \geq \beta(\frac{1-\beta}{2})^\ell, \qquad |g_0(x) - g_0(y)| \leq (\frac{1-\alpha}{2})^\ell = (\frac{1-\beta}{2})^{(k+\delta)\ell}.$$

As a result we get

$$|g_0(x) - g_0(y)| \leq C(\frac{1-\beta}{2})^{\delta\ell}|x - y|^k = o(|x - y|^k),$$

hence Whitney's Theorem yields a C^k extension of g as required.

It is also clear that $g' = 0$ in C_β, so that every point of C_α is a critical value of g, and (5.6) holds.

Step 2. We modify the above construction to produce a function $f \in C^k([0, 1])$ such that, for a fixed $\mu < \frac{1}{k}$, we have

$$\mathcal{H}^\mu(f(\Sigma)) \geq 0 \qquad \text{where } \Sigma := \{x \in [0, 1] : f' \text{ changes sign near } x\}. \qquad (5.7)$$

To do this, we fix α, β as above and define $f_0 : C_\beta \to C_\alpha$, characterized by

$$f_0(C_\alpha \cap C_\alpha(i)) = C_\beta \cap C_\beta(i^*) \qquad \text{for every } \ell \in \mathbb{N} \text{ and } i \in \{0, 1\}^\ell,$$

where for every k and every $i \in \{0, 1\}^k$, we define i^* by

$$i_j^* = i_j \quad \text{if } j \text{ is odd}, \qquad i_j^* = i_j + 1 \mod 2 \quad \text{if } j \text{ is even}.$$

Then, as in the classical argument described above, f_0 extends to a C^k function $f : [0, 1] \to \mathbb{R}$. In addition, we have the inclusion $C_\beta \subset \Sigma$, since every interval $C_\alpha(i)$ for i odd contains points such that $x < y$ and $f(x) < f(y)$, whereas for i even $C_\alpha(i)$ contains points such that $x < y$ and $f(x) > f(y)$.

Step 3. For every $m > k$, let f_m be a function satisfying (5.7) with $\mu = \frac{1}{k} - \frac{1}{m}$, and extend f_m so that $f'_m = 0$ on $\mathbb{R} \setminus [0, 1]$. We define

$$f(x) := \sum_{k=1}^{\infty} h_m f_m \left(2^{m+1}(x - 2^{-m})\right)$$

for a sequence (h_m) decreasing to zero fast enough so that the series converges in C^k. Then $f \in C^k$ and satisfies (5.1). $\qquad \square$

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