Representation, relaxation and convexity for variational problems in Wiener spaces

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Abstract

We show convexity of solutions to a class of convex variational problems in the Gauss and in the Wiener space. An important tool in the proof is a representation formula for integral functionals in this infinite dimensional setting, that extends analogous results valid in the classical Euclidean framework.

1 Introduction

The aim of this paper is to study the convexity of the minimizers of some variational problems in Wiener spaces. In the Euclidean setting convexity is a widely discussed issue [24]. Recently, following previous work by Korevaar [25] and Alvarez, Lasry and Lions [2], Alter, Caselles and Chambolle [1, 11] showed the convexity of solutions to variational problems involving functionals with linear growth and in particular to the prescribed curvature problem. Using quite different techniques, Figalli and Maggi [18] proved the convexity of small mass minimizers of this problem.

The main goal of this paper, is to extend these results to the (finite dimensional) Gauss space and to the (infinite dimensional) Wiener space. In this setting, very few results are currently available. To the best of our knowledge, the only result in this direction is contained in [12], where the authors proved the convexity of the solutions of the isoperimetric problem in convex domains. More explicitly they prove the following:

Theorem 1.1. [12] *Let C be a convex set of positive (Gaussian) measure and of finite (Gaussian) perimeter, then there exists $\alpha > 0$ such that for every $v \in [\alpha, \gamma(C)]$, the solution of the constrained isoperimetric problem*

$$\min \{P_\gamma(E) \ : \ E \subseteq C \text{ and } \gamma(E) = v\}$$

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has a unique solution which is convex.

We are interested in the convexity of solutions of the problem

$$\min_{\gamma(E)=v} P_\gamma(E) - \int_E g(x)\,d\gamma(x), \tag{1}$$

where g is a convex function.
The idea is to follow the approach of Caselles and Chambolle [11] in the Euclidean case. We will thus be naturally led to consider the variational problem

$$\min_{BV_\gamma \cap L^2_\gamma(X)} \int_X |D_\gamma u|_H + \frac{1}{2}\int_X |u-g|^2 d\gamma \tag{2}$$

for which we will show convexity of the minimizers. More generally, we will prove that minimizers of

$$\min_{L^2_\gamma(X)} \int_X F(D_\gamma u)\,d\gamma + \frac{1}{2}\int_X |u-g|^2 d\gamma \tag{3}$$

are convex if F and g are convex (see Theorems 4.1 and 5.1).
Extending the variational methods from Euclidean to Wiener spaces is now a quite active field. In particular extending the theory of functions of bounded variation to this setting started with the work of Fukushima [19] and Fukushima and Hino [20]. Since then the properties of BV functions and sets of finite perimeter have been investigated by Ambrosio and his collaborators, see [5] in particular but also [6] and [4]. We also refer to the paper [22] where relaxation of the perimeter, isoperimetry and symmetrization are investigated with application to a kind of Modica-Mortola result. We must point out that this theory of BV functions is strongly linked with older works of Ledoux and Malliavin [26], [27].

The plan of the paper is the following. In Section 2 we recall some notation about the Wiener space and functions of bounded variation. In Section 3 we prove a useful representation formula for integral functionals on Wiener spaces. In Section 4 we show the convexity of the minima of (2) in finite dimension, and in Section 5 we investigate the convexity of the minimizers in the infinite dimensional Wiener space.

2 Notation and preliminary results

A clear and comprehensive reference on the Wiener space is the book by Bogachev [7] (see also [27]). We follow here closely the notation of [5]. Let X be a separable Banach space and let X^* be its dual. We say that X is a Wiener space if it is endowed with a non-degenerate centered Gaussian probability measure γ. That amounts to say that γ

is a probability measure for which $x^*_\sharp\gamma$ is a centered Gaussian measure on $\mathbb{R}$ for every $x^* \in X^*$. The non-degeneracy hypothesis means that γ is not concentrated on any proper subspace of X.

As a consequence of Fernique's Theorem [7, Theorem 2.8.5], for every $x^* \in X^*$, the function $R^*x^*(x) = \langle x^*, x\rangle$ is in $L^2_\gamma(X) = L^2(X, \gamma)$. Let $\mathcal{H}$ be the closure of R^*X^* in $L^2_\gamma(X)$; the space $\mathcal{H}$ is usually called the reproducing kernel Hilbert space of γ. Let R, the operator from $\mathcal{H}$ to X, be the adjoint of R^* that is, for $\hat{h} \in \mathcal{H}$,

$$R\hat{h} = \int_X x\hat{h}(x)\, d\gamma$$

where the integral is to be intended in the Bochner sense. It can be shown that R is a compact and injective operator [7]. We will let $Q = RR^*$ so that for every x^*, $y^* \in X^*$,

$$\langle Qx^*, y^*\rangle = \int_X \langle x^*, x\rangle\langle y^*, x\rangle\ d\gamma.$$

We denote by H the space $R\mathcal{H} \subset X$. This space is called the Cameron-Martin space. It is a separable Hilbert space with the scalar product given by

$$[h_1, h_2]_H = \langle \hat{h}_1, \hat{h}_2\rangle_{L^2_\gamma(X)}$$

if $h_i = R\hat{h}_i$. We will denote by $|\cdot|_H$ the norm in H. The space H is a dense subspace of X, with compact embedding, and $\gamma(H) = 0$ if X is of infinite dimension.

For $x^*_1, .., x^*_m \in X^*$ we denote by $\Pi_{x^*_1,..,x^*_m}$ the projection from X to $\mathbb{R}^m$ given by

$$\Pi_{x^*_1,..,x^*_m}(x) = (\langle x^*_1, x\rangle, .., \langle x^*_m, x\rangle).$$

We will also denote it by Π_m when specifying the points x^*_i is unnecessary. Two elements x^*_1 and x^*_2 of X^* will be called orthonormal if the corresponding $h_i = Qx^*_i$ are orthonormal in H (or equivalently if x^*_1 and x^*_2 are orthonormal in $L^2_\gamma(X)$). We will fix in the following an orthonormal basis of H given by $h_i = Qx^*_i$.

We also denote by $H_m = \mathrm{span}(h_1, .., h_m)$ and $X^\perp_m = \mathrm{Ker}(\Pi_m) = \overline{H^\perp_m}^X$, so that $X \cong \mathbb{R}^m \oplus X^\perp_m$. The map Π_m induces the decomposition $\gamma = \gamma_m \otimes \gamma^\perp_m$, with γ_m, $\gamma^\perp_m$ Gaussian measures on $\mathbb{R}^m$, $X^\perp_m$ respectively.

Proposition 2.1. [7] *Let $\hat{h}_1, .., \hat{h}_m$ be in $\mathcal{H}$ then the image measure of γ under the map*

$$\Pi_{\hat{h}_1,..,\hat{h}_m}(x) = (\hat{h}_1(x), .., \hat{h}_m(x))$$

is a Gaussian in $\mathbb{R}^m$. If the $\hat{h}_i$ are orthonormal, then such measure is the standard Gaussian measure on $\mathbb{R}^m$.

Given $u \in L^2_\gamma(X)$, we will consider the canonical cylindrical approximation $\mathbb{E}_m$ given by

$$\mathbb{E}_m u(x) = \int_{X_m^\perp} u(\Pi_m(x), y)\, d\gamma_m^\perp(y).$$

Notice that $\mathbb{E}_m u$ depends only on the first m variables (we call such function a cylindrical function) and $\mathbb{E}_m u$ converges to u in $L^2_\gamma(X)$.
We will denote by $\mathcal{F}\mathcal{C}^1_b(X)$ the space of all cylindrical $\mathcal{C}^1$ bounded functions that is the functions of the form $v(\Pi_m(x))$ with v a $\mathcal{C}^1$ bounded function from $\mathbb{R}^n$ to $\mathbb{R}$. We denote by $\mathcal{F}\mathcal{C}^1_b(X,H)$ the space generated by all functions of the form Φh, with $\Phi \in \mathcal{F}\mathcal{C}^1_b(X)$ and $h \in H$.
We now give the definitions of gradients, Sobolev spaces functions of bounded variation. Given $u : X \to \mathbb{R}$ and $h = R\hat{h} \in H$, we define

$$\frac{\partial u}{\partial h}(x) = \lim_{t \to 0} \frac{u(x+th) - u(x)}{t}$$

whenever the limit exists, and

$$\partial_h^* u = \frac{\partial u}{\partial h} - \hat{h}u.$$

We define $\nabla_H u : X \to H$, the gradient of u by

$$\nabla_H u = \sum_{i=1}^{+\infty} \frac{\partial u}{\partial h_i}\, h_i$$

and the divergence of $\Phi : X \to H$ by

$$\operatorname{div}_\gamma \Phi = \sum_{i=1}^{+\infty} \partial_{h_i}^* [\Phi, h_i]_H.$$

The operator $\operatorname{div}_\gamma$ is the adjoint of the gradient so that for every $u \in \mathcal{F}\mathcal{C}^1_b(X)$ and every $\Phi \in \mathcal{F}\mathcal{C}^1_b(X,H)$, the following integration by parts holds:

$$\int_X u \operatorname{div}_\gamma \Phi \, d\gamma = -\int_X [\nabla_H u, \Phi]_H d\gamma. \tag{4}$$

The ∇_H operator is closable in $L^2_\gamma(X)$ and we will denote by $H^1_\gamma(X)$ its closure in $L^2_\gamma(X)$. Formula (4) still holds for $u \in H^1_\gamma(X)$ and $\Phi \in \mathcal{F}\mathcal{C}^1_b(X,H)$. Analogously, we define the Sobolev spaves $W^{1,p}_\gamma(X)$ for $p \geq 1$ (these spaces are denoted by $\bar{D}^{1,p}(X,\gamma)$ in [5]).
Following [19, 5], given $u \in L^1_\gamma(X)$ we say that $u \in BV_\gamma(X)$ if

$$\int_X |D_\gamma u|_H = \sup\left\{ \int_X u \operatorname{div}_\gamma \Phi \, d\gamma;\ \Phi \in \mathcal{F}\mathcal{C}^1_b(X,H),\ |\Phi|_H \leq 1\ \forall x \in X \right\} < +\infty.$$

We will also denote by $|D_\gamma u|(X)$ the total variation of u. If $u = \chi_E$ is the characteristic function of a set E we will denote $P_\gamma(E)$ its total variation and say that E is of finite perimeter if $P_\gamma(E)$ is finite.
As shown in [5] we have the following properties of $BV_\gamma(X)$ functions.

Theorem 2.2. *Let $u \in BV_\gamma(X)$ then the following properties hold:*

- *$D_\gamma u$ is a countably additive measure on X with finite total variation and values in H such that for every $\Phi \in \mathcal{FC}_b^1(X)$ we have:*

$$\int_X u\,\partial^*_{h_j}\Phi\,d\gamma = -\int_X \Phi d\mu_j \qquad \forall j \in \mathbb{N}$$

where $\mu_j = [h_j, D_\gamma u]_H$.

- $|D_\gamma u|(X) = \inf \underline{\lim}\{\int_X |\nabla_H u_i|_H d\gamma \ :\ u_j \in W^{1,1}_\gamma(X),\ u_j \to u \text{ in } L^1_\gamma(X)\}$.

We next introduce the the Ornstein-Uhlenbeck semigroup. Let $u \in L^1_\gamma(X)$ then

$$T_t u(x) := \int_X u\left(e^{-t}x + \sqrt{1-e^{-2t}}y\right)\,d\gamma(y).$$

Proposition 2.3. *The Ornstein-Uhlenbeck semigroup satisfies:*

- *if $u \in L^1_\gamma(X)$ then $T_t u \in W^{1,1}_\gamma(X)$,*
- *if $u \in L^p_\gamma(X)$ then $T_t u$ converges in $L^p_\gamma(X)$ to u when t goes to zero,*
- *for every $\Phi \in \mathcal{FC}_b^1(X,H)$, and $u \in L^2_\gamma(X)$,*

$$\int_X T_t u\,\mathrm{div}_\gamma\,\Phi\,d\gamma = e^{-t}\int_X u\,\mathrm{div}_\gamma\,T_t\Phi\,d\gamma, \tag{5}$$

- *if $\Phi \in \mathcal{FC}_b^1(X,H)$ then $T_t\Phi \in \mathcal{FC}_b^1(X,H)$*
- *for every convex function $F : H \to \mathbb{R}\cup\{+\infty\}$, and every Φ,*

$$\int_X F(T_t\Phi)d\gamma \le \int_X F(\Phi)\,d\gamma. \tag{6}$$

The proof of this proposition can be found in [5]. The only additional property here is (6) which follows from Jensen's inequality and the rotation invariance of the measure γ.

Remark 2.4. Notice that (5) holds more generally for u in the Orlicz space $L\log^{\frac{1}{2}} L$ but not for a general u in $L^1_\gamma(X)$ (see [5]).

Proposition 2.5. *Let $u = v(\Pi_m)$ be a cylindrical function then $u \in BV_\gamma(X)$ if and only if $v \in BV_{\gamma_m}(\mathbb{R}^m)$. We then have*

$$\int_X |D_\gamma u|_H = \int_{\mathbb{R}^m} |D_\gamma v|.$$

Proposition 2.6 (Coarea formula [3])**.** *If $u \in BV_\gamma(X)$ then for every Borel set $B \subset X$,*

$$|D_\gamma u|(B) = \int_{\mathbb{R}} P_\gamma(\{u > t\}, B)\, dt. \tag{7}$$

The following result can be found in [7, Corollary 4.4.2].

Proposition 2.7. *Let u be a convex function from X to $\mathbb{R} \cup \{+\infty\}$, let $F(t) = \gamma(\{u \leq t\})$ and $t_0 = \inf\{t \ : \ F(t) > 0\}$, then F is continuous on $\mathbb{R} \backslash \{t_0\}$. As a consequence $\gamma(\{u = t\}) = 0$ for every $t \neq t_0$.*

In the finite dimensional setting, we will keep the same notations as in the infinite dimensional one. Notice that in $\mathbb{R}^m$, the following equality holds:

$$\operatorname{div}_\gamma \Phi = \operatorname{div} \Phi - \langle x, \Phi \rangle.$$

We see that functions in $BV_{\gamma_m}(\mathbb{R}^m)$ are in $BV_{\text{loc}}(\mathbb{R}^m)$ and that $D_{\gamma_m} u = \gamma_m Du$ so that most of the properties of classical BV functions extend to $BV_{\gamma_m}(\mathbb{R}^m)$ (see [3]).

For $F : H \to \mathbb{R}$ a convex function we denote by F^* its convex conjugate defined for $\Phi \in H$ by

$$F^*(\Phi) := \sup_{h \in H} \, [\Phi, h]_H - F(h)$$

and by F^∞ its recession function defined for $h \in H$ as:

$$F^\infty(h) := \lim_{t \to +\infty} \frac{F(th)}{t}.$$

For the main properties of these functions we refer to [29]. The main assumptions we will make are:

(H1) $F : H \to \mathbb{R} \cup \{+\infty\}$ is a proper (i.e. F is not identically plus infinity) convex lower semi-continuous (lsc), bounded from below and attains its minimum.

(H2) F has $p \geq 1$ growth i.e. there exists α_1, β_1, α_2 and β_2 real positive such that

$$\alpha_1 |h|_H^p + \beta_1 \geq F(h) \geq \alpha_2 |h|_H^p - \beta_2 \qquad \forall h \in H.$$

Notice that of course (H2) implies (H1). Notice also that hypothesis (H2) includes the limiting case $p = 1$ which is of particular interest for us. Under hypothesis (H1), it is not restrictive to assume that $F(0) = 0$ and $F \geq 0$.

By Hahn-Banach Theorem, for every proper convex lsc function $F : H \to \mathbb{R} \cup \{+\infty\}$, there exists $q \in H$ such that $F'(h) := F(h) - [q, h]_H$ satisfies (H1).

3 Representation formula and relaxation of integral functionals

We extend in this section a representation formula for integral functionals. We start by proving it for functionals with linear growth.

Proposition 3.1. *Let $F : H \to \mathbb{R}$ be a convex function satisfying*

$$\alpha|h| + \beta \geq F(h) \geq 0 \qquad \forall h \in H$$

For $\mu \in \mathcal{M}(X, H)$, with $\mu = \mu^a\gamma + \mu^s$ its Radon-Nikodym decomposition, let

$$\int_X F(\mu) := \int_X F(\mu^a)d\gamma + \int_X F^\infty\left(\frac{d\mu^s}{d|\mu^s|}\right)d|\mu^s|,$$

then there holds

$$\int_X F(\mu) = \sup_{\Phi \in \mathcal{F}\mathcal{C}_b^1(X,H)} \int_X [\Phi, d\mu]_H - \int_X F^*(\Phi)d\gamma. \tag{8}$$

Proof. For $\mu \in \mathcal{M}(X, H)$, with $\mu = \mu^a\gamma + \mu^s$ its Radon-Nikodym decomposition let $\mathcal{D}_F := \{\Phi = \sum_{i=1}^n \chi_{A_i} h_i \ /n \in \mathbb{N},\ A_i$ disjoint Borel sets, $h_i \in H,\ F^*(h_i) < +\infty\}$. Then we start by proving

$$\int_X F(\mu) = \sup_{\Phi \in \mathcal{D}_F} \int_X [\Phi, d\mu]_H - \int_X F^*(\Phi)\, d\gamma. \tag{9}$$

The proof is adapted from [14] and is divided into three steps.
Step 1. Let

$$M(\mu) := \sup_{\Phi \in \mathcal{D}_F} \int_X [\Phi, d\mu]_H - \int_X F^*(\Phi)d\gamma.$$

We will show that for every $h \in L^1_\gamma(X)$,

$$M(h\gamma) = \int_X F(h)d\gamma. \tag{10}$$

By definition of convex conjugate, it is readily checked that $M(h\gamma) \leq \int_X F(h)d\gamma$. We thus turn to the other inequality. By definition of the integral, for every $\delta > 0$, there exists $h_i \in H$ and $A_i \subset X$ with A_i disjoints Borel sets and $i \in [1,n]$ such that if we set

$$\theta = \sum_{i=1}^n \chi_{A_i} h_i$$

then $|\theta - h|_{L^1_\gamma} \leq \delta$. As F is of linear growth it is Lipschitz continuous and thus we can assume that also

$$|F(h) - F(\theta)|_{L^1_\gamma(X)} \leq \delta.$$

For every i, by definition of convex conjugate, there exists $\xi_i \in H$ such that

$$F(h_i) \leq [\xi_i, h_i]_H - F^*(\xi_i) + \delta.$$

Notice that since F is of linear growth, the ξ_i are uniformly bounded. From this, setting $\Phi = \sum_{i=1}^n \chi_{A_i}\xi_i$ we have

$$\begin{aligned}
\int_X F(h)d\gamma &\leq \int_X F(\theta)d\gamma + \delta \\
&= \sum_{i=1}^n \int_{A_i} F(h_i)d\gamma + \delta \\
&\leq \sum_{i=1}^n \int_{A_i} [\xi_i, h_i]_H - F^*(\xi_i)d\gamma + 2\delta \\
&= \int_X [\Phi, \theta]_H - F^*(\Phi)d\gamma + 2\delta \\
&\leq \int_X [\Phi, h]_H - F^*(\Phi)d\gamma + C\delta \\
&\leq M(h) + C\delta.
\end{aligned}$$

Since δ is arbitrary we have $M(h\gamma) = \int_X F(h)d\gamma$.

Step 2. By reproducing the proof with F^∞ instead of F, $\frac{d\mu^s}{d|\mu^s|}$ instead of h and $|\mu^s|$ instead of γ we find, using that $\mathcal{D}_{F^\infty} = \mathcal{D}_F$ (since dom $F^* =$ dom $(F^\infty)^*$ by [29, Thm. 13.3]) and $(F^\infty)^* = 0$ in its domain,

$$M_\infty(\mu^s) := \sup_{\Phi \in \mathcal{D}_F} \int_X [\Phi, d\mu^s] = \int_X F^\infty\left(\frac{d\mu^s}{d|\mu^s|}\right) d|\mu^s|.$$

Step 3. It remains to show that

$$M(\mu^a\gamma + \mu^s) = M(\mu^a\gamma) + M_\infty(\mu^s).$$

One inequality is easily obtained, since

$$\begin{aligned} M(\mu^a\gamma + \mu^s) &= \sup_{\Phi\in\mathcal{D}_F} \int_X [\Phi, \mu^a]_H d\gamma + \int_X [\Phi, d\mu^s] - \int_X F^*(\Phi) d\gamma \\ &\leq \left(\sup_{\Phi\in\mathcal{D}_F} \int_X [\Phi, \mu^a]_H - F^*(\Phi) d\gamma \right) + \left(\sup_{\Phi\in\mathcal{D}_F} \int_X [\Phi, d\mu^s] \right) \\ &= M(\mu^a\gamma) + M_\infty(\mu^s). \end{aligned}$$

For the opposite inequality, let $\delta > 0$ be fixed then there exists Φ_1 and Φ_2 such that

$$\begin{aligned} M(\mu^a\gamma) &\leq \int_X [\Phi_1, \mu^a]_H - F^*(\Phi_1) d\gamma + \delta \\ M_\infty(\mu^s) &\leq \int_X [\Phi_2, d\mu^s]_H + \delta. \end{aligned}$$

Taking Φ equal to Φ_2 on a sufficiently small neighborhood of the support of μ^s and equal to Φ_1 outside this neighborhood, we get

$$\begin{aligned} M(\mu^a\gamma) + M_\infty(\mu^s) &\leq \int_X [\Phi, \mu^a]_H - F^*(\Phi) d\gamma + \int_X [\Phi, d\mu^s]_H + C\delta \\ &\leq M(\mu^a\gamma + \mu^s) + C\delta \end{aligned}$$

which gives the opposite inequality and shows (9).

For $\Phi \in \mathcal{D}_F$, the image of Φ, being a finite number of vectors of H, is included in a finite dimensional vector space V of H. If we now consider K the convex hull of these vectors then K is a convex polytope of V. We can then write $\Phi = \sum_{i=1}^N \theta_i \tilde{h}_i$ with $\tilde{h}_i$ the extremal points of K and $\theta_i \in L^1_\gamma(X) \cap L^1_\mu(X)$ with $\theta_i \geq 0$ and $\sum_{i=1}^N \theta_i \leq 1$. Arguing as in [5, Section 2.1], $\gamma + |\mu|$ being tight we can approximate θ_i in $L^1_\gamma(X) \cap L^1_\mu(X)$ with $\theta_i^k \in \mathcal{FC}_b^1(X)$ in such a way that $\theta_i^k \geq 0$ and $\sum_{i=1}^N \theta_i^k \leq 1$. As F^* is bounded and continuous on K, letting $\Phi^k := \sum_{i=1}^N \theta_i^k \tilde{h}_i$ we have $\Phi^k \in \mathcal{D}_F$ and

$$\lim_{k\to+\infty} \int_X [\Phi^k, d\mu] - \int_X F^*(\Phi^k) d\gamma = \int_X [\Phi, d\mu] - \int_X F^*(\Phi) d\gamma.$$

□

We then deduce the following corollary:

Theorem 3.2. *For $F : H \to \mathbb{R} \cup \{+\infty\}$ a proper lsc convex function and $\mu \in \mathcal{M}(X, H)$, with $\mu = \mu^a \gamma + \mu^s$, then again*

$$\int_X F(\mu) = \sup_{\Phi \in \mathcal{FC}_b^1(X,H)} \int_X [\Phi, d\mu]_H - \int_X F^*(\Phi) d\gamma.$$

Proof. Case 1. First assume that (H1) holds. For $n \in \mathbb{N}$ let

$$F_n(p) := \sup_{|\Phi|_H \le n} [\Phi, p]_H - F^*(\Phi).$$

Then F_n is of linear growth and F_n is a nondecreasing sequence converging pointwise to F and thus by the monotone convergence theorem,

$$\int_X F(\mu^a)\, d\gamma = \lim_{n \to \infty} \int_X F_n(\mu^a)\, d\gamma.$$

Analogously, $(F_n)^\infty$ converges monotonically to F^∞. Indeed, since F_n is nondecreasing, $(F_n)^\infty$ is clearly nondecreasing and

$$(F_n)^\infty(p) = \lim_{t \to +\infty} \frac{F_n(tp)}{t} \le \lim_{t \to +\infty} \frac{F(tp)}{t} = F^\infty(p).$$

On the other hand, for every $\Phi \in \text{dom } F^* = \text{dom } (F^\infty)^*$, if $n \ge |\Phi|_H$, for every $p \in H$ and $t > 0$,

$$\frac{F_n(tp)}{t} \ge [\Phi, p]_H - \frac{F^*(\Phi)}{t}$$

and thus letting t goes to infinity and then n goes to infinity as well, we find

$$\lim_{n \to \infty} (F_n)^\infty(p) \ge \sup_{\Phi \in \text{dom } F^*} [\Phi, p]_H = F^\infty(p).$$

We thus have

$$\int_X F\left(\frac{d\mu^s}{d|\mu^s|}\right) d|\mu^s| = \lim_{n \to \infty} \int_X F_n\left(\frac{d\mu^s}{d|\mu^s|}\right) d|\mu^s|.$$

By Proposition 3.1, for every $n \in \mathbb{N}$,

$$\int_X F_n(\mu^a) d\gamma + \int_X F_n\left(\frac{d\mu^s}{d|\mu^s|}\right) d|\mu^s| = \sup_{\substack{\Phi \in \mathcal{FC}_b^1(X,H) \\ |\Phi|_\infty \le n}} \int_X [\Phi, d\mu] - \int_X F^*(\Phi)\, d\gamma. \tag{11}$$

Passing to the limit when n tends to infinity we get

$$\int_X F(\mu) = \sup_{\Phi \in \mathcal{FC}_b^1(X,H)} \int_X [\Phi, d\mu] - \int_X F^*(\Phi)\, d\gamma.$$

Case 2. Let now F be a generic proper lsc convex function and $q \in H$ be such that $F'(h) := F(h) - [q, h]$ satisfies (H1). It is readily seen that $(F')^\infty(h) = F^\infty(h) - [q, h]_H$ and $(F')^*(\Phi) = F^*(\Phi + q)$. Since (9) holds for F',

$$
\begin{aligned}
\int_X F(\mu) - \int_X [q, d\mu]_H &= \int_X F'(\mu) \\
&= \sup_{\Phi \in \mathcal{F}\mathcal{C}_b^1(X,H)} \int_X [\Phi, d\mu]_H - \int_X F^*(\Phi + q) d\gamma \\
&= \sup_{\Phi \in \mathcal{F}\mathcal{C}_b^1(X,H)} \int_X [\Phi - q, d\mu]_H - \int_X F^*(\Phi) d\gamma \\
&= \sup_{\Phi \in \mathcal{F}\mathcal{C}_b^1(X,H)} \left\{ \int_X [\Phi, d\mu]_H - \int_X F^*(\Phi) d\gamma \right\} - \int_X [q, d\mu]_H .
\end{aligned}
$$

□

Remark 3.3. An important example of functionals covered by the Theorem is given by the functionals with $p \geq 1$ growth.

For F a proper lsc convex function, we can define the functional on $L^2_\gamma(X)$

$$
\int_X F(D_\gamma u) := \sup_{\Phi \in \mathcal{F}\mathcal{C}_b^1(X,H)} \int_X -u \operatorname{div}_\gamma \Phi - F^*(\Phi) \; d\gamma. \tag{12}
$$

The functional defined in this way is thus lsc in $L^2_\gamma(X)$. By (8), we have

$$
\int_X F(D_\gamma u) = \int_X F(\nabla u) d\gamma + \int_X F^\infty \left(\frac{dD^s_\gamma u}{d|D^s_\gamma u|} \right) d|D^s_\gamma u| \tag{13}
$$

for $u \in BV_\gamma(X)$ with $D_\gamma u = \nabla u \gamma + D^s_\gamma u$ its Radon-Nikodym decomposition.

We then have the following relaxation result:

Proposition 3.4. *Let F be a proper lsc convex function then the functional $\int_X F(D_\gamma u)$ is the relaxation of the functional defined as $\int_X F(\nabla_H u) d\gamma$ for $u \in W^{1,1}_\gamma(X)$. If F satisfies also (H2) then is is also the relaxation of the functional $\int_X F(\nabla_H u) d\gamma$ defined on the smaller class $\mathcal{F}\mathcal{C}_b^1(X)$.*

Proof. Case 1. Assume first that F satisfies (H1). We start by proving that

$$
\int_X F(D_\gamma u) = \inf \underline{\lim} \left\{ \int_X F(\nabla_H u_n) \, d\gamma, \; u_n \in W^{1,1}_\gamma(X) \quad u_n \to u \text{ in } L^2_\gamma(X) \right\}. \tag{14}
$$

Thanks to Proposition 3.1, the inequality '$\le$' is obvious. To prove the opposite inequality, we proceed as in [5, Th. 4.1] by using the Ornstein-Uhlenbeck semigroup. For $u \in L^2_\gamma(X)$ and $t > 0$, thanks to Proposition 2.3,

$$\begin{aligned}
\int_X F(D_\gamma T_t u) &= \sup_{\Phi \in \mathcal{FC}^1_b(X,H)} \int_X -T_t u \operatorname{div}_\gamma \Phi - F^*(\Phi)\, d\gamma \\
&= \sup_{\Phi \in \mathcal{FC}^1_b(X,H)} \int_X -e^{-t} u \operatorname{div}_\gamma T_t\Phi - F^*(\Phi)\, d\gamma \\
&\le \sup_{\Phi \in \mathcal{FC}^1_b(X,H)} \int_X -e^{-t} u \operatorname{div}_\gamma T_t\Phi - F^*(T_t\Phi)\, d\gamma \\
&\le e^{-t} \sup_{\Phi \in \mathcal{FC}^1_b(X,H)} \int_X -e^{-t} u \operatorname{div}_\gamma T_t\Phi - F^*(T_t\Phi)\, d\gamma \\
&\le e^{-t} \int_X F(D_\gamma u)
\end{aligned}$$

where, as $F(0) = 0$ we have $F^* \ge 0$ and thus $e^{-t}F^* \le F^*$. This inequality shows that

$$\int_X F(D_\gamma u) \ge \inf \underline{\lim} \left\{ \int_X F(\nabla_H u_n)\, d\gamma,\ u_n \in W^{1,1}_\gamma(X) \quad u_n \to u \text{ in } L^2_\gamma(X) \right\}.$$

Case 2. Let F be a proper lsc convex function and $q \in H$ be such that $F'(h) = F(h) - [q, h]$ satisfies (H1) then for $u \in L^2_\gamma(X)$,

$$\int_X F(D_\gamma u) = \int_X F'(D_\gamma u) - \int_X u \operatorname{div}_\gamma p\ d\gamma.$$

Therefore, by Case 1 applied to F' we get that

$$\int_X F(D_\gamma u) = \inf \underline{\lim} \left\{ \int_X F(\nabla_H u_n)\, d\gamma,\ u_n \in W^{1,1}_\gamma(X) \quad u_n \to u \text{ in } L^2_\gamma(X) \right\}.$$

Case 3. If now F satisfies (H2), by the density of $\mathcal{FC}^1_b(X)$ in $W^{1,p}_\gamma(X)$ for $p \ge 1$, for every $u \in W^{1,p}_\gamma(X)$ there exists $u_n \in \mathcal{FC}^1_b(X)$ tending to u in $W^{1,p}_\gamma(X)$ and almost everywhere. Then as $F(\nabla_H u_n) \le \alpha_2 |\nabla_H u_n|^p_H + \beta_2$, by the dominated convergence theorem,

$$\int_X F(\nabla_H u_n)\ d\gamma \to \int_X F(\nabla_H u)\ d\gamma.$$

Thus starting from $W^{1,p}_\gamma(X)$ or $\mathcal{FC}^1_b(X)$ gives the same relaxation for $\int_X F(\nabla_H u) d\gamma$. $\square$

4 The finite dimensional case

In this section we focus on the finite dimensional problem. Let $F : \mathbb{R}^m \to \mathbb{R}$ be a convex function satisfying for $p \geq 1$,

$$(H_2') \qquad \alpha_2 |h|^p + \beta_2 \geq F(h) \geq \alpha |h|^p - \beta \qquad \forall h \in \mathbb{R}^m.$$

As before we set

$$\int_{\mathbb{R}^m} F(D_{\gamma_m} u)\, d\gamma_m := \sup_{\Phi \in \mathcal{C}_b^1(\mathbb{R}^m)} \int_{\mathbb{R}^m} \left(-u \operatorname{div}_\gamma \Phi - F^*(\Phi)\right)\, d\gamma_m.$$

By Theorem 3.2 and Proposition 3.4,

$$\int_X F(D_{\gamma_m} u) = \int_{\mathbb{R}^m} F(\nabla u) d\gamma_m + \int_{\mathbb{R}^m} F^\infty \left(\frac{dD^s_{\gamma_m} u}{d|D^s_{\gamma_m} u|} \right) d|D^s_{\gamma_m} u|$$

and this functional also coincides with the relaxation for the $L^2_{\gamma_m}(\mathbb{R}^m)$ topology of the functional classically defined on Lipschitz functions u by $\int_{\mathbb{R}^m} F(\nabla u) d\gamma_m$. In this finite dimensional setting this representation formula is not new (see [8] and [10]).
We show in this section the convexity of the solutions of

$$\min_{u \in L^2_{\gamma_m}(\mathbb{R}^m)} \int_{\mathbb{R}^m} F(D_{\gamma_m} u) + \frac{(u-g)^2}{2}\, d\gamma_m. \tag{15}$$

Formally the Euler-Lagrange equation of this problem reads

$$-\operatorname{div} \nabla F(\nabla u) \; + \; x \cdot \nabla F(\nabla u) \; + \; u \; = \; g. \tag{16}$$

Theorem 4.1. *Let $F : \mathbb{R}^m \to \mathbb{R}$ be a convex function satisfying (H2') and $g \in L^2_{\gamma_m}(\mathbb{R}^m)$ be a convex function. The minimizer of* (15) *is then convex.*

Proof. We consider $F_n \to F$ a sequence of smooth, uniformly convex functions, with quadratic growth which converge locally uniformly to F. The functional $\int_{\mathbb{R}^m} F_n(\nabla u) d\gamma_m$ is then finite if and only if $u \in H^1_{\gamma_m}(\mathbb{R}^m)$.
We consider for $\varepsilon > 0$ the approximation

$$g_\varepsilon(x) \; = \; \max\{g(x), -\frac{1}{\varepsilon}\} + \varepsilon x^2 + \frac{1}{\varepsilon} F_n^*(\varepsilon x)$$

so that $g_\varepsilon \to g$ locally uniformly as $\varepsilon \to 0$. Indeed, it follows from the uniform convexity of F_n that F_n^* is differentiable, hence

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} F_n^*(\varepsilon x) \; = \; \nabla F_n^*(0) \cdot x \; = \; 0.$$

Since $F_n(p) \geq C(|p|^2+1)$, $F_n^*(q) \leq C(|q|^2-1)$ and $g_\varepsilon \in L^2_{\gamma_m}(\mathbb{R}^m)$.
In particular, letting

$$u_\varepsilon(x) \;=\; \frac{F_n^*(\varepsilon x)}{\varepsilon} + m\varepsilon - \frac{1}{\varepsilon} \;\in\; L^2_{\gamma_m}(\mathbb{R}^m)\,,$$

we have

$$-\mathrm{div}\,\nabla F_n(\nabla u_\varepsilon) \;+\; x\cdot\nabla F_n(\nabla u_\varepsilon) \;+\; u_\varepsilon \;=\; -m\varepsilon \;+\; \varepsilon x^2 \;+\; \frac{F_n^*(\varepsilon x)}{\varepsilon} \;+\; m\varepsilon \;-\; \frac{1}{\varepsilon} \;\leq\; g_\varepsilon(x)$$

hence u_ε is a classical subsolution of the approximate problem. We observe that both g_ε and u_ε have superlinear growth at infinity.
We now consider the solution $\bar{u}$ of

$$\min_{u\geq u_\varepsilon}\int_{\mathbb{R}^m} F_n(\nabla u) + \frac{(u-g_\varepsilon)^2}{2}\;d\gamma_m \tag{17}$$

which by definition is above u_ε.
We must show that it is a supersolution of

$$-\mathrm{div}\,\nabla F_n(\nabla u) \;+\; x\cdot\nabla F_n(\nabla u) \;+\; u \;=\; g_\varepsilon. \tag{18}$$

Let us first notice that by [28], the function $\bar{u}$ is Hölder continuous. Assume that $\bar{u}$ is not a supersolution of (17) then there exists $x_0 \in \mathbb{R}^m$ and a smooth function ϕ such that $\phi < \bar{u}$ in $\mathbb{R}^m\backslash\{x_0\}$, $\phi(x_0) = \bar{u}(x_0)$ and

$$-\mathrm{div}\,\nabla F_n(\nabla\phi) \;+\; x\cdot\nabla F_n(\nabla\phi) \;+\; \phi \;-\; g_\varepsilon < 0 \quad\text{ at } x_0. \tag{19}$$

By the smoothness of ϕ we can assume that inequality (19) holds for $\phi+\delta$ in a neighborhood of x_0 for δ small. Replacing ϕ by $\phi - \eta|x-x_0|^2$ we can further assume that (19) holds on the open set $\{\phi+\delta > \bar{u}\}$. As $v = \max(\phi+\delta, \bar{u}) \geq u_\varepsilon$, we have

$$\int_{\mathbb{R}^m} F_n(\nabla v) + \frac{(v-g_\varepsilon)^2}{2}d\gamma_m \geq \int_{\mathbb{R}^m} F_n(\nabla\bar{u}) + \frac{(\bar{u}-g_\varepsilon)^2}{2}d\gamma_m$$

and thus

$$\int_{\{\phi+\delta>\bar{u}\}} F_n(\nabla\phi) + \frac{(\phi+\delta-g_\varepsilon)^2}{2}d\gamma_m \geq \int_{\{\phi+\delta>\bar{u}\}} F_n(\nabla\bar{u}) + \frac{(\bar{u}-g_\varepsilon)^2}{2}d\gamma_m.$$

Using that $F_n(\nabla\bar{u}) - F_n(\nabla\phi) \geq \nabla F_n(\nabla\phi)\cdot(\nabla\bar{u}-\nabla\phi)$ by convexity of F_n and

$$\frac{(\bar{u}-g_\varepsilon)^2}{2} - \frac{(\phi+\delta-g_\varepsilon)^2}{2} \geq \frac{(\phi+\delta-\bar{u})^2}{2} + (\phi+\delta-g_\varepsilon)(\bar{u}-\phi-\delta)$$

we get

$$
\begin{aligned}
0 &\geq \int_{\{\phi+\delta>\bar{u}\}} \nabla F_n(\nabla\phi)\cdot(\nabla\bar{u}-\nabla\phi) + \frac{(\phi+\delta-\bar{u})^2}{2} + (\phi+\delta-g_\varepsilon)(\bar{u}-\phi-\delta)d\gamma_m \\
&= \int_{\{\phi+\delta>\bar{u}\}} \left[-\mathrm{div}\,\nabla F_n(\nabla\phi) + x\cdot\nabla F_n(\nabla\phi) + \phi+\delta-g_\varepsilon\right](\bar{u}-\phi-\delta) + \frac{(\phi+\delta-\bar{u})^2}{2}d\gamma_m \\
&> 0
\end{aligned}
$$

and thus a contradiction. The integration by part used above is justified by the fact that $\{\phi+\delta>\bar{u}\}$ is an open set on the boundary of which $\phi+\delta$ and $\bar{u}$ agree.
Notice that using the same arguments it can be shown that there is no contact between u_ε and $\bar{u}$ so that $\bar{u}$ is in fact an unconstrained minimizer of the energy.
Now, thanks to [2, Proposition 3], given any supersolution u of (18), with superlinear growth, the convex envelope u^{**} is still a supersolution. Moreover, if $u \geq u_\varepsilon$, then clearly $u^{**} \geq u_\varepsilon$ (which is convex).
Hence, if we define $\tilde{u} \leq \bar{u}$ as the infimum of all supersolutions of (18) which are larger than u_ε, it is also the infimum of their convex envelopes (hence it is a locally uniform limit of convex supersolutions) and therefore is convex. It is also a supersolution.
Let us now show that $\tilde{u}$ is a viscosity solution. If it were not, there would exist a smooth ϕ and $x\in\mathbb{R}^m$ with $\tilde{u}(x)=\phi(x)$, and $\tilde{u}<\phi$ in $\mathbb{R}^m\setminus\{x\}$, with

$$
-\mathrm{div}\,\nabla F_n(\nabla\phi(x)) \;+\; x\cdot\nabla F_n(\nabla\phi(x)) \;+\; \phi(x) \;>\; g_\varepsilon(x).
$$

In particular, $\tilde{u}(x) > u_\varepsilon(x)$, otherwise x would also be a local maximum of $u_\varepsilon-\phi$ and the reverse inequality should hold. Now, by standard arguments, we check that $\min\{\tilde{u},\phi-\delta\}$ is still a supersolution, larger than u_ε, if $\delta>0$ is small enough, a contradiction.
Hence $\tilde{u}$ is a solution of (18). By [23, Theorem 4], $\tilde{u}$ is a $C^{1,1}$ function and thus by [2, Lemma 2], $\tilde{u}$ satisfies (18) almost everywhere (and also weakly). The function $\tilde{u}$ is therefore a critical point of the (strictly convex) energy, hence the unique solution to (15) (with F replaced with F_n and g with g_ε). Denote now this solution by u_ε^n.
Let us now show that we can send $\varepsilon\to 0$ and then $n\to\infty$.
Comparing the energy of u_ε^n with the energy of 0, we find that

$$
\|u_\varepsilon^n\|_{L^2_{\gamma_m}(\mathbb{R}^m)} \;\leq\; 2\|g_\varepsilon\|_{L^2_{\gamma_m}(\mathbb{R}^m)} \;\leq\; 2\|g\|_{L^2_{\gamma_m}(\mathbb{R}^m)} + 2\left\|\varepsilon x^2 + \frac{1}{\varepsilon}F_n^*(\varepsilon x)\right\|_{L^2_{\gamma_m}(\mathbb{R}^m)} \tag{20}
$$

so that $\|u_\varepsilon^n\|_{L^2_{\gamma_m}(\mathbb{R}^m)}$ is uniformly bounded. Hence, we can send $\varepsilon\to 0$ and will find that $u_\varepsilon^n \rightharpoonup u^n$. By a Theorem of Dudley [15], u_ε^n converges locally uniformly to u^n which is thus convex. By the lower-semicontinuity of the energy, u^n is the solution of problem (15) with F replaced with F_n.

Analogously, $u^n \to u$ locally uniformly since by (20), $\|u^n\|_{L^2_{\gamma_m}(\mathbb{R}^m)} \leq 2\|g\|_{L^2_{\gamma_m}(\mathbb{R}^m)}$ and thus u is convex. Let us show that u is the minimizer of (15). We start by proving that

$$\varliminf_{n\to\infty} \int_{\mathbb{R}^m} F_n(\nabla u^n) d\gamma_m \geq \int_{\mathbb{R}^m} F(\nabla u) d\gamma_m. \tag{21}$$

Since u^n is a sequence of convex functions converging to $u \in L^2_{\gamma_m}(\mathbb{R}^m)$ then, up to subsequence, ∇u^n converges to ∇u almost everywhere. Moreover, for all $R > 0$ there exists $C = C(R, v)$ such that $\|\nabla u^n\|_{L^\infty(B_R)} \leq C$ for all $n \in \mathbb{N}$. This is a general property of convex functions and we refer to [11, Theorem 3] for further details. By the dominated convergence Theorem, we then get

$$\varliminf_{n\to\infty} \int_{\mathbb{R}^m} F_n(\nabla u^n) d\gamma_m \geq \varliminf_{n\to\infty} \int_{B_R} F_n(\nabla u^n) d\gamma_m = \int_{B_R} F(\nabla u) d\gamma_m.$$

Letting $R \to +\infty$ we obtain (21).
Now if v is a Lipschitz function in $L^2_{\gamma_m}(\mathbb{R}^m)$, as F_n converges locally uniformly to F,

$$\lim_{n\to\infty} \int_{\mathbb{R}^m} F_n(\nabla v) d\gamma_m = \int_{\mathbb{R}^m} F(\nabla v) d\gamma_m$$

and thus, by the minimality of u^n and (21),

$$\begin{aligned}
\int_{\mathbb{R}^m} F(\nabla v) + \frac{(v-g)^2}{2}\, d\gamma_m &= \lim_{n\to\infty} \int_{\mathbb{R}^m} F_n(\nabla v) + \frac{(v-g)^2}{2}\, d\gamma_m \\
&\geq \varliminf_{n\to\infty} \int_{\mathbb{R}^m} F_n(\nabla u^n) + \frac{(u^n-g)^2}{2}\, d\gamma_m \\
&\geq \int_{\mathbb{R}^m} F(\nabla u) + \frac{(u-g)^2}{2}\, d\gamma_m.
\end{aligned}$$

Since Lipschitz functions are dense in energy in $L^2_{\gamma_m}(\mathbb{R}^m)$, we obtain that u is a minimizer of (15). □

Remark 4.2. The proof directly extends to variational problems of the form

$$\min_{u \in L^2(\mu)} \int_{\mathbb{R}^m} F(\nabla u) + \frac{(u-g)^2}{2}\, d\mu$$

for measures $d\mu = \mu(x)\, dx$, with $\mu(x) = e^{-(Ax,x)}$ and $A > 0$.

Remark 4.3. An other possible approach for proving convexity of the minimizers of (15) is to adapt the ideas of Korevaar [25]. See [21] for further details.

Remark 4.4. Arguing as in the Theorem 5.1 of the next section, we see that this result extends to generic proper lsc convex functions F.

5 The infinite dimensional case

In this final section we turn to the infinite dimensional problem.

Theorem 5.1. *Let $F : H \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc convex function and $g \in L^2_\gamma(X)$ be a convex function then the minimizer of*

$$J(u) := \int_X F(D_\gamma u) + \frac{1}{2}\int_X (u-g)^2 d\gamma$$

is convex.

Proof. Case 1. We start by assuming that F satisfies also (H2) .

Let $g_m = \mathbb{E}_m(g)$ then g_m is a convex function. Let also $\bar{u}_m$ be the minimizer of

$$\min_{u \in L^2_\gamma(X):\, u=\mathbb{E}_m u} J_m(u) := \int_X F(D_\gamma u) + \frac{1}{2}\int_X |u-g_m|^2 d\gamma.$$

Thanks to (13), if u depends only on the first m variables then

$$\int_X F(D_\gamma u) = \int_X F_m(D_\gamma u) = \int_{\mathbb{R}^m} F_m(D_{\gamma_m} u)$$

where $F_m(h) = F(\Pi_m h)$. By Theorem 4.1, u_m is thus a convex function. As $J_m(\bar{u}_m) \le J_m(0)$ and since $g_m \to g$ in $L^2_\gamma(X)$, $\bar{u}_m$ is bounded in $L^2_\gamma(X)$ and is thus weakly converging to $\bar{u}$ which is therefore convex by [17, Theorem 4.4].

We now show that $\bar{u}$ is the minimizer of J.

If u_m is a weakly converging sequence to $u \in L^2_\gamma(X)$, then by strong convergence of g_m to g we have

$$\varliminf_{m\to\infty} \frac{1}{2}\int_X |u_m - g_m|^2 d\gamma \ge \frac{1}{2}\int_X |u-g|^2 d\gamma.$$

By the lower semicontinuity of $\int_X F(D_\gamma u)$ (which comes from (12)) we then have

$$\varliminf_{m\to\infty} J_m(u_m) \ge J(u).$$

Thus if $u \in \mathcal{FC}^1_b(X)$, by minimality of $\bar{u}_m$,

$$J(u) = \lim_{m\to+\infty} J_m(u) \ge \varliminf_{m\to+\infty} J_m(\bar{u}_m) \ge J(\bar{u}). \tag{22}$$

Since we assumed that F satisfies (H2), by Proposition 3.4, the space $\mathcal{FC}_b^1(X)$ is dense in energy in $L^2_\gamma(X)$ and thus inequality (22) proves that $\bar{u}$ is the minimizer of J in $L^2_\gamma(X)$.

Case 2. If F is a proper lsc convex function, we can approximate it by a convex function F_δ with linear growth

$$F_\delta(p) := \delta |p|_H + \inf_{q\in H} \left(\frac{1}{\delta}|p-q|_H + F(q) \right).$$

By Case 1, the minimizer u_δ of the functional with F_δ instead of F is convex. As before, we have that u_δ weakly converges to a convex function u in $L^2_\gamma(X)$. As $W^{1,1}_\gamma(X)$ is dense in energy in $L^2_\gamma(X)$, in order to conclude, it is sufficient to prove that for every $v \in W^{1,1}_\gamma(X) \cap L^2_\gamma(X)$,

$$\int_X F(\nabla_H v) d\gamma \geq \overline{\lim_{\delta\to 0}} \int_X F_\delta(\nabla_H v) \tag{23}$$

and

$$\underline{\lim_{\delta\to 0}} \int_X F_\delta(D_\gamma u_\delta) \geq \int_X F(D_\gamma u). \tag{24}$$

For inequality (23) we can assume that $\int_X F(\nabla_H v) d\gamma < +\infty$ then as for the Moreau regularization, $\lim_{\delta\to 0} F_\delta(p) = F(p)$ for every $p \in H$ so that for every $v \in W^{1,1}_\gamma(X)$, $F_\delta(\nabla_H v)$ converges almost everywhere to $F(\nabla_H v)$ and since $F_\delta(\nabla_H v) \leq \delta|\nabla_H v|_H + F(\nabla_H v)$, by the dominated convergence Theorem, inequality (23) follows.

For inequality (24), we start by noticing that by calculus on inf-convolutions and convex conjugates, we have,

$$F^*_\delta(q) = \inf_{\substack{|p|_H \leq \frac{1}{\delta} \\ |p-q|_H \leq \delta}} F^*(p),$$

where we take as a convention that $F^*_\delta(q) = +\infty$ if $B_{\frac{1}{\delta}} \cap B_\delta(q) = \emptyset$. Therefore, for every $q \in H$, as soon as $|q|_H \leq \frac{1}{\delta}$, we have $F^*_\delta(q) \leq F^*(q)$ and thus

$$\overline{\lim_{\delta\to 0}} F^*_\delta(q) \leq F^*(q) \qquad \forall q \in H.$$

If now $\Phi \in \mathcal{FC}_b^1(X, H)$ with $F^*(\Phi)$ integrable, we have $F^*_\delta(\Phi) \leq F^*(\Phi)$ for δ small enough and thus by the reverse Fatou lemma,

$$\overline{\lim_{\delta\to 0}} \int_X F^*_\delta(\Phi)\, d\gamma \leq \int_X \overline{\lim_{\delta\to 0}} F_\delta(\Phi) d\gamma \leq \int_X F^*(\Phi)\, d\gamma. \tag{25}$$

We can now conclude since for every $\Phi \in \mathcal{FC}_b^1(X,H)$ with $\int_X F^*(\Phi)\,d\gamma$ we have using (25),

$$\begin{aligned}\varliminf_{\delta\to 0}\int_X F_\delta(D_\gamma u_\delta) &\geq \varliminf_{\delta\to 0}\int_X -u_\delta \operatorname{div}_\gamma \Phi - F_\delta^*(\Phi)\,d\gamma\\ &\geq \int_X -u \operatorname{div}_\gamma \Phi - F(\Phi)\,d\gamma\end{aligned}$$

Taking then the supremum on all $\Phi \in \mathcal{FC}_b^1(X,H)$ and using (8), we get (24).

□

Remark 5.2. Notice that, by taking $F(h) = |h|^p$ with $p \geq 1$, Theorem 5.1 applies in particular to the p-Dirichlet problems

$$\min_{L^2_\gamma(X)} \int_X |\nabla_H u|_H^p\,d\gamma + \frac{1}{2}\int_X |u-g|^2 d\gamma.$$

Remark 5.3. When X is an Hilbert space, there is another definition of the gradient due to Da Prato which gives an alternative definition of Sobolev and BV spaces (see [5, Section 5]). Roughly speaking it corresponds to $Du := Q^{-\frac{1}{2}}\nabla_H u$. Theorem 5.1 then applies to the associated total variation since it is given by the choice

$$F(h) = \left(\sum_{i=1}^{+\infty} \frac{1}{\lambda_i}|h_i|^2\right)^{\frac{1}{2}}$$

where the λ_i's are the eigenvalues of Q.

Remark 5.4. Notice that in the proofs of Theorem 4.1 and 5.1 we made standard Γ-convergence arguments (see [9]).

We can now use these convexity results to show the convexity of solutions of (1).

Theorem 5.5. *Let g be a convex function in $L^2_\gamma(X)$ and let u be the minimizer of* (2). *Let $\overline{\lambda} = \inf\{\lambda \,:\, \gamma(u \leq \lambda) > 0\}$. If $\overline{v} = \gamma(\{u \leq \overline{\lambda}\})$ then for every $v > \overline{v}$ there exists a unique solution to* (1) *and this solution is convex.*

Proof. The proof follows quite standard arguments so that we only sketch it (see [12] and [1] for details). Let us first consider the problem

$$\min P_\gamma(E) + \int_E (g-\lambda)d\gamma. \qquad (P_\lambda)$$

Then as in Proposition 34 of [12], by the direct method of the calculus of variations and by the co-area formula it is not difficult to show that (P_λ) has a minimum E_λ. By [12, Lemma 8] we have $E_{\lambda_1} \subset E_{\lambda_2}$ if $\lambda_1 \leq \lambda_2$.

Setting $w(x) = \inf\{\lambda \,:\, x \in E_\lambda\}$, it is not hard to see that $w \in BV_\gamma \cap L^2_\gamma(X)$ and that w solves (2) (see [12] again or Lemma 3.5 in [13]). By the uniqueness of minimizers of (2), $w = u$ and $E_\lambda = \{u < \lambda\}$ for almost every λ (and then for every λ by an approximation procedure).
By Proposition 2.7, the function $\lambda \to \gamma(E_\lambda)$ is continuous on $]\overline{\lambda}, +\infty[$ and nondecreasing. Together with the inclusion property of the E_λ this implies the uniqueness of the minimizers of (P_λ). Moreover, the sets E_λ solve the problem:

$$\min_{\gamma(E_\lambda)=\gamma(E)} P_\gamma(E) + \int_E g d\gamma.$$

Vice-versa, if E_v solves (1) and $v > \overline{v}$ then there exists $\lambda > \overline{\lambda}$ such that $\gamma(E_\lambda) = v$ and as E_v solves (P_λ) we get $E_v = E_\lambda$. $\square$

Remark 5.6. If $F : H \to \mathbb{R}$ is homogeneous of degree one and such that

$$c|h|_H \leq F(h) \leq C|h|_H \qquad \forall h \in H,$$

then F satisfies (H2) and we can define the anisotropic perimeter P_F by

$$P_F(E) := \int_X F(D_\gamma \chi_E).$$

Repeating verbatim the proof of [16, Section 5.5], (and using that smooth cylindrical functions are dense in $BV_\gamma(X)$ by Proposition 3.4), we still have a coarea formula,

$$\int_X F(D_\gamma u) = \int_{\mathbb{R}} P_F(\{u < t\})\; dt \qquad \forall u \in BV_\gamma(X).$$

Using Theorem 5.1, it is then not difficult to extend Theorem 1.1 and Theorem 5.5 to these anisotropic perimeters P_F.

We can finally state a simple corollary.

Corollary 5.7. *Let g be a convex function in $L^2_\gamma(X)$ and let*

$$F(E) = P_\gamma(E) + \int_E g\, d\gamma.$$

Then two situations can occur:

- *If $\min F < 0$ then there exists a unique non-empty minimizer of F. Moreover this minimizer is convex.*
- *If $\min F = 0$ then there exists at most one non-empty minimizer of F which is then convex.*

Proof. The two possibilities corresponds respectively to $\overline{\lambda} < 0$ and $\overline{\lambda} \geq 0$. $\square$

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