Regularity results for some 1-homogeneous functionals

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Abstract

We consider local minimizers for a class of 1-homogeneous integral functionals defined on $BV_{\rm loc}(\Omega)$, with $\Omega \subset \mathbb{R}^2$. Under general assumptions on the functional, we prove that the boundary of the subgraph of such minimizers is (locally) a lipschitz graph in a suitable direction. The proof of this statement relies on a regularity result holding for boundaries in $\mathbb{R}^2$ which minimize an anisotropic perimeter. This result is applied to the boundary of sublevel sets of a minimizer $u \in BV_{\rm loc}(\Omega)$.

We also provide an example which shows that such regularity result is optimal.

Keywords: Variational problems; Nonsmooth analysis; Regularity of solutions; Crystals

1 Introduction

In this paper we study the regularity properties of local minimizers of functionals of the type

$$F_\varphi(u) = \int_\Omega \varphi(Du) + \int_\Omega f u \,\mathrm{d}x, \tag{1}$$

where $\Omega \subset \mathbb{R}^2$, $u \in BV_{\rm loc}(\Omega)$, $f \in L^\infty_{\rm loc}(\Omega)$ and $\varphi : \mathbb{R}^2 \to \mathbb{R}^+$ is a generic positively 1-homogeneous convex function. Since Du is in general only a measure we shall give a precise meaning to the integral $\int \varphi(Du)$.

A serious difficulty in considering minimizers of (1) comes from the fact that the functional F_φ is not strictly convex in u and has only a linear growth, hence we cannot apply the usual techniques [4], [9] which lead to $C^{1,\alpha}$ regularity, as for example in the prescribed mean curvature problem, i.e. when $\varphi(Du) = \sqrt{1 + |\nabla u|^2}$ [10], [3]. In the case $\varphi(Du) = |Du|$, it is easy to find minimizers of (1) which are not even continuous. Nevertheless, in the case $f = 0$ and $\varphi(Du) = |Du|$ it has been proved [11] that, if we prescribe sufficiently regular boundary conditions, a lipschitz minimum always exists.

However, the homogeneity property of the functional allows us to conclude that if u is a minimizer of F_φ then $\chi_{\{u<t\}}$ is also a minimizer for any $t \in \mathbb{R}$. This implies that each sublevel set of u is itself a minimum of an anisotropic prescribed curvature problem [6], [7], [12]. Such minimizers have been considered in [2], [13] and it is known (in dimension $n = 2$) that their boundary is locally the graph of a lipschitz function.

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Using this information we are able to conclude that the boundary of the subgraph of u, as a subset of $\Omega \times \mathbb{R}$ is itself locally a lipschitz graph, even if not necessarily in the vertical direction. To perform this step we need an assumption on the convex set $\{\varphi < 1\}$, which we call *fatness* condition (see Section 2), and we show with an explicit example that such condition is necessary to get this kind of regularity. Moreover, such an example shows that in some cases the minimizers of a crystalline perimeter (in dimension $n = 3$) are not locally lipschitz graphs.

We conjecture that, without any assumption on φ, the graph of a minimizer is locally parameterizable by means of a bilipschitz map.

The study of the functional (1) is a first step in the analysis of the regularity of the minimizers of

$$\int_\Omega f(Du) \qquad u \in BV_{\rm loc}(\Omega),$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a generic convex function. Our result gives a suggestion on which kind of regularity can be expected in the general case.

The plan of the paper is the following. In Section 2 we describe the notation that we shall use in the sequel. In Section 3 we introduce the class of local minimizers for F_φ, and we prove a compactness results for sequences in this class (Theorem 3.4). Moreover, in Theorem 3.12 we prove the main result of the paper, which states that the graph of a local minimizer is itself (locally) the grapf of a lipschtz function, when φ satisfies the *fatness* condition. Finally, in Section 4 we give an example which shows that this condition is really necessary to conclude that the graph of the minimizer is locally a lipschitz graph.

2 Notation

Let $\varphi \colon \mathbb{R}^2 \to \mathbb{R}^+$ be a function such that

1. $\varphi(x) = 0 \Leftrightarrow x = 0$ (coercivity);
2. $\varphi(tx) = t\varphi(x) \qquad \forall t \geq 0$ (positive 1-homogenity);
3. $\varphi(x + y) \leq \varphi(x) + \varphi(y)$ (convexity).

We define $\varphi^o \colon \mathbb{R}^2 \to \mathbb{R}^+$ as

$$\varphi^o(v) = \sup_{\xi \neq 0} \frac{\langle \xi, v \rangle}{\varphi(\xi)}$$

where $\langle \cdot, \cdot \rangle$ is the usual scalar product of $\mathbb{R}^2$. It is not difficult to check that φ^o satisfies the same properties of φ and that

$$\varphi(\xi) = \sup_{v \neq 0} \frac{\langle \xi, v \rangle}{\varphi^o(v)}.$$

We call *Wulff shape* the set $W_\varphi := \{x \in \mathbb{R}^2 \colon \varphi^o(x) < 1\}$ and *Frank diagram* the set $F_\varphi := \{\xi \in \mathbb{R}^2 \colon \varphi(\xi) < 1\}$. We define the following (multivalued) duality maps

$$\begin{aligned} v^* &:= \{\xi \in \mathbb{R}^2 \colon \varphi(\xi) = \varphi^o(v),\ \langle \xi, v \rangle = \varphi(\xi)\varphi^o(v)\} \\ \xi^* &:= \{v \in \mathbb{R}^2 \colon \varphi^o(v) = \varphi(\xi),\ \langle \xi, v \rangle = \varphi(\xi)\varphi^o(v)\}. \end{aligned}$$

We say that W_φ is *slim* if there exists an edge $l \subset \partial W_\varphi$ such that the sum of the two angles of W_φ adjacent to l is less than or equal to π. We say that W_φ *fat* if it is not slim.

For example, all triangles and quadrilaterals are slim, whereas strictly convex Wulff shapes are fat.

Given $E \subseteq \mathbb{R}^n$, we let $\chi_E : \mathbb{R}^n \to \mathbb{R}$ be the characteristic function of E, i.e. $\chi_E(x) = 1$ if $x \in E$, and $\chi_E(x) = 0$ otherwise.

We will denote with $\mathcal{H}^k$, $k > 0$, the k-dimensional Hausdorff measure in $\mathbb{R}^n$, and we let $|E| := \mathcal{H}^n(E)$ be the Lebesgue measure of the set $E \subseteq \mathbb{R}^n$.

Given $v \in \mathbb{R}^3 \setminus \{0\}$ we say that a set $S \subset \mathbb{R}^3$ is a *graph along* v, if it is not possible to find two different points $x, y \in S$ such that $x - y = \lambda v$ for some $\lambda \in \mathbb{R}$.

The anisotropic perimeter of a set E in the open set $A \subset \mathbb{R}^2$ is defined by

$$P_\varphi(E, A) := \sup \left\{ \frac{\pi}{|W_\varphi|} \int_E \operatorname{div} \psi(x)\, \mathrm{d}x \colon \psi \in \mathcal{C}^1_c(A; \mathbb{R}^2), \varphi^o(\psi(y)) \leq 1 \quad \forall y \in A \right\}.$$

The usual notion of perimeter of E in A will still be denoted by $P(E, A)$.

We let $B_\rho(x) := \{y \colon \|x - y\| < \rho\}$ be the usual euclidean ball of $\mathbb{R}^2$ and we set for simplicity $B_\rho := B_\rho(0)$.

Given a set $E \subseteq \mathbb{R}^n$ of locally finite perimeter, we define

$$\begin{aligned}
\partial E &:= \{x \in \mathbb{R}^n \colon \forall \rho > 0 \quad |E \cap B_\rho(x)| \in]0, |B_\rho(x)|[\,\},\\
\overline{E} &:= \{x \in \mathbb{R}^n \colon \forall \rho > 0 \quad |E \cap B_\rho(x)| \neq 0\},\\
\mathring{E} &:= \{x \in \mathbb{R}^n \colon \forall \rho > 0 \quad |E \cap B_\rho(x)| = |B_\rho(x)|\}.
\end{aligned}$$

It holds, as usual, that $\overline{E}, \partial E$ are closed sets, whereas $\mathring{E} = \overline{E} \setminus \partial E$ is an open set. Notice that if $|E \triangle F| = 0$ then $\partial E = \partial F$ (where $E \triangle F := (E \setminus F) \cup (F \setminus E)$).

We let $\partial^* E$ be the reduced boundary in the sense of De Giorgi [5], and

$$\nu_E(x) := \lim_{\rho \to 0^+} -\frac{D\chi_E(B_\rho(x))}{|D\chi_E(B_\rho(x))|}$$

be the exterior unit normal to ∂E in $x \in \partial^* E$.

Given a vector-valued Radon measure μ on $\Omega \subseteq \mathbb{R}^2$, we define the measure $\varphi(\mu)$ as the φ-total variaton of μ:

$$\varphi(\mu)(B) := \sup \sum_i \varphi(\mu(B_i))$$

where the supremum is taken over the family of all partitions $\{B_i\}_{i \in I}$ of the Borel set $B \subseteq \Omega$. With this definition, if $u \in BV_{\mathrm{loc}}(\Omega)$ then $\varphi(Du)$ is a positive measure on Ω and the integral $\int_\Omega \varphi(Du)$ makes sense.

For $u \in BV_{\mathrm{loc}}(\Omega)$ we define $S = S_u := \{(x, t) \in \Omega \times \mathbb{R} \colon u(x) < t\}$ and let $\Gamma = \Gamma_u := \partial S_u \cap (\Omega \times \mathbb{R})$. Notice that Γ_u is always a closed set in $\Omega \times \mathbb{R}$ and $\Gamma_u = \Gamma_v$ whenever $u = v$ a.e.

For $t \in \mathbb{R}$ we define $E_t := \{x \in \Omega \colon u(x) > t\}$ and $F_t := \{x \in \Omega \colon u(x) \geq t\}$.

3 Local minimizers and regularity result

As explained in the Introduction, we shall consider local minimizers of the functional

$$F_\varphi(u) = \int_\Omega \varphi(Du) + \int_\Omega f u \,\mathrm{d}x$$

where Ω is an open subset of $\mathbb{R}^2$, $u \in BV_{\mathrm{loc}}(\Omega)$, $f \in L^\infty_{\mathrm{loc}}(\Omega)$ and φ is a convex positive 1-homogeneous function.

Remark 3.1 *Observe that for any $u \in BV_{\mathrm{loc}}(\Omega)$ there holds*

$$F_\varphi(u) = \sup \left\{ -\int_\Omega (u \operatorname{div} \psi + f u)\ dx :\ \psi \in C^1_c(\Omega),\ \varphi^o(\psi) \leq 1 \right\} \tag{2}$$

In particular, from (2) it follows that F_φ is lower semicontinuous in $L^1_{\mathrm{loc}}(\Omega)$, i.e.

$$\int_B \varphi(Du) + \int_B fu\,\mathrm{d}x \leq \liminf_{k\in\mathbb{N}} \int_B \varphi(D(u+\psi_k)) + \int_B f(u+\psi_k)\,\mathrm{d}x,$$

whenever $\psi_k \to 0$ in $L^1(B)$, $\psi_k \in BV(B)$, $B \Subset \Omega$.

Definition 3.2 *We say that $u \in BV_{\mathrm{loc}}(\Omega)$ is a (local) minimizer for F_φ if*

$$\int_{B_\rho(x)} \varphi(Du) + \int_{B_\rho(x)} fu\,\mathrm{d}x \leq \int_{B_\rho(x)} \varphi(Dv) + \int_{B_\rho(x)} fv\,\mathrm{d}x,$$

whenever $B_\rho(x) \Subset \Omega$, $v \in BV_{\mathrm{loc}}(\Omega)$ and $\overline{\{x \in \Omega : u(x) \neq v(x)\}} \subset B_\rho(x)$.

Notice that, by approximation, we can restrict the class of test functions to the functions $v = u + \psi$, where $\psi \in C^\infty_c(B_\rho(x))$.

We denote by $\mathcal{M}(\Omega)$ the family of all local minimizers of F_φ in Ω. With a little abuse of notation, when $E \subset \Omega$ is a measurable set, we write $E \in \mathcal{M}(\Omega)$ instead of $\chi_E \in \mathcal{M}(\Omega)$.

Remark 3.3 *The semicontinuity of F_φ guarantees that (when Ω is bounded) minimizers do exist in the closure of any nonempty subset of $BV(\Omega)$ which is bounded in L^1. In particular, given $u_0 \in BV(\Omega)$, there always exists a minimizer for F_φ, among the functions $u \in BV(\Omega)$ which coincide with u_0 outside of a set $B \Subset \Omega$. However, it may be convenient for the minimizers to have jumps on ∂B*

In the following theorem we state a fundamental compactness property of the class $\mathcal{M}(\Omega)$.

Theorem 3.4 (compactness) *If $u_k \in \mathcal{M}(\Omega)$, $u \in L^1_{\mathrm{loc}}(\Omega)$ and $u_k \to u$ in $L^1_{\mathrm{loc}}(\Omega)$ then $u \in \mathcal{M}(\Omega)$.*

Proof.
Let $B_\rho(x) \Subset \Omega$ and $v \in BV_{\mathrm{loc}}(\Omega)$ be such that $K = \overline{\{y \in \Omega : u(y) \neq v(y)\}} \subset B_\rho(x)$. Let moreover $\rho' \in]0, \rho[$ be such that $K \subset B'_\rho(x)$. Suppose for simplicity of notation $x = 0$. Since the proof does not change significantly we shall also assume $f = 0$.

We claim that it is possible to find a sequence of radii $\eta_k, \eta \in]\rho', \rho[$ with $\eta_k \nearrow \eta$ such that it holds

$$\forall k \quad \int_{\partial B_{\eta_k}} |Du_k| = 0, \tag{3}$$

$$\lim_{k\to\infty} \int_{B_\eta \setminus B_{\eta_k}} |Du_k| = 0, \tag{4}$$

$$\liminf_{k\to\infty} \int_{\partial B_{\eta_k}} \left| D\big((u-u_k)\chi_{B_{\eta_k}}\big)\right| = 0. \tag{5}$$

If this is true we can conclude the proof by considering the functions $v_k = (v-u_k)\chi_{B_{\eta_k}} + u_k$ which are variations of u_k in B_η and coincide with v in B_{η_k} so that, by the semicontinuity of $u \mapsto \int_U \varphi(Du)$, the minimality of u_k with respect to v_k, the locality of $u \mapsto \int_U \varphi(Du)$, we get

$$\begin{aligned}\int_{B_\eta} \varphi(Du) &\leq \liminf_{k\to\infty} \int_{B_\eta} \varphi(Du_k) \leq \liminf_{k\to\infty} \int_{B_\eta} \varphi(Dv_k) \\ &= \liminf_{k\to\infty} \left[\int_{B_{\eta_k}} \varphi(Dv) + \int_{\partial B_{\eta_k}} \varphi(Dv_k) + \int_{B_\eta \setminus \overline{B_{\eta_k}}} \varphi(Du_k) \right]\end{aligned}$$

$$\leq \int_{B_\eta} \varphi(Dv) + C \liminf_{k\to\infty} \left[\int_{\partial B_{\eta_k}} \left| D\big((u-u_k)\chi_{B_{\eta_k}} + u_k\big) \right| + \int_{B_\eta \setminus B_{\eta_k}} |Du_k| \right]$$
$$\leq \int_{B_\eta} \varphi(Dv) + C \liminf_{k\to\infty} \int_{\partial B_{\eta_k}} \left[\left| D\big((u-u_k)\chi_{B_{\eta_k}}\big) \right| + |Du_k| \right]$$
$$= \int_{B_\eta} \varphi(Dv),$$

where $C > 0$ is such that $\varphi(\xi) \leq C\,|x|$.

Let us prove the claim. Let $T_1 = \bigcap_k \{t \in]\rho', \rho[: \int_{\partial B_t} |Du_k| = 0\}$. Since $u_k \in BV(B_\rho)$ the set T_1 is an intersection of countably many sets with measure $\rho - \rho'$ that is T_1 has itself measure $\rho - \rho'$. So, for (3) to hold, we just need $\eta_k \in T_1$ for all k.

Regarding (5), we notice that [1, Sec. 3.7] for a.e. $t > 0$ we have

$$\int_{\partial B_t} \left| D\big((u-u_k)\chi_{B_t}\big) \right| \leq \int_{\partial B_t} |u(x) - u_k(x)| \, \mathrm{d}\mathcal{H}^1(x),$$

where the second integral must be intended in the Lebesgue sense.

Consider the functions $f_k(t) = \int_{\partial B_t} |u(x) - u_k(x)| \, \mathrm{d}\mathcal{H}^1(x)$ defined for $t \in]\rho', \rho[$. By Fubini-Tonelli formula we know that $f_k \in L^1(]\rho', \rho[)$ and

$$||f_k||_{L^1(]\rho',\rho[)} = |u - u_k|_{L^1(B_\rho \setminus \overline{B_{\rho'}})} \to 0 \qquad \mathit{for}\ k \to 0.$$

So, applying Egoroff theorem, there exists $T_2 \subset]\rho', \rho[$ with $|T_2| > (\rho - \rho')/2$ and a subsequence of f_k which converges to 0 uniformily on T_2. Let now T_3 be the set of points $\eta \in T_2 \cap T_1$ such that there exists an increasing sequence $\eta_j \nearrow \eta$ with $\eta_j \in T_2 \cap T_1$. Since $T_2 \cap T_1$ is uncountable also T_3 is uncountable and in particular not empty. Therefore there exist $\eta \in T_3$ and $\eta_j \nearrow \eta$, with $\eta_j \in T_2 \cap T_1$, such that

$$\liminf_{k\to\infty} \sup_j \int_{\partial B_{\eta_j}} |u(x) - u_k(x)| \, \mathrm{d}\mathcal{H}^1(x) \leq \liminf_{k\to\infty} \sup_{t \in T_2} |f_k(t)| = 0,$$

since (up to a subsequence) $f_k \to 0$ uniformily on T_2.

Concerning (4), we simply note that if $\eta_j \nearrow \eta$, then for all $k \in \mathbb{N}$ we get

$$\lim_{j\to\infty} \int_{B_\eta \setminus B_{\eta_j}} |Du_k| = 0$$

since $\bigcap_j (B_\eta \setminus B_{\eta_j}) = \emptyset$. So, given $k \in \mathbb{N}$, we can find $j(k)$ such that $\int_{B_\eta \setminus B_{\eta_j}} |Du_k| < 1/k$. By letting $\eta_k = \eta_{j(k)}$ we have thus determined the sequence which satisfies (3), (4) and (5). □

The following lemma, which strictly depends on the homogeneity property of F_φ, allows us to prove that the characteristic function of a sublevel set of a minimizer is also a minimizer for F_φ (see [8]).

Lemma 3.5 *Let $u \in \mathcal{M}(\Omega)$. Then $u \vee C, u \wedge C, \lambda u \in \mathcal{M}(\Omega)$ for any $C \in \mathbb{R}$ and $\lambda > 0$.*

Proof.
It is clear that $u + C \in \mathcal{M}(\Omega)$ and $\lambda u \in \mathcal{M}(\Omega)$ since $D(u+C) = Du$ and $\varphi(D(\lambda u)) = \lambda\varphi(Du)$. Write now $u = u^+ + u^-$, where $u^+ := u \vee 0$ (resp. $u^- := u \wedge 0$) is the positive (resp. negative) part of u. Given $U \Subset \Omega$, $\psi \in C_c^\infty(U)$, we have

$$\int_U \varphi(Du^+) + \int_U f u^+ \, \mathrm{d}x + \int_U \varphi(Du^-) + \int_U f u^- \, \mathrm{d}x$$

$$
\begin{aligned}
&= \int_U \varphi(Du) + \int_U fu\,\mathrm{d}x \le \int_U \varphi(Du + D\psi) + \int_U f(u+\psi)\,\mathrm{d}x \\
&\le \int_U \varphi(Du^+ + D\psi) + \int_U f(u^+ + \psi)\,\mathrm{d}x + \int_U \varphi(Du^-) + \int_U fu^-\,\mathrm{d}x,
\end{aligned}
$$

hence

$$
\int_U \varphi(Du^+) + \int_U fu^+\,\mathrm{d}x \le \int_U \varphi(Du^+ + D\psi) + \int_U f(u^+ + \psi)\,\mathrm{d}x.
$$

□

Theorem 3.6 *Let $u \in \mathcal{M}(\Omega)$. Then for any $t \in \mathbb{R}$ we have $E_t, F_t \in \mathcal{M}(\Omega)$.*

Proof.
Given $\varepsilon > 0$ consider $u_\varepsilon(x) = (u(x) - t)/\varepsilon \wedge 1 \vee 0$. An easy check assures that $u_\varepsilon \to \chi_{E_t}$ pointwise as $\varepsilon \to 0^+$. Since u_ε are dominated by the constant 1, by Lebesgue convergence theorem $u_\varepsilon \to \chi_{E_t}$ in $L^1_{\mathrm{loc}}(\Omega)$. So, by compactness we get $E_t \in \mathcal{M}(\Omega)$.

If we instead consider $u_\varepsilon(x) = (u(x)-t+\varepsilon)/\varepsilon\wedge 1\vee 0$ we conclude that $F_t \in \mathcal{M}(\Omega)$.
□

We point out that, as a consequence of Theorem 3.6, from [2, Corollary 3.6] it follows that $|\partial E_t \setminus \partial^* E_t| = 0$.

Lemma 3.7 *Let $u \in BV_{\mathrm{loc}}(\Omega)$. Given $(x,t) \in \partial^* S_u$ then $x \in \partial E_t$ or $\nu_S(x,t) = (0,1)$ (and both conditions can hold). Moreover, for a.e. $t \in \mathbb{R}$ and for $\mathcal{H}^1$-a.e. $x \in \partial^* E_t$ it holds*

$$
\nu_S(x,t) = \frac{(\lambda\nu, 1)}{\sqrt{1+\lambda^2}} \quad \text{or} \quad \nu_S(x,t) = \nu \tag{6}
$$

for some $\lambda \in \mathbb{R}$ and for $\nu = \nu_{E_t}(x)$.

Proof.
Suppose that $x \notin \partial E_t$. If $x \notin \overline{E_t}$ then for some $\rho > 0$ we have

$$
|\{y \in B_\rho(x) \colon u(y) < t\}| = |B_\rho(x)|.
$$

On the other hand $(x,t) \in \partial^* S_u$ means that

$$
\frac{S_u - (x,t)}{\varepsilon} \to H(\nu_{S_u}(x,t)) = \{(y,s) \colon \langle (y,s), \nu_{S_u}(x,t)\rangle < 0\} \qquad \text{in } L^1_{\mathrm{loc}},
$$

for $\varepsilon \to 0$. Since in this case $\varepsilon^{-1}(S_u - (x,t)) \cap B_{\rho/\varepsilon} \times \{t\} \subset H((0,1))$, we have necessarily $\nu_{S_u} = (0,1)$. The proof is similar when $x \in \mathring{E}_t$.

In order to prove the second statement, let us consider the orthogonal projection $\Pi(y,s) = y$. We recall that [1, Ch. 3] for a.e. $t \in \mathbb{R}$ and for $\mathcal{H}^1$-a.e. $x \in \partial^* E_t$ there holds $\Pi(\nu_{S_u}(x,t)) = s\nu_{E_t}(x)$ for some $s \in \mathbb{R}$, which implies (6). □

Lemma 3.8 *Let $E, F \subset \mathbb{R}^2$ be Caccioppoli sets with lipschitz boundary such that $E \subseteq F$. Assume that $\partial E \cap B_{\frac{\rho}{2}} \neq \emptyset$, $\partial F \cap B_{\frac{\rho}{2}} \neq \emptyset$, for some $\rho > 0$, and let $K_1, K_2 \subset \mathbb{R}^2$ be two convex cones (i.e. $\lambda K_1 = K_1$, $\lambda K_2 = K_2$ for any $\lambda > 0$) such that for $\mathcal{H}^1$-a.e. $x \in \partial E \cap B_\rho$ and $y \in \partial F \cap B_\rho$ it holds $-\nu_E(x) \in K_1$ and $-\nu_F(y) \in K_2$. Then the following estimate holds*

$$
\mathrm{dist}(K_1 \cap \partial B_1, K_2 \cap \partial B_1) \le \frac{2}{\rho}\mathrm{dist}(\partial E \cap \overline{B_{\frac{\rho}{2}}}, \partial F \cap \overline{B_{\frac{\rho}{2}}}). \tag{7}
$$

Proof.
Let $u \in \partial E \cap \overline{B_{\frac{\rho}{2}}}$, $v \in \partial F \cap \overline{B_{\frac{\rho}{2}}}$ be the points for which the minimum on the right hand side of the estimate is reached, and let $D = |u-v|$. Let also d be the minimum value of the left hand side.

Let $K_1^\perp, K_2^\perp$ be the convex cones "orthogonal" to K_1, K_2 (respectively) defined as

$$K_i^\perp = \{n \in \mathbb{R}^2 \colon \langle n, v\rangle \le 0, \text{ for all } v \in K_i\} \qquad i \in \{1,2\}.$$

It is not difficult to show that $K_1^\perp \cap (-K_2^\perp)$ is a (convex) cone of angle 2α such that $\sin(\alpha) = \frac{d}{2}$. Moreover, since $\overline{E} \supset (-K_1^\perp + u) \cap B_\rho$ and $(K_2^\perp + v) \cap B_\rho \subset B_\rho \setminus \mathring{F}$, it follows that the two cones $-K_1^\perp + u$ and $K_2^\perp + v$ do not intersect within the ball B_ρ. On the other hand, they must intersect in a ball of radius $R \le \frac{D}{2\sin(\alpha)} + \frac{\rho}{2} = \frac{D}{d} + \frac{\rho}{2}$, therefore

$$\rho \le R \le \frac{D}{d} + \frac{\rho}{2},$$

which gives (7). □

We recall the following regularity result from [2, Theorem 6.19].

Theorem 3.9 *Assume that W_φ is not a triangle, and let $\Omega \subset \mathbb{R}^2$ be an open set. Then, for any $E \in \mathcal{M}(\Omega)$, $x \in \partial E \cap \Omega$ and $\rho > 0$ such that $B_\rho(x) \Subset \Omega$, there exists a lipschitz graph Γ, whose lipschitz constant depends only on F_φ and ρ, such that $\partial E \cap B_\rho(x) \subset \Gamma$.*

Moreover, there exists a lipschitz function $v \colon \Gamma \to \partial W_\varphi$ such that

$$v(y) \in \left(\frac{-\nu_E(y)}{\varphi(-\nu_E(y))}\right)^* \qquad \forall y \in \partial^* E \cap B_\rho(x).$$

Lemma 3.10 *Let $\Omega \subset \mathbb{R}^n$ be a connected open set and $E \subset \mathbb{R}^n$ be a Caccioppoli set. If there exist $v \in \mathbb{R}^n, |v| = 1$ and $\lambda \in]0,1[$ such that $\langle v, \nu_E(x)\rangle \le -\lambda$ for $\mathcal{H}^{n-1}$-a.e. $x \in \partial^* E \cap \Omega$, then ∂E is an L-lipschitz graph in the direction v, with $L = \sqrt{1/\lambda^2 - 1}$.*

Proof.
Let us choose mollifiers function $\rho_\varepsilon \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ such that $\text{spt}\rho_\varepsilon \subset B_\varepsilon$ and consider the functions $u_\varepsilon := \chi_E * \rho_\varepsilon$. Since E has locally finite perimeter, for sufficiently small ε the integral $\alpha_\varepsilon^x = \int_{\mathbb{R}^n} \rho_\varepsilon(y-x)\, \mathrm{d}\,|D\chi_E|(y)$ is finite. Hence the measures μ_ε^x defined by $\mathrm{d}\mu_\varepsilon^x(y) := (\rho_\varepsilon(y-x)/\alpha_\varepsilon^x)\, \mathrm{d}\,|D\chi_E|(y)$ are probability measures. By the hypothesis we know that for $\mathcal{H}^{n-1}$–a.e. $y \in \partial E$ and hence for μ_ε^x–a.e. y the vector $-\nu_E(y)$ lies in the convex set $K = \{\xi \in \mathbb{R}^n \colon \langle \xi, v\rangle \ge \lambda\}$. As $\mathrm{d}D\chi_E(y) = -\nu_E(y)\, \mathrm{d}\,|D\chi_E|(y)$, we obtain $\nabla u_\varepsilon(x) = -\alpha_\varepsilon^x \int_{\mathbb{R}^n} \nu_E(y)\, \mathrm{d}\mu_\varepsilon^x(y)$ that is $\nabla u_\varepsilon(x)/\alpha_\varepsilon^x$ is a weighted mean value of $\nu_E(y)$ and hence $\nabla u_\varepsilon \in \alpha_\varepsilon^x K$.

Suppose now by simplicity that $v = (0,1) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and let (z,t) be the two variables of $\mathbb{R}^{n-1} \times \mathbb{R}$. Since $\nabla u_\varepsilon(x) \in \alpha_\varepsilon^x K$ we notice that $\frac{\partial u_\varepsilon}{\partial t}(x) > 0$ for all $x \in \Omega$. By the Implicit Function Theorem we obtain that $\{u_\varepsilon = 1/2\} \cap \Omega$ is the graph of a function $f_\varepsilon \colon \mathbb{R}^{n-1} \to \mathbb{R}$ and we know that (letting $x = (z, f_\varepsilon(z))$)

$$|\nabla_z f_\varepsilon| = \left| -\nabla_z u_\varepsilon / \left(\frac{\partial u_\varepsilon}{\partial t}\right)\right| \le \frac{\alpha_\varepsilon^x}{(\lambda \alpha_\varepsilon^x)} = \frac{1}{\lambda}.$$

Up to a subsequence, we may suppose that there exists a lipschitz function $f \colon \mathbb{R}^{n-1} \to \mathbb{R}$ such that locally $f_\varepsilon \to f$ uniformily. On the other hand since $u_\varepsilon \to \chi_E$ in L^1_{loc} we have $\{u_\varepsilon \le 1/2\} \to \mathbb{R}^n \setminus E$ locally in measure, that is the subgraphs of f_ε converge in measure to $\mathbb{R}^n \setminus E$, hence ∂E is the graph of f.

Moreover, since the relation $\langle \nu_E, v\rangle = -\frac{1}{\sqrt{1+(\nabla f)^2}}$ holds almost everywhere, we get the estimate $|\nabla f| \le \sqrt{1/\lambda^2 - 1}$, which gives the value of the lipschitz constant. □

The following lemma provides a characterization of fat Wulff shapes in $\mathbb{R}^2$.

Lemma 3.11 *The set W_φ is fat if and only if the following property holds: there exist $\varepsilon_0 > 0$, $\delta_0 > 0$ and $0 < \delta < 1$ such that given any $X \subset \partial W_\varphi$ the following property holds:*

$$\mathrm{dist}(K_1, K_2) \le \delta_0 \quad \forall v_1, v_2 \in X \qquad \Longrightarrow \qquad \exists p \in \partial W_\varphi \text{ s.t. } \langle p, \xi\rangle > \varepsilon_0 \quad \forall \xi \in X^*, \tag{8}$$

where $K_i := \{\xi \in \partial\mathcal{F}_\varphi : \langle \xi, v_i\rangle \ge 1 - \delta\}$ and $X^* = \bigcup_{v\in X} v^*$.

Proof.
Notice that W_φ is fat if and only if there exists $\varepsilon_0 > 0$ such that for all $v_1, v_2 \in \partial W_\varphi$ with $v_1^* \cap v_2^* \neq \emptyset$ (that is v_1, v_2 belong to the same edge) there exists $p \in \partial W_\varphi$ such that given any $\xi \in v_1^* \cup v_2^*$ it holds $\langle \xi, p\rangle > \varepsilon_0$. The "if" part of the statement is then simply proved (take $X = \{v_1, v_2\}$).

For the "only if" part, reasoning by contradiction with $\delta_0 = \delta = 1/k$, we can find a sequence of sets $X_k \subset \partial W_\varphi$ such that (8) is false. Up to a subsequence we may assume that $X_k \to X$ (in the sense of Kuratowski). We claim that there exist $v_1, v_2 \in X$ such that $X^* = v_1^* \cup v_2^*$. In fact given any couple $v_1, v_2 \in X$ there exist $v_i^k \in X_k$ $(i = 1, 2)$ such that $v_i^k \to v_i$. By (8) we can find $\xi_i^k \in K_i^k = \{\xi \in \partial\mathcal{F}_\varphi : \langle \xi, v_i^k\rangle \ge 1 - 1/k\}$ such that $|\xi_1^k - \xi_2^k| \le 1/k$. Up to a subsequence we may suppose that $\xi_i^k \to \xi_i$, hence

$$\langle \xi_i, v_i\rangle = \lim_k \langle \xi_i^k, v_i^k\rangle \ge \lim_k (1 - 1/k) = 1$$

that is $\xi_i \in v_i^*$. On the other hand it holds $\xi_1 = \xi_2$, hence we have proved that $v_1^* \cap v_2^* \neq \emptyset$ and this is true for any couple $v_1, v_2 \in X$. Since W_φ is different from a triangle (because the triangle is *slim*) and $n = 2$ the claim is proved.

Let now $p \in \partial W_\varphi$ and choose $v_i \in X$, $v_i^k \in X_k$, $\xi_i^k \in K_i^k$, $\xi_1 = \xi_2 \in v_1^* \cap v_2^*$ as before. Since we have supposed that (8) is false, there exist $\xi_p^k \in X_k^*$ such that $\langle \xi_k^p, p\rangle \le \varepsilon_0$.

Again up to a subsequence we may assume that $\xi_p^k \to \xi_p \in X^* = v_1^* \cup v_2^*$ (indeed if $\xi_k \to \xi$ and $\xi_k \in X_k^*$ then $\xi \in X^*$). It holds $\langle \xi_p, p\rangle = \langle \xi_p - \xi_p^k, p\rangle + \langle \xi_p^k, p\rangle \le \varphi(\xi_p - \xi_p^k) + \varepsilon_0 \to \varepsilon_0$, which contradicts the characterization of *fatness* given at the beginning of the proof. □

We can now prove the main result of the paper.

Theorem 3.12 *Assume that W_φ is fat and let $u \in \mathcal{M}(\Omega)$. Then $\Gamma_u \subset \Omega \times \mathbb{R}$ is locally a lipschitz graph. Moreover, the lipschitz constant depends only on φ, on f and on the distance from $\partial\Omega$.*

Proof.
Let $x_0 \in \Omega$ and let $\rho_2 = \mathrm{dist}(x_0, \partial\Omega)/2$. Let ε_0, δ_0 and δ be the constants given by Lemma 3.11 and set $\varepsilon = \delta_0/3$. Let L be the lipschitz constant given by Theorem 3.9 with respect to ρ_2. Choose $C \ge 1$ such that for all ξ it holds $|\xi|/C \le \varphi(\xi) \le C|\xi|$ and set $\rho_1 := \delta/(2CL)$, $\rho_0 := \min\{\varepsilon\rho_1/(4C), \rho_1/2\}$.

We consider the family of subsets of Ω defined by

$$\mathcal{F} := \Big\{E_t : t \in \mathbb{R},\ \partial E_t \cap B_{\rho_0}(x_0) \neq \emptyset\Big\} \cup \Big\{F_t : t \in \mathbb{R},\ \partial F_t \cap B_{\rho_0}(x_0) \neq \emptyset\Big\}.$$

By Theorem 3.6 we have $\mathcal{F} \subset \mathcal{M}(\Omega)$, and by Theorem 3.9 given $E \in \mathcal{F}$ there exists an L-lipschitz function $v_E : \partial E \cap B_{\rho_2}(x_0) \to \partial W_\varphi$ such that $\langle \nu_E(x), v_E(x)\rangle = -\varphi(-\nu_E(x))$ for $\mathcal{H}^1$-a.e. $x \in \partial E$.

Consider also the set

$$X := \Big\{v_E(x) : x \in \partial E \cap B_{\rho_0}(x_0),\ E \in \mathcal{F}\Big\} \subseteq \partial W_\varphi,$$

we want to apply Lemma 3.11 to prove that there exists $p \in \partial W_\varphi$ such that $\langle \xi, p \rangle > \varepsilon_0$ for all $\xi \in X^*$.

Given $v_1, v_2 \in X$ we let $x_1, x_2 \in B_{\rho_0}(x_0)$ and $E_1, E_2 \in \mathcal{F}$ be such that (from now on $i = 1,2$) $x_i \in \partial E_i$ and $\nu_i = \nu^\varphi_{E_i}(x_i)$ (we let $\nu^\varphi_E(x) := \nu_E(x)/\varphi(-\nu_E(x))$), so that $\nu_i \in v_i^*$. Define also $K_i := \{\xi \in \mathbb{R}^2 : \langle \xi, v_i \rangle \geq (1-\delta)\varphi(\xi)\}$. By the L-lipschitz continuity of v_E we notice that for $x \in B_{\rho_1}(x_0) \cap \partial E_i$ it holds $\langle \nu^\varphi_{E_i}(x), v_i \rangle = \langle \nu^\varphi_{E_i}(x), v_{E_i}(x) \rangle + \langle \nu^\varphi_{E_i}(x), v_i - v_{E_i}(x) \rangle \geq 1 - CL|x - x_i| \geq 1 - 2L\rho_1 = 1 - \delta$. We have proved that $-\nu_{E_i}(x) \in K_i^\delta$ for $\mathcal{H}^1$-a.e. $x \in B_{\rho_1}(x_0) \cap \partial E_i$, and we can apply Lemma 3.8 in the ball $B_{\rho_1}(x_0)$ in order to obtain $\operatorname{dist}(K_1^\delta \cap \partial B_1, K_2^\delta \cap \partial B_2) \leq 4\rho_0/\rho_1 \leq \varepsilon/C$. So there exist $\xi_i \in K_i^\delta \cap \partial \mathcal{F}_\varphi$ such that $|\xi_1 - \xi_2| \leq \varepsilon$. By Lemma 3.11 we can find a vector $p \in \partial W_\varphi$ such that $\langle p, \xi \rangle > \varepsilon_0$ for all $\xi \in X^*$, and in particular for $\xi = \nu^\varphi_E(x)$ with $E \in \mathcal{F}$ and $x \in B_{\rho_0} \cap \partial E$.

Notice that, given $(x,t) \in \Gamma_u \cap (B_{\rho_0}(x_0) \times \mathbb{R})$, by Lemma 3.7 we get that $x \in \partial E$ for some $E \in \mathcal{F}$ or $\nu_{S_u}(x,t) = (0,1)$. In both cases we can write $\nu_{S_u}(x,t) = (\lambda \nu_E(x), 1)/\sqrt{1+\lambda^2}$ for some $\lambda \geq 0$ and we obtain

$$\langle \left(\frac{p}{|p|}, 1 \right), \nu_{S_u}(x,t) \rangle = \frac{\lambda \langle \nu_E(x), p \rangle + 1}{|p|\sqrt{1+\lambda^2}} \geq \frac{1}{|p|} \langle \nu_E(x), p \rangle > \frac{\varphi(-\nu_E(x))}{|p|} \varepsilon_0 \geq \frac{\varepsilon_0}{C^2},$$

since $\frac{a\lambda+1}{\sqrt{1+\lambda^2}} \geq a$ for all $\lambda \geq 0$, $a \leq 1$.

So, applying Lemma 3.10 we conclude the proof. □

4 Example

In this section we provide an example of a function $u \in \mathcal{M}(\mathbb{R}^2)$ such that the set of points where Γ_u is not locally the graph of a lipschitz function has positive $\mathcal{H}^2$-measure. A point $x \in \mathbb{R}^2$ will be denoted by its coordinates $x = (x_1, x_2)$. We set $\varphi(x) = |x_1| + |x_2|$ so that W_φ is a square. It is not difficult to show that in a similar way we can treat every other slim W_φ.

Given $a, b, t \in \mathbb{R}$, $a < b$, we define the sets $Q(a,b,t) := \{(x_1, x_2) : (3a+b)/4 < x_1 < (a+3b)/4,\ x_2 < t\}$, $L(a,b,t) := \{(x_1,x_2) : a < x_1 \leq (a+b)/2\} \setminus Q(a,b,t)$, $R(a,b,t) := \{(x_1,x_2) : (a+b)/2 < x_1 < b\} \setminus Q(a,b,t)$ and define $u_{a,b,t} = a\chi_{L(a,b,t)} + b\chi_{R(a,b,t)} + \frac{a+b}{2}\chi_{Q(a,b,t)} :]a,b[\times \mathbb{R} \to \mathbb{R}$ (see Figure 1).

Notice that $\Gamma_{u_{a,b,t}}$ is not the graph of a lipschitz function in any direction and in any neighbourhood of the point $((a+b)/2, t, (a+b)/2) \in \mathbb{R}^3$. However it can be proved (for example using a calibration) that $u_{a,b,t} \in \mathcal{M}(]a,b[\times \mathbb{R})$.

We now merge such functions so that the singular points accumulate in a Cantor like set with positive measure.

Let $a_k, b_k \in [0,1]$ be a sequence of points such that the intervals $I_k = [a_k, b_k]$ are disjoint and $K = \bigcap_k ([0,1] \setminus]a_k, b_k[)$ is a set with positive measure such that $K \subseteq \overline{\{b_k : k \in \mathbb{N}\}}$.

We then define $a_{kj} = b_k - \frac{b_k - a_k}{2^j}$, $b_{kj} = b_k - \frac{b_k - a_k}{2^{j+1}}$. The intervals $]a_{kj}, b_{kj}[$ are all disjoint and accumulate in the points b_k. Consider also an enumeration q_j of the rational numbers. The sets $L(a_{kj}, b_{kj}, q_j)$, $R(a_{kj}, b_{kj}, q_j)$ and $Q(a_{kj}, b_{kj}, q_j)$ are all disjoint so that we can define

$$u(x_1, x_2) = \begin{cases} a_{kj} & \text{if } (x_1, x_2) \in L(a_{kj}, b_{kj}, q_j) \\ b_{kj} & \text{if } (x_1, x_2) \in R(a_{kj}, b_{kj}, q_j) \\ (a_{kj} + b_{kj})/2 & \text{if } (x_1, x_2) \in Q(a_{kj}, b_{kj}, q_j) \\ x_1 & \text{elsewhere} \end{cases}$$

Every point $((a_{kj} + b_{kj})/2, q_j, (a_{kj} + b_{kj})/2)$ is a singular point for the function u and the closure of these points contains the whole set $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2) \in K,\ x_3 = x_1\}$ whose $\mathcal{H}^2$-measure is greater than $\mathcal{H}^2(K) > 0$.

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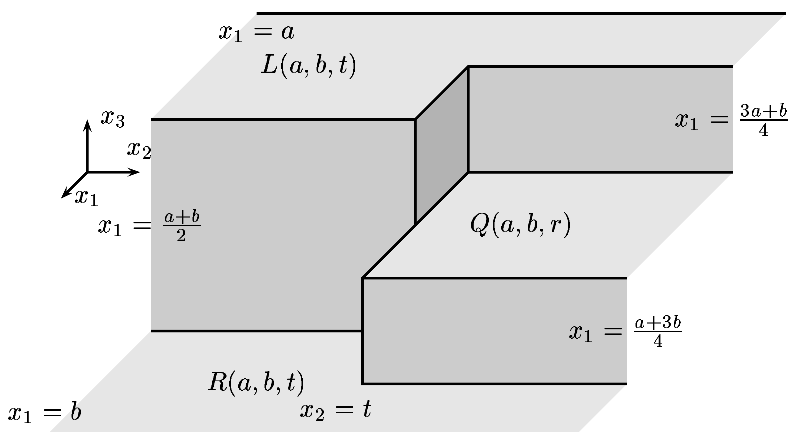

Figure 1: the construction of the function $u_{a,b,t}$