Elastic networks, statics and dynamics

Matteo Novaga and Alessandra Pluda

Abstract.

We consider planar networks minimizing the elastic energy, we state an existence and regularity result, and we discuss some geometric properties of minimal configurations. We also consider the evolution of networks by the gradient flow of the energy, and we give a well–posedness result in the case of natural boundary conditions.

§1. Introduction

We begin by defining the mathematical objects we consider in this paper.

Definition 1.1. *A planar* network $\mathcal{N}$ *is a connected set in* $\mathbb{R}^2$*, finite union of sufficiently smooth regular curves* $\mathcal{N}^i$ *that meet at junctions. In general we allow the presence of loops and* k*–valent vertices (see Figure 1). We say that the network is regular if it has only triple junctions and the unit tangent vectors of the three concurring curves form equal angles at the junction.*

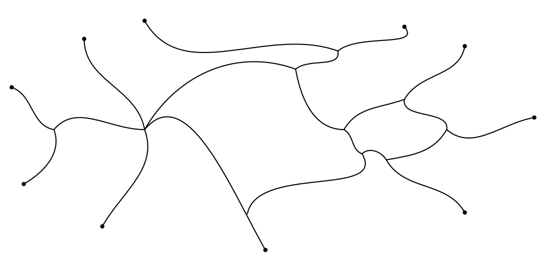

Fig. 1. A planar network

We just defined a network as a set in the Euclidean plane, but in the following it is more convenient to *parametrize* each curve $\mathcal{N}^i$ of a network $\mathcal{N}$ with maps $\gamma^i : I \subset \mathbb{R} \to \mathbb{R}^2$. A curve is of class C^k (or

H^k) if it admits a parametrization of class C^k (or H^k, respectively). We remind that a curve is regular if $|\gamma'(x)| \neq 0$ for every $x \in I$.

Definition 1.2. *Let $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$. The* elastic energy functional *for a network $\mathcal{N}$ is defined as*

$$E(\mathcal{N}) := \int_{\mathcal{N}} \alpha k^2 + \beta \,\mathrm{d}s = \sum_i \int_{\mathcal{N}^i} \alpha (k^i)^2 + \beta \,\mathrm{d}s^i \,, \tag{1}$$

where k^i is the curvature and s^i the arclength parameter of the curve $\mathcal{N}^i$.

Fixed a certain class $\mathcal{C}$ of planar networks, that is fixed the topology of the network in the class (the number of curves and number of junctions) we are interested in the following two problems:

i) finding the minimizers of E among all networks in $\mathcal{C}$;
ii) studying the solutions of the L^2 gradient flow of the energy E with initial datum $\mathcal{N}_0$ in $\mathcal{C}$. In particular, we look for classical solution of the flow requiring that the evolving network satisfies certain boundary conditions for all times.

Looking back to the origin of elasticity theory for objects that can be modeled by one dimensional sets, we see that the first attempts regard rods and strings. That means reducing to the case of the simplest possible network: a (closed or open) curve Γ. Moreover the parameters in (1) are fixed to $\alpha = 1$ and $\beta = 0$. The problem was already considered at the times of Galileo, when clearly scientists had a mechanical point of view: they looked for equations for equilibrium of moments and forces. The idea to relate the curvature of the fiber of the beam to the bending moment came only later when in 1691 Jacob Bernoulli proposed to model the bending energy of thin inextensible elastic rods with a functional involving the curvature. Finally in 1744 Daniel Bernoulli, following the principle that the potential energy of the elastic lamina must be minimal, proposed to attach the problem with variational techniques. The modern formulation of Bernoulli's Problem reads as follow: find

$$\min \Big\{ E(\Gamma) := \int_\Gamma k^2 \,\mathrm{d}s \;\Big|\; \Gamma : [a,b] \subset \mathbb{R} \to \mathbb{R}^2 \,, \\ \Gamma(a) = \alpha, \Gamma(b) = \beta, \Gamma'(a) = \tau_a, \Gamma'(b) = \tau_b \Big\} .$$

A natural approach to this problem is to study the critical points of the functional E, looking at the Euler–Lagrange equation, which involves four derivatives of Γ: indeed it reads as

$$2k_{ss} + k^3 = 0 \,, \tag{2}$$

where k is the scalar curvature and s the arclength parameter. Euler was able to treat these kind of equations and divided the critical points of the functional in nine classes. He called *elastica* a solution of (2). More recently Langer and Singer contributed to the classification of the elasticae [20, 21].

We remark here that the apparently easier problem

$$\min\left\{E(\Gamma) := \int_\Gamma k^2\,\mathrm{d}s \mid \Gamma : [a,b] \subset \mathbb{R} \to \mathbb{R}^2\,,\ \Gamma \text{ closed}\right\}$$

is not well–posed. Indeed the infimum is zero and not attained. To convince yourself of this fact it is enough to consider a sequence of circles $\mathcal{C}_R$ with radius R, then $E(\mathcal{C}_R)$ goes to zero as $R \to \infty$ and the value zero is not attained. To overcome the bad–posedness, in the literature the problem was treated with several different additional constraints. Between the others, it is worth to mention:

- the length penalization (that is one asks for $\beta > 0$ in (1));
- restrict the ambient space from $\mathbb{R}^2$ to a bounded domain (confined elastica, see [14, 15]);
- defining the class $\mathcal{C}$ as the set of all embedded closed curves that enclosed a fixed area. To be more precise consider the class of all smooth, simply connected and bounded open set $\Omega \subset \mathbb{R}^2$, bounded by a Jordan curve γ, and call $A(\Omega)$ the area of Ω. One looks for

  $$\min\{E(\partial\Omega) := E(\gamma)\;\Big|\; A(\Omega) = A_0\}\,.$$

  Apart from proving that the unique minimizer is the disc, it has also been established (see [7, 17]) the isoperimetric inequality

  $$E^2(\partial\Omega)A(\Omega) \geq 4\pi^3\,.$$

Let us now consider the "opposite" case, i.e., letting $\alpha = 0$ and $\beta = 1$ in the definition of the energy functional (1). With such choice of the parameters we are looking for a network of minimal length. If we do not fix some points of the network the problem is not well–posed: indeed the length goes to zero and the network reduces to a point. So it is meaningful to consider the class of networks with n end–points of order one fixed in the plane. If we require the networks to be connected but we do not fix the topology, this gives the well–know Steiner Problem: called S a collection of n points $p_1, \dots, p_n$ in the Euclidean plane, find

$$\min\left\{L(K) = \int_K 1\,\mathrm{d}s \;\Big|\; K \text{ connected network such that } S \subset K\right\}. \tag{3}$$

The minimal networks are composed of straight segments meeting at triple junctions forming equal angles. The proof of existence of minimizers is considered nowadays very classical, but finding explicitly solutions is extremely challenging (even numerically) because of the lack of uniqueness due to the strong non convexity of the problem. For this reason every method to determine solutions is welcome. Motivated by the research for fast algorithms, the resolution of the Steiner problem by variational methods has recently aroused an increasing interest (see for instance [1, 2, 3, 6, 23]).

Another celebrated problem related to the minimization of the length is the isoperimetric problem. In this case we do not fix points of the network but we fix the area of the regions in which the network divides the plane, in a certain sense the topology of its complementary. More precisely an embedded compact network in $\mathbb{R}^2$ define a partition of the plane in finitely many bounded sets $\mathcal{E}_1, \dots, \mathcal{E}_n$ and an unbounded one $\mathcal{E}_0 := \mathbb{R}^2 \setminus \cup_{i=1}^n \mathcal{E}_i$. We ask the sets $\mathcal{E}_i$ to be open and we fix their area to m_i. We call $\mathcal{E}_i$ chambers and $\mathcal{E} = (\mathcal{E}_1, \dots, \mathcal{E}_n)$ cluster. Thus one want to find

$$\min \left\{ L(\Gamma) \,\middle|\, \Gamma^{\mathrm{c}} = \mathcal{E} = (\mathcal{E}_1, \dots, \mathcal{E}_n) \text{ with } |\mathcal{E}_i| = m_i \right\} .$$

For a detailed analysis of this problem see [22, 24].

§2. Minimization problems

As in the case of the length, also the minimization of the elastic energy of networks presents several possible variants. We shall present in particular three problems.

First suppose that we fix n points in $\mathbb{R}^2$ and $\alpha, \beta > 0$ in (1) and in the same spirit of problem (3) we search for

$$\min \left\{ E(K) \,\middle|\, K \text{ connected network such that } S \subset K \right\} .$$

The expected minimizers (for every $\alpha > 0$ and $\beta > 0$) coincide with the Steiner networks (solution of the Steiner Problem (3)), as the length is minimal and the term that involves the curvature is zero.

The second option is clearly inspired by the isopetrimetric problem. This time we want to minimize the functional E, defined in (1), among all networks that give an n–cluster with the area of each chamber fixed. This question is widely open, the only known case is $n = 1$, proven in [7, 17].

Last but not least, we look for minimizers of the elastic energy among networks with fixed topology. For simplicity we restrict to networks of three curves that meet at two junctions, namely Theta–networks, and we fix (non zero) angles at the junctions points. More precisely we are interested in:

$$\inf\{E\,(\Gamma)\,|\,\Gamma \text{ is a Theta–network }_{(\alpha_1,\alpha_2,\alpha_3)}\,\}\,. \tag{4}$$

where a Theta–network $_{(\alpha_1,\alpha_2,\alpha_3)}$ is a network composed of three curves that meet at two triple junctions with fixed angles α_1 between γ^1 and γ^2, α_2 between γ^2 and γ^3 and, as a consequence, α_3 between γ^3 and γ^1 with $\sum_{i=1}^{3}\alpha_i = 2\pi$. Moreover each curve is regular and of class H^2.

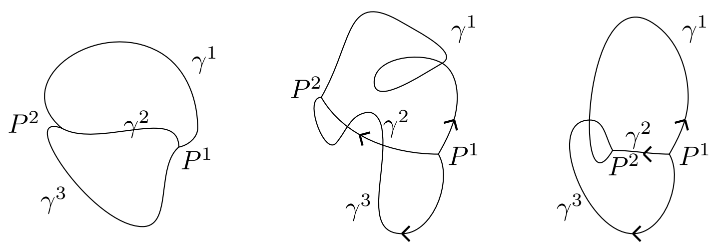

Fig. 2. Examples of Theta-networks.

Why do we fix the angles at the junctions? Suppose instead that we minimize E (with $\alpha, \beta > 0$) on the set of Theta–networks without fixing the angles. Then the infimum is zero and is clearly not attained in the class of regular networks (think for instance of a minimizing sequence given by three equal and overlapped segments whose length goes to zero). Fixing non–zero angles at the junctions we overcome this issue, since it implies a lower bound on the energy. An embedded Theta–network can be seen as two closed curve (with a common piece). Consider two of the three curves of the Theta network that meet forming an (internal) angle of θ_1 at the junction P^1 (θ_2 at P^2 respectively), and call γ the union of these two curves. Combining the Gauss–Bonnet Theorem (in the slightly generalized version proved in [10, Appendix A]) with Hölder inequality we get

$$c := 2\pi - (\pi - \theta_1) - (\pi - \theta_2) \leq \int_\gamma |k|\,\mathrm{d}s \leq \left(\int_\gamma k^2\,\mathrm{d}s\right)^{1/2} L^{1/2}(\gamma)\,,$$

then

$$E(\gamma) = \alpha \int_\gamma k^2 \,\mathrm{d}s + \beta L(\gamma) \geq \alpha \frac{c^2}{L(\gamma)} + \beta L(\gamma) \geq 2\sqrt{\alpha\beta}\, c\,.$$

Hence, by fixing the angles $\theta_i \neq 0$, we get an uniform bound from below for the energy.

2.1. Existence of the minimizers

First of all we notice that we can reduce to the case $\alpha = \beta = 1$. Indeed, letting

$$E_{\alpha,\beta}(\mathcal{N}) = \int_{\mathcal{N}} \alpha k^2 + \beta \,\mathrm{d}s \qquad \text{and} \qquad E_{1,1}(\mathcal{N}) = \int_{\mathcal{N}} k^2 + 1 \,\mathrm{d}s\,,$$

and recalling the rescaling properties of the energy, we have

$$E_{\alpha,\beta}(\mathcal{N}) = \sqrt{\alpha\beta}\, E_{1,1}\left(\sqrt{\frac{\beta}{\alpha}}\mathcal{N}\right)\,,$$

so if $\mathcal{N}_{\min}$ is a minimizer for $E_{1,1}$, then the rescaled network $\frac{\beta}{\alpha}\mathcal{N}_{\min}$ is a minimizer for $E_{\alpha,\beta}$, and vice versa. From now on for simplicity we write E for $E_{1,1}$.

We introduce the notion of "degenerate" Theta–network$_{(\alpha_1,\alpha_2,\alpha_3)}$: a network composed by two regular curves of class H^2, forming angles in pairs of α_i and $\pi - \alpha_i$ degrees, for some $i \in \{1,2,3\}$, and by a curve of length zero. Moreover we define the functional $\overline{E}$ on networks Γ of three curves of class H^2 as follows:

$$\overline{E}(\Gamma) = \begin{cases} \displaystyle\sum_{i=1}^{3} \frac{1}{L^3(\gamma^i)} \int_0^1 (\gamma^i_{xx})^2 \,\mathrm{d}x + L(\gamma^i) & \text{if } \Gamma \text{ is Theta–network,} \\[2em] \displaystyle\sum_{i=1}^{2} \frac{1}{L^3(\gamma^i)} \int_0^1 (\gamma^i_{xx})^2 \,\mathrm{d}x + L(\gamma^i) & \text{if } \Gamma \text{ is degenerate,} \\[2em] +\infty & \text{otherwise.} \end{cases}$$

The two functionals E and $\overline{E}$ coincide on Theta–networks, indeed E is geometrical, invariant by reparametrization and here we simply parametrized each curve proportionally to arclength.

Theorem 1 (see [11, 10])**.** *The functional $\overline{E}$ is the relaxation of the functional E in H^2. Moreover there exists $\Gamma_{\min}$ Theta–network or* *"degenerate" Theta–network minimizer of the functional $\overline{E}$ among 3–networks of class H^2. Each curve $\widetilde{\gamma}$ of $\Gamma_{\min}$ is of class C^∞ and solves the equation* (2)*.*

Without loss of generality we can assume $\alpha_1 \leq \alpha_2 \leq \alpha_3$.

Theorem 2 (see [11, 10])**.** *Suppose that $\alpha_1 \leq \alpha_2 \leq \frac{4\pi}{3}$, then the minimizers of the functional $\overline{E}$ are Theta–networks with injective curvers, that is there exists minimizers for Problem* (4)*.*

§3. Gradient flow of the energy in the class of networks

We pass now to consider the dynamical counterpart of the problem. The questions we want to answer are the following: we fix a certain class of networks $\mathcal{C}$ and we take a network $\mathcal{N}_0$ in $\mathcal{C}$. Does it exists an evolution law that for all times $t \in [0, +\infty)$ transform the network $\mathcal{N}_0$ into a network $\mathcal{N}_t$ (still in the class C) decreasing the elastic energy along the flow? And if we want that the energy not only decrease, but that it decreases as efficiently as possible?

The evolution we have in mind can be understood as the L^2-gradient flow of the energy E where the normal velocity induces the steepest descent of the energy.

3.1. The evolution equation

How do we find an explicit expression for the velocity? Formally we derive the evolution equation by computing the first variation of E. We take a map $\gamma^i : [a,b] \to \mathbb{R}^2$ that parametrized the curve $\mathcal{N}^i$ of the network and we compute how the energy changes if we vary this parametrization by adding $\tau\psi^i$ with $\psi^i \in C^\infty([a,b], \mathbb{R}^2)$. We call the modified parametrization $\widetilde{\gamma}^i := \tau\psi^i$. If we think of ψ^i encoding the evolution in time, we can now answer the question what equations an evolution has to satisfy that decreases E most efficiently. The Cauchy Schwarz inequality implies that the normal component of the velocity $\boldsymbol{v}^i$ of each curve $\mathcal{N}^i$ of the network is given by

$$\left\langle \boldsymbol{v}^i, \nu^i \right\rangle = \alpha(-2k^i_{ss} - (k^i)^3) + \beta k^i \qquad \text{on } [0,\infty) \times [a,b], , \tag{5}$$

where ν^i is the unit normal vector of the curve γ^i. These equations are coupled with conditions at the end points of the curves $\mathcal{N}^i$. To maintain the structure (topology) of the network we have to require something on the ψ^i. For instance if two curves $\mathcal{N}^i, \mathcal{N}^j$ (parametrized by γ^i, γ^j) meet at their end points in a junction (let say $\gamma^i(a) = \gamma^j(a)$) then

$\psi^i(a) = \psi^j(a)$. In the end, at each junction point we get

$$k^i = 0 \quad \text{and} \quad \sum_{i=1} \alpha(-2k^i_s \nu^i - \left(k^i\right)^2 \tau^i) + \beta\tau^i = 0 \tag{6}$$

where ν^i and τ^i are the unit normal and tangent vector, respectively. Instead at the (fixed) end point of order one the curvature has to be zero (see [4, 11, 19]).

Once we have established what the system of equations is, we want to show that there exists a solution, (at least for a shot time). The strategies to prove existence of classical solutions for all $t \in [0, +\infty)$ can be divided into two steps: local (in time) and global existence. The techniques are very different.

Before proceeding with our analysis we point out that:

- if $\alpha = 1$, $\beta = 0$ we have the Willmore flow;
- if $\alpha > 0$, $\beta > 0$ the flow is called "curve straight–shortening flow";
- if $\alpha = 0$, $\beta = 1$ we have the curve shortening flow (one dimensional Mean Curvature Flow).

Several authors considered the first two cases for the simplest network: a (closed or open) curve (see for instance [8, 12, 13, 16, 25, 26]).

3.2. Short–time existence

We shall now state the existence result for a solution of the flow.

Theorem 3 (see [19]). *Let $\mathcal{N}_0$ be a geometrically admissible initial network for the curve straight–shortening flow. Then there exists a positive time T such that within the interval $[0,T]$ the flow admits a unique solution.*

We have seen that our evolution problem is geometric, indeed both the motion equations (5) and the boundary conditions (6) are related only to geometric quantities and independent on the parametrization of the curves. But unfortunately the motion equations (5) are degenerate. Although we consider variations in any direction *only the normal movement* is specified. To turn these degenerate equations into a well–posed parabolic system of quasilinear PDEs one has to specify a suitable tangential movement. Equations (5) have the following structure:

$$\left\langle \gamma^i_t, \nu^i \right\rangle \nu^i = -A^i \nu^i = -\left\langle 2\frac{\gamma^i_{xxxx}}{|\gamma^i_x|^4} + g(\gamma^i_x, \gamma^i_{xx}, \gamma^i_{xxx}), \nu^i \right\rangle \nu^i \,.$$

Choosing as tangential velocity

$$T^i := \left\langle 2\frac{\gamma^i_{xxxx}}{|\gamma^i_x|^4} + g(\gamma^i_x, \gamma^i_{xx}, \gamma^i_{xxx}), \tau^i \right\rangle ,$$

we get

$$\gamma_t = -2\frac{\gamma_{xxxx}}{|\gamma_x|^4} + \tilde{f}(\gamma_{xxx}, \gamma_{xx}, \gamma_x) .$$

One notices that even the boundary conditions of order two have a tangential degree of freedom. Proceeding as for the motion equations we arrive to the following system:

$$(7) \quad \begin{cases} \gamma^i_t(t,x) = -A^i(t,x)\nu^i(t,x) - T^i(t,x)\tau^i(t,x) & \\ \gamma^1(t,y) = \gamma^2(t,y) = \gamma^3(t,y) & \text{for } y \in \{0,1\} \\ \gamma^i_{xx}(t,y) = 0 & \text{for } y \in \{0,1\} \\ \sum_{i=1}^3 \left(2k^i_s\nu^i - \mu\tau^i\right)(t,y) = 0 & \text{for } y \in \{0,1\} \\ \gamma^i(0,x) = \varphi^i(x) & \end{cases}$$

for every $t \in [0,T), x \in [0,1]$ and for $i \in \{1,2,3\}$ with φ^i admissible initial parametrization. Once solved in a unique way system (7), the last step is to pass from the parametrizations back to the curve and the degenerate geometric problem. A crucial point is to check that the tangential velocity does not change the geometry of the problem, and can always be obtained by reparametrizing the curves in the right way.

3.3. Long time behavior

Recently also the long time behavior of the elastic flow of networks has been considered (see [18, 9]).

Theorem 4. *Let $\mathcal{N}_0$ be a geometrically admissible initial network. Suppose that $(\mathcal{N}(t))_{t\in[0,T_{\max})}$ is a maximal solution to the elastic flow with initial datum $\mathcal{N}_0$ in the maximal time interval $[0,T_{\max})$ with $T_{\max} \in (0,\infty) \cup \{\infty\}$. Then*

$$T_{max} = \infty$$

or at least one of the following happens:

(i) *the inferior limit of the length of at least one curve of $\mathcal{N}(t)$ is zero as $t \nearrow T_{max}$.*

(ii) *at one of the triple junctions*

$$\liminf_{t\nearrow T_{max}} \max\left\{\left|\sin\alpha^1(t)\right|, \left|\sin\alpha^2(t)\right|, \left|\sin\alpha^3(t)\right|\right\} = 0 ,$$

where $\alpha^1(t)$, $\alpha^2(t)$ and $\alpha^3(t)$ are the angles at the respective triple junction.

§4. Open problems

We list below a few open questions which we find interesting.

4.1. Statics

- Is the minimizer of Theorem 2 unique? Is it an embedded, symmetric Theta–network?
- Does Theorem 2 hold if two angles are greater then $\frac{3\pi}{4}$ or instead there exists a minimizer of Problem (4) with a 4–point in the class of Theta-networks$_{(\alpha_1,\alpha_2,\alpha_3)}$?
- Generalize Problem (4) to networks composed of N curves; in particular find the lower semicontinuous envelope of the energy in general situations (for result in this direction we refer to [5] in the case $N = 1$ in the more complicated setting of elastic clusters).

4.2. Dynamics

- There are self–similar solutions of the elastic flow?
- Analysis of the possible singularities of the flow listed in Theorem 4 and definition of the flow past singularities.
- Definition of a global weak solution.

References

[1] S. Amato, G. Bellettini, and M. Paolini, Constrained BV functions on covering spaces for minimal networks and Plateau's type problems, Adv. Calc. Var., **10** (2017), 25–47.

[2] L. Ambrosio, and A. Braides, Functionals defined on partitions in sets of finite perimeter. I. Integral representation and Γ-convergence, J. Math. Pures Appl. (9), **69** (1990), 285–305.

[3] L. Ambrosio, and A. Braides, Functionals defined on partitions in sets of finite perimeter. II. Semicontinuity, relaxation and homogenization, J. Math. Pures Appl. (9), **69** (1990), 307–333.

[4] J. W. Barrett, H. Garcke, and R. Nürnberg, Elastic flow with junctions: variational approximation and applications to nonlinear splines, Math. Models Methods Appl. Sci., **22** (2012), 1250037, 57 pp.

[5] G. Bellettini, and L Mugnai, Characterization and representation of the lower semicontinuous envelope of the elastica functional, Ann. Inst. H. Poincaré Anal. Non Linéaire, **21** (2004), 839–880.

[6] M. Bonafini, G. Orlandi, and E. Oudet, Variational approximation of functionals defined on 1-dimensional connected sets: the planar case, to appear: SIAM J. Math. Anal.

[7] D. Bucur, and A. Henrot, A new isoperimetric inequality for elasticae, J. Eur. Math. Soc. (JEMS), **19** (2017) 3355–3376.

[8] A. Dall'Acqua, C. C. Lin, and P. Pozzi, Evolution of open elastic curves in $\mathbb{R}^n$ subject to fixed length and natural boundary conditions, Analysis (Berlin), **34** (2014), 209–222.

[9] A. Dall'Acqua, C. C. Lin, and P. Pozzi, Flow of elastic networks: long-time existence result, arXiv:1812.11367.

[10] A. Dall'Acqua, M. Novaga and A. Pluda, Minimal elastic networks, to appear: Indiana Univ. Math. J.

[11] A. Dall'Acqua, and A. Pluda, Some minimization problems for planar networks of elastic curves, Geom. Flows **3** (2017), 105–124.

[12] A. Dall'Acqua, and P. Pozzi, A Willmore-Helfrich L^2-flow of curves with natural boundary conditions, Comm. Anal. Geom., **22**, (2014), 617–669.

[13] A. Dall'Acqua, P. Pozzi, and A. Spener, The Lojasiewicz- Simon gradient inequality for open elastic curves, J. Differential Equations, **261** (2016), 2168–2209.

[14] F. Dayrens, S. Masnou, and M. Novaga, Existence, regularity and structure of confined elasticae, ESAIM Control Optim. Calc. Var., **24**, (2018), 25–43.

[15] P. W. Dondl, L. Mugnai and M. Röger, Confined elastic curves, SIAM J. Appl. Math., **71** (2011), 2205–2226.

[16] G. Dziuk, E. Kuwert, and R. Schätzle, Evolution of elastic curves in $\mathbb{R}^n$: existence and computation, SIAM J. Math. Anal., **33** (2002) 1228–1245.

[17] V. Ferone, B. Kawohl, and C. Nitsch, The elastica problem under area constraint, Math. Ann., **365** (2016), 987–1015.

[18] H. Garcke, J. Menzel, and A. Pluda, Long Time Existence of Solutions to an Elastic Flow of Networks arXiv:1901.03246.

[19] H. Garcke, J. Menzel, and A. Pluda, Willmore flow of planar networks, to appear: J. Differential Equations.

[20] J. Langer, and D. A. Singer, The total squared curvature of closed curves, J. Differential Geom., **20** (1984), 1–22.

[21] J. Langer, and D. A. Singer, Curve straightening and a minimax argument for closed elastic curves, Topology, **24** (1985), 75–88.

[22] F. Maggi, Sets of Finite Perimeter and Geometric Variational Problems. An introduction to geometric measure theory, Cambridge University Press, Cambridge (2012).

[23] A. Marchese, and A. Massaccesi, The Steiner tree problem revisited through rectifiable G-currents, Adv. Calc. Var., **9** (2016), 19–39.

[24] F. Morgan, Geometric measure theory, A beginner's guide, Academic Press, Inc., San Diego, CA, (2000).

[25] M. Novaga, and S. Okabe, Curve shortening-straightening flow for non-closed planar curves with infinite length, J. Differential Equations, **256** (2014), 1093–1132.

[26] A. Polden, Curves and Surfaces of Least Total Curvature and Fourth-Order Flows, PhD Thesis, Universität Tübingen, (1996).

Dipartimento di Matematica, Università di Pisa, Largo Pontecorvo 5, 56127 Pisa, Italy
E-mail address: matteo.novaga@unipi.it

Fakultät für Mathematik, Universität Regensburg, Universitätsstrasse 31, 93053 Regensburg, Germany
E-mail address: alessandra.pluda@ur.de