Existence and uniqueness for planar anisotropic and crystalline curvature flow

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Abstract.

We prove short-time existence of φ-regular solutions to the planar anisotropic curvature flow, including the crystalline case, with an additional forcing term possibly unbounded and discontinuous in time, such as for instance a white noise. We also prove uniqueness of such solutions when the anisotropy is smooth and elliptic. The main tools are the use of an implicit variational scheme in order to define the evolution, and the approximation with flows corresponding to regular anisotropies.

Contents

§1. Introduction

In this paper we consider the anisotropic curvature flow of planar curves, corresponding to the evolution law

$$V \;=\; \kappa_\varphi + \frac{\partial G}{\partial t} \tag{1}$$

in the Cahn-Hoffmann direction n_φ. We shall assume that the forcing term G has the form $G = G_1 + G_2$ with G_1, G_2 satisfying:

i) $G_1 \in C^0([0,\infty))$ does not depend on x;
ii) $\partial_t G_2 \in Lip([0,\infty) \times \mathbb{R}^2)$.

Observe that (1) is only formal, as $\partial G_1/\partial t$ does not necessarily exist, however the motion can still be defined in an appropriate way (see Definition 2). Notice also that we include the case of G being a typical path of a Brownian motion, which is necessary to take into account a stochastic forcing term as in [18, 31].

In the smooth anisotropic case, the first existence and uniqueness results of a classical evolution in can be found in [5], where S. Angenent showed existence, uniqueness and comparison for a class of equations which include (1) in the case $G = G_2$ and φ regular. The existence and uniqueness of a weak solution for the forced flow, with a Lipschitz continuous forcing term, follows from standard viscosity theory [16, 17].

The crystalline curvature flow was mathematically formalized by J. Taylor in a series of papers (see for instance [35, 36]). In two-dimensions, when the driving force G is constant the existence

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of the flow reduces to the analysis of a system of ODEs. It was first shown by F.J. Almgren and J. Taylor in [1], together with a proof of consistency of a variational scheme similar to the one introduced in Section 3. The uniqueness and comparison principle in this case were established shortly after by Y. Giga and M.E. Gurtin in [29]. We also refer to [23] for a well-posedness result based on the theory of maximal monotone operators. The forced crystalline flow was studied in [8], however with strong hypotheses on the forcing to ensure the preservation of the facets. A theory for weak solution, in 2D, has been developed by Giga and Giga in the past recent years [25]. In this framework, existence and uniqueness for quite general weak motions have been established, however in general with constant forcing terms.

It is only in relatively recent work that the flow has been studied with a quite general forcing term; in particular, in [27, 28, 26] (see also [33, 34]) a Lipschitz forcing is considered. The papers [27, 26] consider the evolution of graphs, with a quite general mobility, while [28] considers only rectangular anisotropies, and assumes that the initial datum is close to the Wulff shape. The paper [15] deals with quite general forcing terms (slightly less regular than in this paper), but requires the anisotropy to be smooth. It shows the consistency of the variational scheme and a comparison for regular evolutions. In [7], the authors show the existence of convex crystalline evolutions (extending their results of [6]) with time-dependent (bounded) forcing terms and apply it to show the existence of volume preserving flows.

We show here a general existence result for the two-dimensional crystalline curvature flow, only with "natural" mobility, but which holds in the two following cases: for a general forcing $G = G_1$ depending only on time, and for a regular forcing $G = G_2$ with $\partial G_2/\partial t$ Lipschitz continuous in space and time. Our proof relies on estimates for the variational scheme introduced in [2, 32], which show that, if the initial curve has a strong regularity (expressed in terms of an internal and external Wulff shape condition), then this regularity is preserved for some time which depends only on the initial set. This allows us to approximate a general anisotropic flow with evolutions corresponding to smooth anisotropies, in such a way that the anisotropic curvature stays bounded for a uniform time interval. Extending these proofs to higher dimension would require quite strong regularity results for nonlinear elliptic PDEs, which do not seem available at a first glance.

Stability results for anisotropic evolutions have been proved in [24, 25] in the context of viscosity solutions. We also mention the paper [30], where a similar approximation argument is applied to the diffuse interface case.

The paper is organized as follows: in the Section 2 we define the "anisotropy" and introduce our notion of a "regular" curvature flow for smooth and nonsmooth anisotropies. In Section 3 we study the time-discrete implicit scheme of [2], and extend some regularity results of [6] to the flow with forcing. We then show in Section 4 the main existence result for smooth anisotropies. The fundamental point is that the time of existence only depends on the (intrinsic) $C^{1,1}$-regularity of the initial curve. In the smooth case, we also show uniqueness of regular evolutions. Eventually, in Section 5 we extend the existence result to the crystalline case. This follows from an elementary approximation result (see Lemma 1) and the fact that the time of existence is uniformly controlled in this approximation.

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§2. RW_φ-condition and φ-regular flows

We call *anisotropy* a function φ which is convex, one-homogeneous and coercive on $\mathbb{R}^2$. We will also assume that φ is even, i.e. φ is a norm, although we expect that the results of this paper still hold in the general case (but some proofs become more tedious to write).

We will always assume that there exists $c_0 > 0$ such that

$$c_0|x| \;\le\; \varphi(x) \;\le\; c_0^{-1}|x| \qquad \forall x \in \mathbb{R}^2. \tag{2}$$

We denote by φ° the polar of φ, defined as

$$\varphi^\circ(\nu) := \sup_{x:\,\varphi(x)\le 1} \nu\cdot x \qquad \nu\in\mathbb{R}^2,$$

it is obviously also a convex, one-homogeneous and even function on $\mathbb{R}^2$. Notice that from (2) it easily follows

$$c_0|\nu| \ \le\ \varphi^\circ(\nu)\ \le\ c_0^{-1}|\nu| \qquad \forall\nu\in\mathbb{R}^2.$$

We denote by $W_\varphi := \{\varphi\le 1\}$ the unit ball of φ, which is usually called the *Wulff shape*.

We say that φ is *smooth* if $\varphi\in C^2(\mathbb{R}^2\setminus\{0\})$ and φ is *elliptic* if φ^2 is strictly convex, that is $\nabla^2(\varphi^2)\ge c\,\mathrm{Id}$ in the distributional sense, for some $c>0$. It is easy to check that φ is smooth and elliptic iff φ° is smooth and elliptic.

Given a set $E\subset\mathbb{R}^2$ we let d^E_φ be the signed φ-distance function to ∂E defined as

$$d^E_\varphi(x) := \min_{y\in E}\varphi(x-y) - \min_{y\notin E}\varphi(y-x)\,,$$

We let $\nu^E_\varphi := \nabla d^E_\varphi$ be the exterior φ-normal to ∂E, $n_\varphi\in\partial\varphi^\circ(\nu_\varphi)$ be the so-called *Cahn-Hoffmann* vector field (where ∂ denotes the subdifferential), and $\kappa_\varphi := \mathrm{div} n_\varphi$ be the φ-curvature of ∂E, whenever they are defined. We also set $E^c := \mathbb{R}^2\setminus E$.

Following [11] we give the following definition:

Definition 1 (**RW_φ-condition**)**.** *We say that a set E satisfies the inner RW_φ-condition for some $R>0$ if*

$$\overline{E} = \bigcup_{x:\,d^E_\varphi(x)\le -R} (x+RW_\varphi) \tag{3}$$

and for any $r<R$ and $x\in\mathbb{R}^2$, $(x+rW_\varphi)\cap E^c$ is connected.

We say that E satisfies the outer RW_φ-condition if its complementary E^c satisfies the inner RW_φ-condition.

We say that E satisfies the RW_φ-condition if it satisfies both the inner and outer RW_φ-conditions.

Remark 1. Notice that, if E satisfies the RW_φ-condition for some $R>0$, then ∂E is locally a Lipschitz graph. Moreover, when φ is smooth, the RW_φ-condition implies that ∂E is of class $C^{1,1}$ and $|\kappa_\varphi|\le 1/R$ a.e. on ∂E. In this case the connectedness condition in Definition 1 is automatically satisfied whenever (3) holds. However, in the nonsmooth case one can have some pathological examples if one removes the connectedness condition, as the one depicted in Fig. 1 when the Wulff shape is a square.

Remark 2. It is not difficult to show that E satisfies the inner RW_φ-condition iff (3) holds and the following property holds: for all x such that $d^E_\varphi(x) = -R' > -R$, the set $\partial E\cap(x+R'\partial W_\varphi)$ is connected.

By (3) it follows that $\partial E\cap(x+R\partial W_\varphi)$ is either a segment (possibly a point), or the union of two segments. In particular, if φ is elliptic, this is equivalent to say that there exists a unique point in ∂E minimizing the φ-distance from x.

Definition 2 (**φ-regular flows**)**.** *We say that a map $[0,T]\ni t\to\mathcal{P}(\mathbb{R}^2)$ defines a φ-regular flow for* (1) *if*

1. *$E(t)$ satisfies the RW_φ-condition for all $t\in[0,T]$ and for some $R>0$;*
2. *there exist an open set $U\subset\mathbb{R}^2$ and a vector field $z\in L^\infty([0,T]\times U;\mathbb{R}^2)$ such that*
   - (a) *$\partial E(t)\subset U$ for all $t\in[0,T]$,*
   - (b) *$d^E_\varphi(t,x) := d^{E(t)}_\varphi(x)\in C^0([0,T];\mathrm{Lip}(U))$,*
   - (c) *$z\in\partial\varphi^\circ(\nabla d^E_\varphi)$ a.e. in $[0,T]\times U$,*
   - (d) *$\mathrm{div}z\in L^\infty([0,T]\times U)$;*

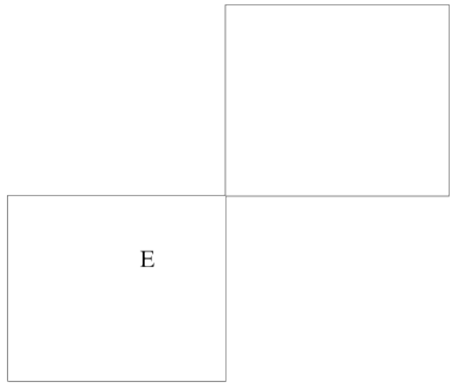

Fig. 1. The pathological set described in Remark 1.

(3) *there exists $\lambda > 0$ such that, for any t, s with $0 \le t < s \le T$ and a.e. $x \in U$, there holds*

$$\left| d_\varphi^E(s,x) - d_\varphi^E(t,x) - \int_t^s \operatorname{div} z(\tau, x)\, d\tau - G(s,x) + G(t,x) \right| \le \lambda(s-t) \max_{t \le \tau \le s} |d_\varphi^E(x,\tau)| \,. \tag{4}$$

Observe that (4) implies that $(d - G)$ is Lipschitz continuous, so that (4) can be rewritten as

$$\left| \frac{\partial (d_\varphi^E - G)}{\partial t}(t,x) - \operatorname{div} z(t,x) \right| \;\le\; \lambda |d_\varphi^E(t,x)| \,. \tag{5}$$

for a.e. $(t,x) \in [0,T] \times U$. In case G is C^1 in time, equation (5) expresses the fact that $\partial E(t)$ evolves with speed given by (1).

2.1. An approximation result.

We now show that, given any set E satisfying the RW_φ-condition for a general anisotropy φ, there exist smooth and elliptic anisotropies $\varphi_\varepsilon \to \varphi$ and sets $E_\varepsilon \to E$, as $\varepsilon \to 0$, such that E_ε satisfies the RW_{φ_ε}-condition.

Lemma 1. *Let φ be a general anisotropy and let φ_ε be smooth and elliptic anisotropies converging to φ, with $\varphi_\varepsilon \ge \varphi$. Let $E \subseteq \mathbb{R}^2$ satisfy the RW_φ-condition for some $R > 0$. Then there exist sets E_ε, with $\partial E_\varepsilon \to \partial E$ as $\varepsilon \to 0$ in the Hausdorff sense, such that each E_ε satisfies the RW_{φ_ε}-condition.*

Proof. Let

$$\begin{aligned} \widetilde{E}_\varepsilon &:= \bigcup \{ (x + RW_{\varphi_\varepsilon}) :\, (x + RW_{\varphi_\varepsilon}) \subset \overline{E} \} \\ E_\varepsilon &:= \mathbb{R}^2 \setminus \bigcup \left\{ (x + RW_{\varphi_\varepsilon}) :\, (x + RW_{\varphi_\varepsilon}) \subset \overline{\widetilde{E}_\varepsilon^c} \right\}. \end{aligned}$$

Notice that, by definition, $\widetilde{E}_\varepsilon$ satsifies the innner RW_{φ_ε}-condition and E_ε satisfies the outer RW_{φ_ε}-condition, so that we have to prove that E_ε also satisfies the inner RW_{φ_ε}-condition.

Step 1. Let us show that $\partial E_\varepsilon \to \partial E$ as $\varepsilon \to 0$, in the Hausdorff sense. In fact, this is obvious from the construction: since $W_{\varphi_\varepsilon} \subset W_\varphi$ and for any $x \in E$, there exists $z \in E$ with $x \in z + RW_\varphi \subset E$, we see that the distance from x to $\widetilde{E}_\varepsilon$ (and then E_ε) is bounded by the Hausdorff distance between RW_φ and RW_{φ_ε}. An estimate for the complement can be derived in the same way, so that $d_{\mathcal{H}}(\partial E_\varepsilon, \partial E) \le R d_{\mathcal{H}}(W_\varphi, W_{\varphi_\varepsilon})$.

Step 2. We now prove that $\widetilde{E}_\varepsilon$ satisfies the outer RW_φ-condition. We first show that, for all $x \in \partial \widetilde{E}_\varepsilon$, there exists y such that

$$(y + RW_\varphi) \subset \overline{\widetilde{E}_\varepsilon^c} \qquad \text{and} \qquad x \in \partial(y + RW_\varphi)\,. \tag{6}$$

Indeed, if $x \in \partial\widetilde{E}_\varepsilon \cap \partial E$, (6) readily follows from the fact that E satisfies the outer RW_φ-condition.

If $x \in \partial\widetilde{E}_\varepsilon \setminus \partial E$, then by definition of $\widetilde{E}_\varepsilon$ there exists $x_1 \in \mathbb{R}^2$ such that

$$(x_1 + RW_{\varphi_\varepsilon}) \subset \widetilde{E}_\varepsilon \qquad \text{and} \qquad x \in \partial(x_1 + RW_{\varphi_\varepsilon})\,.$$

Let ℓ_x be the maximal arc of $\partial(x_1 + RW_{\varphi_\varepsilon})$ containing x and contained in the interior of E, and let $y_1, y_2 \in \partial E$ be the endpoints of ℓ_x. Notice that $\varphi_\varepsilon(y_1 - y_2) < 2R$. Let $y_3 := (y_1 + y_2)/2$, $R' := \varphi(y_1 - y_2)/2 \le \varphi_\varepsilon(y_1 - y_2)/2 < R$. As E satisfies the inner RW_φ-condition, the set $(y_3 + R'W_\varphi)$ has connected intersection with E^c, so that the set $(\partial E \setminus \partial\widetilde{E}_\varepsilon) \cap (y_3 + R'W_\varphi)$ contains a connected arc $\tilde{\ell}_x$ joining y_1 and y_2 (see Figure 2).

Let S_x be the subset of E such that $\partial S_x = \ell_x \cup \tilde{\ell}_x$ and set $E' := \widetilde{E}_\varepsilon \cap (y_3 + R'W_\varphi)$. Note that $S_x \subset \overline{E'^c}$. As E' is a convex set, there exists y such that $x \in \partial(y + RW_\varphi)$ and $E' \subset (y + RW_\varphi)^c$. Moreover, since E satisfies the outer RW_φ-condition, the set $E \cap \mathrm{int}(y + RW_\varphi) \supseteq S_x \cap \mathrm{int}(y + RW_\varphi)$ is connected. This implies that $(y + RW_\varphi) \subset \overline{\widetilde{E}_\varepsilon^c}$ and proves (6). Notice that from (6) it follows that

$$\overline{\widetilde{E}_\varepsilon^c} = \bigcup_{y:\, d_\varphi^{\widetilde{E}_\varepsilon}(y) \ge R} (y + RW_\varphi)\,.$$

In order to prove that $\widetilde{E}_\varepsilon$ satisfies the outer RW_φ-condition, by Remark 2 it remains to show that, given $\bar{x}$ with $d_\varphi^{\widetilde{E}_\varepsilon} = R' < R$, the set $\partial\widetilde{E}_\varepsilon \cap (\bar{x} + R'\partial W_\varphi)$ is connected.

If $\partial\widetilde{E}_\varepsilon \cap (\bar{x} + R'\partial W_\varphi) \subset \partial E$ this follows directly from the fact that E satisfies the outer RW_φ-condition. Otherwise, there exists $x \in (\partial\widetilde{E}_\varepsilon \setminus \partial E) \cap (\bar{x} + R'\partial W_\varphi)$. In this case we claim that $\partial\widetilde{E}_\varepsilon \cap (\bar{x} + R'\partial W_\varphi) = \{x\}$. Indeed, since ℓ_x is a strictly convex arc, we have $\ell_x \cap (\bar{x} + R'\partial W_\varphi) = \{x\}$. Hence, if $\partial\widetilde{E}_\varepsilon \cap (\bar{x} + R'\partial W_\varphi)$ contains another point $y \neq x$, then $y \notin \overline{S_x}$. As $S_x \cap (\bar{x} + R'W_\varphi) \neq \emptyset$, it follows that $E \cap (\bar{x} + (R' + \delta)W_\varphi)$ contains at least two connected components for $\delta > 0$ sufficiently small, contradicting the fact that E satisfies the outer RW_φ-condition. Hence the set $(\partial E \setminus \partial\widetilde{E}_\varepsilon) \cap (y_3 + R'W_{\varphi_\varepsilon})$ is a connected arc $\tilde{\ell}_x$ joining y_1 and y_2.

Step 3. We prove that E_ε satisfies the inner RW_{φ_ε}-condition by reasoning as in *Step 2*, with E replaced by $(\widetilde{E}_\varepsilon)^c$ (and φ replaced by φ_ε). The only difference is due to the fact that $(\widetilde{E}_\varepsilon)^c$ now satisfies inner RW_φ-condition and the outer RW_{φ_ε}-condition. Therefore, letting $R' := \varphi_\varepsilon(y_1 - y_2)/2 < R$, the set $(y_3 + R'W_\varphi) \cap \widetilde{E}_\varepsilon$ is connected, so that $(\partial\widetilde{E}_\varepsilon \setminus \partial E_\varepsilon) \cap (y_3 + R'W_{\varphi_\varepsilon})$ contains a connected arc joining y_1 and y_2. In the rest of the proof one can proceed as in *Step 2*.

Q.E.D.

Lemma 1 has the following direct consequence.

Corollary 1. *Let $E \subseteq \mathbb{R}^2$ satisfy the RW_φ-condition for some $R > 0$. Then E is φ-regular in the sense of* [11]*, that is, there exists a vector field $n_\varphi \in L^\infty(\{|d_\varphi^E| < R\}, \mathbb{R}^2)$ such that $n_\varphi \in \partial\varphi^\circ(\nabla d_\varphi^E)$ a.e. in $\{|d_\varphi^E| < R\}$, and $\mathrm{div} n_\varphi \in L^\infty_{\mathrm{loc}}(\{|d_\varphi^E| < R\})$.*

Proof. Take a sequence φ_ε of smooth and elliptic anisotropies converging to φ, with $\varphi_\varepsilon \ge \varphi$. By Lemma 1 we can approximate E in the Hausdorff distance with sets E_ε satisfyng the RW_{φ_ε}-condition. In particular, letting $n_{\varphi_\varepsilon} = \nabla\varphi_\varepsilon^\circ(\nabla d_{\varphi_\varepsilon}^{E_\varepsilon}) \in L^\infty(\mathbb{R}^2)$ and recalling Remark 1, we have that $\mathrm{div} n_{\varphi_\varepsilon} \in L^\infty_{\mathrm{loc}}(\{|d_{\varphi_\varepsilon}^{E_\varepsilon}| < R\})$. Therefore, any weak* limit n_φ of n_{φ_ε}, as $\varepsilon \to 0$, satisfies the thesis. Q.E.D.

Remark 3. Notice that, given an arbitrary anisotropy φ, it is relatively easy to approximate it with smooth and elliptic anisotropies φ_ε. For instance, one may let $F_\varepsilon := \{\eta_\varepsilon * \varphi^\circ \le 1\} \oplus B(0, \varepsilon)$, with $\eta_r(x) := r^{-d}\eta\left(\frac{x}{r}\right)$, and $\varphi_\varepsilon(x) := \sup_{\nu \in F_\varepsilon} \nu \cdot x$. It is easy to check that the anisotropies φ_ε are smooth and elliptic, and converge locally uniformly to φ as $\varepsilon \to 0$. A similar approximation is defined and used in [24].

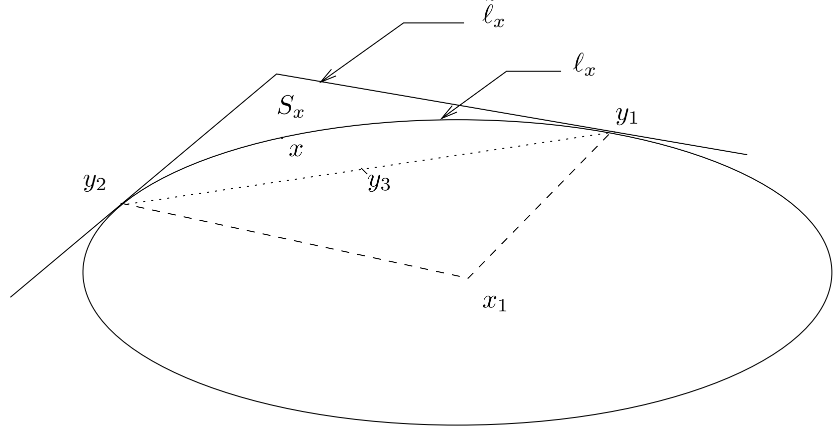

Fig. 2. The configuration in Lemma 1.

§3. The time-discrete implicit scheme

The results of this section hold in any dimension $d \geq 2$ and are stated in this general form. Up to minor improvements, they are essentially stated in [15, 6]. Following [15] we recall the definition and some properties of the implicit scheme introduced in [2, 32]. Given a set $E \subset \mathbb{R}^d$ with compact boundary (we assume without loss of generality that it is bounded), we define for $s > t \geq 0$ a transformation $T_{t,s}$ by letting $T_{t,s}(E) = \{x \in B_R : w(x) < 0\}$, where $B_R = B(0, R)$, R is large and w is the minimizer of

$$\min_{w \in L^2(B_R)} \int_{B_R} \varphi^\circ(Dw) \; + \; \frac{1}{2(s-t)} \int_{B_R} \Big(w(x) - d_\varphi^E(x) - G(s,x) + G(t,x) \Big)^2 \, dx \,, \tag{7}$$

whose existence and uniqueness is shown by standard methods. One checks easily [13, 15, 3] that for R large, the level set $T_{t,s}(E)$ of w does not depend on R, and it is a solution to the variational problem

$$\min P_\varphi(F) \; + \; \frac{1}{s-t} \int_F \big(d_\varphi^E(x) + G(s,x) - G(t,x) \big) \, dx \,, \tag{8}$$

where the minimum is taken among the subsets F of $\mathbb{R}^d$ with finite perimeter, and we set

$$P_\varphi(F) := \int_{\partial^* F} \varphi^\circ(\nu_F(x)) d\mathcal{H}^1(x).$$

It follows that the set $T_{t,s}(E)$ has boundary of class $C^{1,\alpha}$, outside a compact singular set of zero $\mathcal{H}^1$-dimension [2] (when $d = 2$, the set $T_{t,s}(E)$ has boundary of class $C^{1,1}$). The variational problem above is the generalization of the approach proposed in [2, 32], for building mean curvature flows without driving terms, through an implicit time discretization.

For $s = t + h$, the Euler-Lagrange equation for $T_{t,t+h}(E)$ at a point $x \in \partial T_{t,t+h}(E)$ formally reads as

$$d_\varphi^E(x) \; = \; -h \left(\kappa_\varphi(x) + \frac{G(t+h,x) - G(t,x)}{h} \right),$$

with κ_φ being the φ-curvature at x of $\partial T_{t,t+h}(E)$, so that it corresponds to an implicit time-discretization of (1). Observe also that this approximation is trivially monotone: indeed if $E \subseteq E'$

then $d_\varphi^E \ge d_\varphi^{E'}$, which yields $w \ge w'$, w and w' being the solutions of (7) for the distance functions d_φ^E and $d_\varphi^{E'}$ respectively. We deduce that $\{w<0\} \subseteq \{w'<0\}$, that is,

$$E \subseteq E' \implies T_{t,t+h}(E) \subseteq T_{t,t+h}(E'). \tag{9}$$

Consider now the Euler-Lagrange equation for (7), which is

$$-(s-t)\mathrm{div}\, z + w(x) \;=\; d_\varphi^E(x) + G(s,x) - G(t,x) \tag{10}$$

for $x \in B_R$, with $\varphi(z(x)) \le 1$ and $z(x)\cdot\nabla w(x) = \varphi^\circ(\nabla w(x))$ a.e. in B_R (by elliptic regularity one knows that w is Lipschitz).

We show that if E is regular enough, then we have an estimate on the quantity $\mathrm{div}\, z + (G(s,x) - G(t,x))/(s-t)$ near the boundary of E. The technique is adapted from [6].

Lemma 2. *Assume that E is a bounded set which satisfies the $\delta\mathcal{W}_\varphi$-condition for some $\delta>0$. Let $a<b$ be such that $X_{a,b} := \{\max\{w, d_\varphi^E\} \ge a\} \cap \{\min\{w, d_\varphi^E\} \le b\} \subseteq \{|d_\varphi^E| < \delta\}$. Then* $\mathrm{div}\, z \in L^\infty(X_{a,b})$ *and*

$$\left\| \mathrm{div}\, z + \frac{G(s,\cdot)-G(t,\cdot)}{s-t} \right\|_{L^\infty(X_{a,b})} \;\le\; \left\| \mathrm{div}\, n_\varphi^E + \frac{G(s,\cdot)-G(t,\cdot)}{s-t} \right\|_{L^\infty(X_{a,b})}. \tag{11}$$

Proof. Let $f : \mathbb{R} \to [0,+\infty)$ be a smooth increasing function with $f(t)=0$ if $t \le 0$. Since (w,z) solves (10), we find

$$\begin{aligned}
\int_{X_{a,b}} (w-d_\varphi^E) f(w-d_\varphi^E)\,dx \;&=\; \int_{X_{a,b}} ((s-t)\mathrm{div}\, z + G(s,x) - G(t,x))\, f(w-d_\varphi^E)\,dx \\
&=\; (s-t)\int_{X_{a,b}} (\mathrm{div}\, z - \mathrm{div}\, n_\varphi^E) f(w-d_\varphi^E)\,dx \\
&\quad+ \int_{X_{a,b}} \big((s-t)\mathrm{div}\, n_\varphi^E + G(s,x) - G(t,x)\big)\, f(w-d_\varphi^E)\,dx \;=:\; \mathrm{I}+\mathrm{II}.
\end{aligned}$$

We have, observing that $X_{a,b}$ has Lipschitz boundary (for a.e. choice of a,b),

$$\begin{aligned}
\mathrm{I} = \;&-\;(s-t)\int_{X_{a,b}} (z-n_\varphi^E)\cdot\nabla(w-d_\varphi^E) f'(w-d_\varphi^E)\,dx \\
&+\;(s-t)\int_{\partial X_{a,b}} f(w-d_\varphi^E)(z-n_\varphi^E)\cdot \nu^{X_{a,b}}\,d\mathcal{H}^1 =: \mathrm{I}_1 + \mathrm{I}_2.
\end{aligned}$$

First of all, $\mathrm{I}_1 \le 0$ since $z\cdot\nabla w = \varphi^\circ(\nabla w)$ and $z \cdot \nabla d_\varphi^E \le \varphi^\circ(\nabla d_\varphi^E)$. We claim that also $\mathrm{I}_2 \le 0$. Indeed, on one hand, when $f(w-d_\varphi^E)>0$, we have $w > d_\varphi^E$, hence $\nu^{X_{a,b}} = \nu^{\{d_\varphi^E \le b\}} = \nabla d_\varphi^E/|\nabla d_\varphi^E|$ $\mathcal{H}^1$-almost everywhere on $\{\min\{w,d_\varphi^E\}=b\}$, while $\nu^{X_{a,b}} = \nu^{\{w\ge a\}} = -\nabla w/|\nabla w|$ $\mathcal{H}^1$-almost everywhere on $\{\max\{w, d_\varphi^E\}=a\}$. It follows that $f(w-d_\varphi^E)(z-n_\varphi^E)\cdot\nu^{X_{a,b}} \le 0$ on both $\{\min\{w,d_\varphi^E\}=b\}$ and $\{\max\{w,d_\varphi^E\}=a\}$, so that $\mathrm{I}_2 \le 0$. We conclude that $\mathrm{I}\le 0$, hence

$$\int_{X_{a,b}} (w-d_\varphi^E) f(w-d_\varphi^E)\,dx \;\le\; \int_{X_{a,b}} \big((s-t)\mathrm{div}\, n_\varphi^E + G(s,x)-G(t,x)\big)\; f(w-d_\varphi^E)\,dx. \tag{12}$$

Let $q>2$, let $r^+ := r \vee 0$, and let $\{f_n\}$ be a sequence of smooth increasing nonnegative functions such that $f_n(r) \to r^{+(q-1)}$ uniformly as $n\to\infty$. From (12) we obtain

$$\begin{aligned}
\int_{X_{a,b}} ((w-d_\varphi^E)^+)^q\,dx \;&\le\; \int_{X_{a,b}} \big((s-t)\mathrm{div}\, n_\varphi^E + G(s,x)-G(t,x)\big)\; ((w-d_\varphi^E)^+)^{q-1}\,dx \\
&\le\; \int_{X_{a,b}} \big((s-t)\mathrm{div}\, n_\varphi^E + G(s,x)-G(t,x)\big)^+\; ((w-d_\varphi^E)^+)^{q-1}\,dx.
\end{aligned}$$

Applying Young's inequality we obtain

$$\|(w-d_\varphi^E)^+\|_{L^q(X_{a,b})} \;\le\; \left\| \left((s-t)\mathrm{div}\, n_\varphi^E + G(s,\cdot)-G(t,\cdot)\right)^+ \right\|_{L^q(X_{a,b}\cap\{w>d_\varphi^E\})}.$$

A similar proof, reverting the signs, shows that

$$\|(w-d_\varphi^E)^-\|_{L^q(X_{a,b})} \;\le\; \left\| \left((s-t)\mathrm{div}\, n_\varphi^E + G(s,\cdot)-G(t,\cdot)\right)^- \right\|_{L^q(X_{a,b}\cap\{w<d_\varphi^E\})}$$

It follows that

$$\|(s-t)\mathrm{div}\, z + G(s,\cdot)-G(t,\cdot)\|_{L^q(X_{a,b})} \;\le \|(s-t)\mathrm{div}\, n_\varphi^E + G(s,\cdot)-G(t,\cdot)\|_{L^q(X_{a,b})}\,,$$

and letting $q\to\infty$ we obtain (11). Observe that the estimate we may obtain is a bit more precise, in fact we have shown:

$$\begin{aligned}(13)\qquad &\operatorname*{ess\,inf}_{X_{a,b}\cap\{w<d_\varphi^E\}} \mathrm{div}\, n_\varphi^E + \frac{G(s,\cdot)-G(t,\cdot)}{s-t}\\ &\qquad\le\; \mathrm{div}\, z(x) + \frac{G(s,x)-G(t,x)}{s-t} \;\le\; \operatorname*{ess\,sup}_{X_{a,b}\cap\{w>d_\varphi^E\}} \mathrm{div}\, n_\varphi^E + \frac{G(s,\cdot)-G(t,\cdot)}{s-t}\end{aligned}$$

for a.e. $x\in X_{a,b}$. Q.E.D.

We also recall Lemma 3.2 from [15]:

Lemma 3. *Let $x_0\in B_R$ and $\rho>0$, and let $t\ge 0$. Let $\tilde w$ solve*

$$(14)\qquad \min_{\tilde w\in L^2(B_R)} \int_{B_R} \varphi^\circ(D\tilde w) \;+\; \frac{1}{2h}\int_{B_R} (\tilde w(x) - (\varphi(x-x_0)-\rho) - G(x,t+h) + G(x,t))^2\,dx\,.$$

Then

$$(15)\qquad \tilde w(x) \;\le\; \begin{cases} \varphi(x-x_0) + h\dfrac{1}{\varphi(x-x_0)} + \Delta_h(t) - \rho & \text{if } \varphi(x-x_0)\ge\sqrt{2h}\\ 2\sqrt{2h} + \Delta_h(t) - \rho & \text{otherwise,}\end{cases}$$

where $\Delta_h(t) := \|G(\cdot,t+h)-G(\cdot,t)\|_{L^\infty(B_R)}$.

We deduce an estimate on $w-d_\varphi^E$, if E has an inner ρW_φ-condition: indeed, in this case, if $h=s-t$,

$$d_\varphi^E(x) \;\le\; \inf\left\{\varphi(x-x_0)-\rho \,:\, d_\varphi^E(x_0) = -\rho\right\}$$

with, in fact, equality in $\{-\rho\le d_\varphi^E\le\rho'\}$, where $\rho'\ge 0$ is the radius of an outer $\rho' W_\varphi$-condition. It follows from (15) that

$$(16)\qquad w(x) \;\le\; \inf\left\{\varphi(x-x_0) - \rho + h\frac{1}{\varphi(x-x_0)} + \Delta_h(t) \,:\, d_\varphi^E(x_0) = -\rho\right\}$$

for x with $d_\varphi^E(x)\ge -\rho+\sqrt{2h}$, and more precisely if $\rho'\ge d_\varphi^E(x)\ge -\rho/2$,

$$(17)\qquad w(x) \;\le\; d_\varphi^E(x) + \frac{2h}{\rho} + \Delta_h(t),$$

as soon as $h\le \rho^2/16$.

§4. Smooth anisotropies

4.1. Existence of φ-regular flows.

We will prove, in dimension $d=2$, an existence result for the forced curvature flow, first in case the anisotropy is smooth and elliptic. For technical reason, we need the forcing term G to be either time-dependent only (case $G_2=0$), or smooth (globally Lipschitz in space and time, case $G_1=0$).

Theorem 1. *Assume $G_1=0$ or $G_2=0$, and let (φ,φ°) be a smooth and elliptic anisotropy and $E_0\subset\mathbb{R}^2$ an initial set with compact boundary, satisfying both an RW_φ-internal and external condition. Then, there exist $T>0$, and a φ-regular flow $E(t)$ defined on $[0,T]$ and starting from $E(0)=E_0$.*

More precisely, there exist $R'>0$ and a neighborhood U of $\bigcup_{0\le t\le T}\partial E(t)$ in $\mathbb{R}^2$ such that the sets $E(t)$ satisfy the $R'W_\varphi$-condition for all $t\in[0,T]$, the φ-signed distance function $d^E_\varphi(t,x)$ from $\partial E(t)$ belongs to $C^0([0,T];\mathrm{Lip}(U))\cap L^\infty([0,T];C^{1,1}(U))$, $(d^E_\varphi-G)\in\mathrm{Lip}([0,T]\times U)$ and

$$\left|\frac{\partial(d^E_\varphi-G)}{\partial t}(t,x)-\operatorname{div}\nabla\varphi^\circ(\nabla d^E_\varphi)(t,x)\right|\ \le\ \lambda|d^E_\varphi(t,x)|\,. \tag{18}$$

for a.e. $(t,x)\in[0,T]\times U$, where λ is a positive constant. Finally, the time T, the radius R', the set U, and the constant λ depend only on R and G.

Theorem 1 will be proved by time-discretization. Before, we need a technical lemma.

Given $E\subset\mathbb{R}^2$ and $\delta>0$ we let

$$E_\delta:=\{d^E_\varphi<-\delta\}\qquad E^\delta:=\{d^E_\varphi\le\delta\}.$$

Lemma 4. *Let φ,φ° be smooth and elliptic, and a set E satisfy a RW_φ-condition for some $R>0$. We also assume that E is simply connected (∂E is a $C^{1,1}$ Jordan curve). Let $\delta\in(0,R)$ and consider a set F (also simply connected), such that $E_\delta\subset F\subset E^\delta$. Assume that $\|\kappa^F_\varphi\|_{L^\infty(\partial F)}\le K$ for a constant $K<1/(2\delta)$. Then F has a $R'W_\varphi$-condition, with $R'=\min\{R-\delta,(1-2\delta K)/K\}$.*

Proof. We assume that ∂F is at least C^2. If the result holds in this case, then given a more general $C^{1,1}$ set we can smooth it slightly, use the result for the approximations, and then pass to the limit.

Step 1. We have $E^\delta\setminus E_\delta=\bigcup_{x\in\partial E}(x+\delta W_\varphi)$, and for any $x\in\partial E$, the set $x+\delta W_\varphi$ is tangent to ∂E_δ (respectively, ∂E^δ) at exactly one point $x-\delta n_\varphi(x)$ (resp., $x+\delta n_\varphi$). We can define Γ^+_x and Γ^-_x as the two arcs on $\partial(x+\delta W_\varphi)$ delimited by the points $x\pm\delta n_\varphi(x)$, the exponent $+$ and $-$ indicating that $\Gamma^\pm_x$ meets ∂E right "after" or "before" x, relative to an arbitrarily chosen orientation of the curve.

A first observation is that $\sharp(\partial F\cap\Gamma^\pm_x)=1$ for all x. Indeed, we check that this value is a continuous function of x. If not, there will exist for instance a point where $\sharp(\partial F\cap\Gamma^+_x)$ has a "jump", that is, where ∂F is tangent to Γ^+_x and contains a small piece of arc which is inside $x+\delta W_\varphi$ and tangent to its boundary: in this case, we deduce that $\kappa^F_\varphi(x)$ is larger than $1/\delta$ or less than $-1/\delta$, a contradiction.

Since this value is continuous, it can only be odd (since $E_\delta\subset F\subset E^\delta$), moreover if it were larger than 1, there would be a connected component of F (as well as one of its complement) in $E^\delta\setminus E_\delta$, a contradiction.

Step 2. Let $\rho<\min\{R-\delta,(1-2\delta K)/K\}$. Assume that there exists $y\in F$ such that $W:=y+\rho W_\varphi\subset\overline{F}$ and $y+\rho W_\varphi$ meets ∂F in at least two points z^-,z^+ (with z^+ "after" z^- with respect to the orientation along ∂E). These points must be isolated (otherwise there would be a point on ∂F with curvature equal to $1/\rho>K$). Observe also that $W\cap(E^\delta\setminus E_\delta)$ is connected (since E^δ has an inner $(R-\delta)W_\varphi$-condition). To z^+, we can associate a unique x^+ such that $z^+\in\Gamma^+_{x^+}$, and to z^- a unique x^- such that $z^-\in\Gamma^-_{x^-}$. Then, the piece of curve Γ of ∂F between z^- and z^+ lies in the region of $E^\delta\setminus E_\delta$ bounded by $\Gamma^-_{x^-}$ and $\Gamma^+_{x^+}$, which contains points at "distance" at most 2δ from W: more precisely, $\Gamma \subset y+(\rho+2\delta)W_\varphi$. Hence, there exists $s \in (\rho, \rho+2\delta]$ such that Γ is contained in $y+sW_\varphi$ and tangent to its boundary, and thus a point of curvature larger than $1/s \geq 1/(\rho+2\delta) > K$ on ∂W_φ, which is a contradiction. Therefore, the Wulff shapes $y+\rho W_\varphi$ which lie inside F can touch its boundary at most in one point, and an inner condition of radius $\min\{R-\delta, (1-2\delta K)/K\}$ easily follows.

The proof of the outer condition is identical. Q.E.D.

Remark 4. We can refine the lemma to consider a situation where E has an inner R_iW_φ-condition and a outer R_oW_φ-condition, for two given radii $R_i, R_o > \delta$. We assume that $-K_o \leq \kappa_\varphi^F \leq K_i$ for two nonnegative constants K_i, K_o, (still less than $1/(2\delta)$). It is then deduced that F has a inner $R_i'W_\varphi$-condition and a outer $R_o'W_\varphi$-condition, with $R_i' = \min\{R_i-\delta, (1-2\delta K_i)/K_i\}$, $R_o' = \min\{R_o-\delta, (1-2\delta K_o)/K_o\}$.

Proof of Theorem 1. From (11) and (13), we will obtain some regularity of the boundary of $T_{t+h,t}(E)$, which will allow to iterate the variational scheme. To simplify (and without loss of generality) we assume that the initial curve is a Jordan curve (E_0 is simply connected). If not, one may evolve separately each connected component of the boundary.
Step 1.a.: The case $G_1 = 0$. In the case $G_1 = 0$, there exists C such that $\Delta_h(t) \leq Ch$. It follows from (17) that if E satisfies the ρW_φ-condition, then, by solving (7) with $s = t+h$,

$$|w(x) - d_\varphi^E(x)| \;\leq\; h\left(C + \frac{2}{\rho}\right) \tag{19}$$

in $\{|d_\varphi^E| < \rho/2\}$. By standard comparison (using for instance Lemma 3 again) one also can check that $w < 0$ if $d_\varphi^E \leq -\rho/2$, and $w > 0$ if $d_\varphi^E \geq \rho/2$, so that the boundary of $T_{t,t+h}(E)$ is at (φ-)distance of order h from ∂E, if $h \leq \rho^2/36$ (Lemma 3). We also observe that the Hausdorff distance between the sets E and $T_{t,t+h}(E)$ is of the same order, or equivalently, $\|d_\varphi^E - d_\varphi^{T_{t,t+h}(E)}\|_{L^\infty(\mathbb{R}^d)} \leq (C+2/\rho)h$.

A further observation is that if E is simply connected, also $T_{t,t+h}(E)$ is. Indeed, if not, there would be a connected component of either $T_{t,t+h}(E)$ or its complement in the set $\{|d_\varphi^E| \leq h(C+2/\rho)\}$. Assume F is a connected component of $T_{t,t+h}(E)$ wich lies in $\{|d_\varphi^E| \leq h(C+2/\rho)\}$, so that $|F| \leq 2hP_\varphi(E)(C+2/\rho)$. One has that (using the isoperimetric inequality)

$$\begin{aligned} P_\varphi(F) + \frac{1}{h}\int_F d_\varphi^E(x) + G(t+h,x) - G(t,x)\,dx & \\ \geq\; 2\sqrt{|W_\varphi||F|} - 2|F|\left(C+\frac{1}{\rho}\right) & \\ \geq\; 2\sqrt{|F|}\left(\sqrt{|W_\varphi|} - \sqrt{|F|}\left(C+\frac{1}{\rho}\right)\right) & \end{aligned}$$

which is positive if h is small enough (depending on $C, \rho, P_\varphi(E)$), showing that $T_{t,t+h}(E) \setminus F$ has an energy strictly lower than $T_{t,t+h}(E)$ in (8), a contradiction.

Sending both a and b to 0, one deduces from (11) that $T_{t,t+h}(E)$ has $C^{1,1}$ boundary, and moreover

$$\begin{aligned} \left\|\operatorname{div} n_\varphi^{T_{t,t+h}(E)} + \frac{1}{h}(G(t+h,\cdot) - G(t,\cdot))\right\|_{L^\infty(\partial T_{t,t+h}(E))} & \\ \leq\; \left\|\operatorname{div} n_\varphi^E + \frac{1}{h}(G(t+h,\cdot) - G(t,\cdot))\right\|_{L^\infty(E \triangle T_{t,t+h}(E))} &. \end{aligned}$$

On one hand, $|\operatorname{div} n_\varphi^E|$ is bounded by $2/\rho$ in $\{|d_\varphi^E| \leq \rho/2\}$, and it follows (see for instance [6]) that in that set, $|\operatorname{div} n_\varphi^E(x) - \operatorname{div} n_\varphi^E(y)| \leq 4d_\varphi^E(x)/\rho^2$ if $y \in \partial E$ is the point which realizes

$\varphi(x-y) = \pm d_\varphi^E(x)$. On the other hand, $(G(t+h,x)-G(t,x))/h$ is L-Lipschitz in (t,x) for some $L>0$. We deduce that (possibly increasing C)

$$\left\|\operatorname{div} n_\varphi^{T_{t,t+h}(E)} + \frac{1}{h}(G(t+h,\cdot)-G(t,\cdot))\right\|_{L^\infty(\partial T_{t,t+h}(E))} \le \left\|\operatorname{div} n_\varphi^{E} + \frac{1}{h}(G(t,\cdot)-G(t-h,\cdot))\right\|_{L^\infty(\partial E)} + h\left(L+\frac{4}{\rho^2}\right)\left(C+\frac{2}{\rho}\right), \tag{20}$$

provided h is small enough (depending on ρ, L, C). Eventually, it follows that the curvature of $\partial T_{t,t+h}(E)$ (since $d=2$, the total and mean curvature coincide) has a global estimate $1/\rho + 2C + O(h)$, and one will deduce from Lemma 4 that for h small enough, this new set also satisfies the $\rho' W_\varphi$-condition, with $\rho' = \rho/(1+(2C+O(h))\rho) > 0$, provided the assumptions of the lemma are fulfilled.

We now consider E_0, R as in Theorem 1, and let for $h>0$ and any $n\ge 1$, $E_n^h = T_{(n-1)h,nh}(E_0)$. We also define $E^h(t) = E^h_{[t/h]}$ for $t\ge 0$. A first observation is that if $x\in (E_0)_R$, $x+RW_\varphi\subset E_0$ so that if $r(t)$ solves $\dot r = -(1/r+C)$ with $r(0)=R$, for any $\eta>0$ (small), $x+(r(t)-\eta)W_\varphi\subset E^h(t)$ for h small enough, as long as $r(t)\ge\eta$. The function $r(t)$ solves $r(t)-R-\ln\left(\frac{1+Cr(t)}{1+CR}\right)/C = -Ct$, and given $\delta\in(0,R)$ (which will be precised later on), there exists $T_1(R,C,\delta)$ such that if $t\le T_1$ and $h>0$ is small enough,

$$(E_0)_\delta \subset E^h(t) \subset (E_0)^\delta . \tag{21}$$

We let $U = \{|d_\varphi^E|\le\delta\}$.

Letting $E_1^h = T_{0,h}(E_0)$, we deduce from (20) that if $h< R^2/36$ is small enough,

$$A_1^h := \left\|\operatorname{div} n_\varphi^{E_1^h} + \frac{1}{h}(G(h,\cdot)-G(0,\cdot))\right\|_{L^\infty(\partial E_1^h)} \le \frac{1}{R} + C + \frac{1}{4}\left(C+\frac{2}{R}\right) =: M_1.$$

For $n\ge 1$, we then define iteratively the sets $E_{n+1}^h = T_{nh,(n+1)h}(E_n^h)$ and let

$$A_{n+1}^h := \left\|\operatorname{div} n_\varphi^{E_{n+1}^h} + \frac{1}{h}(G((n+1)h,\cdot)-G(nh,\cdot))\right\|_{L^\infty(\partial E_{n+1}^h)}.$$

Let now $R_1 = (2M_1+C))^{-1}$. As long as $A_n\le 2M_1$, one can deduce from Lemma 4, using (21) and provided we had chosen $\delta<R_1/2$, that E_{n+1}^h satisfies the $R_1 W_\varphi$-condition, so that (20) holds (with $E=E_n^h$, $\rho=R_1$) and

$$A_{n+1}^h \le A_n^h + h\left(L+\frac{4}{R_1^2}\right)\left(C+\frac{2}{R_1}\right).$$

By induction, we deduce that (letting $B = (L+4/R_1^2)(C+2/R_1)$) $A_{n+1}^h\le M_1+(n+1)hB$ as long as $nh\le\min\{T_1, M_1/B\} := T>0$.

We observe that since $\delta<R_1/2$, as long as $nh\le T$, not only $\partial E_n^h\subset U$, but all the signed distance functions to the boundaries of E_n^h are in $C^{1,1}(U)$. Notice that T and the width δ of the strip U depend only on R, C, L.

Step 1.b.: The case $G_2=0$. We now show that we can obtain a similar control in case of a space independent forcing term, which can be the derivative of a continuous function G (a relevant example is a Brownian forcing). In that case, we can consider the algorithm from a different point of view: given the set E, we first consider the set E' with signed distance function $d_\varphi^{E'} := d_\varphi^E(x) + G(s) - G(t)$, then, we apply to this set E' the algorithm with $G\equiv 0$, that is, we solve (7) for $E=E'$ and $G=0$:

$$\min_{w\in L^2(B_R)} \int_{B_R}\varphi^\circ(Dw) + \frac{1}{2(s-t)}\int_{B_R}\left(w(x)-d_\varphi^{E'}(x)\right)^2 dx,$$

and then let $E'' = \{w < 0\}$. It is clear that this is equivalent to the original algorithm, so that $E'' = T_{t,s}(E)$.

Assume in addition that E has an inner $r_i W_\varphi$-condition and a outer $r_o W_\varphi$ condition, for some radii $r_i, r_o > 0$. If $(s-t)$ is small enough, then E' has the inner $r'_i W_\varphi$-condition and outer $r'_o W_\varphi$ condition with $r'_i = r_i - G(s) + G(t)$ and $r'_o = r_o + G(s) - G(t)$. In particular, $d_\varphi^{E'} = d_\varphi^E(x) + G(s) - G(t)$ is locally $C^{1,1}$ in the strip $\{-r'_i < d_\varphi^{E'} < r'_o\}$ and the surface $\partial E'$ has a curvature which satisfies a.e.

$$-\frac{1}{r'_o} \;\le\; \operatorname{div} n_\varphi^{E'} \;\le\; \frac{1}{r'_i}\,. \tag{22}$$

As before, from (17) we have that, if $h = s - t$ is small enough, then

$$|w(x) - d_\varphi^{E'}(x)| \;\le\; \frac{2h}{\min\{r'_i, r'_o\}}\,, \tag{23}$$

showing that the boundary of $T_{t,t+h}(E)$ remains close to the boundary of E' (provided r'_i, r'_o are controlled from below).

From (13) (with $G = 0, E = E'$) and (22), (23), we obtain that if h is small enough,

$$-\frac{1}{r'_o} - \frac{2h}{(r'_o)^2 \min\{r'_i, r'_o\}} \;\le\; \operatorname{div} n_\varphi^{T_{t,t+h}(E)} \;\le\; \frac{1}{r'_i} + \frac{2h}{(r'_i)^2 \min\{r'_i, r'_o\}}\,,$$

and in particular we can deduce from Lemma 4 and Remark 4 that $T_h(E)$ satisfies the inner $r''_i W_\varphi$ and outer $r''_o W_\varphi$-conditions with

$$r''_i \;\ge\; r'_i - \frac{ch}{r'_i}\,, \qquad r''_o \;\ge\; r'_o - \frac{ch}{r'_o}\,,$$

for some constant $c > 0$.

As in the previous step, we now consider E_0, R as in Theorem 1, we let $E_0^h = E_0$ and define for each $n \ge 0$, $E_{n+1}^h := T_{nh,(n+1)h}(E_n^h)$. Let $r_o^0 = r_i^0 = R$. The previous analysis shows that E_1^h has the inner $r_i^1 W_\varphi$ and the outer $r_o^1 W_\varphi$-conditions with

$$r_i^1 \;\ge\; r_i^0 - G(h) + G(0) - \frac{ch}{R}, \qquad r_o^1 \;\ge\; r_o^0 + G(h) - G(0) - \frac{ch}{R},$$

provided $|G(h) - G(0)| \le R/2$ (for some constant $c > 0$). Now, assuming that n is such that

$$r_i^n \;\ge\; r_i^0 - G(nh) + G(0) - \frac{cnh}{R}, \qquad r_o^n \;\ge\; r_o^0 + G(nh) - G(0) - \frac{cnh}{R},$$

we deduce that

$$r_i^{n+1} \;\ge\; r_i^0 - G((n+1)h) + G(0) - \frac{c(n+1)h}{R},\; r_o^{n+1} \;\ge\; r_o^0 + G((n+1)h) - G(0) - \frac{c(n+1)h}{R},$$

as long as $|G((n+1)h) - G(0)| + c(n+1)h/R \le R/2$. Define T such that $\max_{0\le t\le T} |G(t) - G(0)| + ct/R \le R/4$, and let $U = \{|d_\varphi^{E_0}| < R/4\}$: then, on one hand, $\partial E_n^h \subset U$ for all $n \ge 0$ with $nh \le T$, on the other hand, E_n^h satisfies the $(R/2)W_\varphi$-condition, so that $d_\varphi^{E_n^h} \in C^{1,1}(U)$. Again, U and T depend only on G and R.

Step 2: Conclusion. For $t \in [0,T]$ and h small, we let $E_h(t) = E_{[t/h]}^h$, $d_h(t,x) = d_\varphi^{E(t)}(x)$, and we now send $h \to 0$. Since $d_h - G$ is uniformly Lipschitz in $[0,T] \times U$ (in time, in fact, we have $|d_h(t,x) - G(t,x) - d_h(s,x) + G(s,x)| \le c|t-s|$ if $|t-s| \ge h$, for some constant c), up to a subsequence (h_k) it converges uniformly to some d with $d - G \in \mathrm{Lip}([0,T] \times U)$, moreover, at each $t > 0$, $E_{h_k}(t)$ converges (Hausdorff) to a set $E(t)$ with $d(t,x) = d_\varphi^{E(t)}(x)$. Let us establish (18).

For $n \le T/h - 1$ and $x \in \partial E_{n+1}^h$, by definition of the scheme we have

$$-d_\varphi^{E_n^h}(x) - h \operatorname{div} n_\varphi^{E_{n+1}^h}(x) - G(t+h,x) + G(t,x) \;=\; 0.$$

As $(G(t+h,\cdot)-G(t,\cdot))/h$ is L-Lipschitz in U, there holds

$$\left|(G(t+h,x)-G(t,x))-\Big(G(t+h,\Pi_{\partial E^h_{n+1}}(x))-G(t,\Pi_{\partial E^h_{n+1}}(x))\Big)\right|\le Ch|d_\varphi^{E^h_{n+1}}(x)|$$

where C depends only on L and φ, where we set

$$\Pi_{\partial E^h_{n+1}}(x)=x-d_\varphi^{E^h_{n+1}}(x)n_\varphi^{E^h_{n+1}}(x).$$

Choose now $x\in U$ such that $d_\varphi^{E^h_{n+1}}(x)\ge 0$. In this case, it follows that

$$d_\varphi^{E^h_n}(x)-d_\varphi^{E^h_n}(\Pi_{\partial E^h_{n+1}}(x)) \ \le\ \varphi(x-\Pi_{\partial E^h_{n+1}}(x)) \ =\ d_\varphi^{E^h_{n+1}}(x).$$

Hence,

$$\begin{aligned}
&d_\varphi^{E^h_{n+1}}(x)-d_\varphi^{E^h_n}(x)-h\mathrm{div}\, n_\varphi^{E^h_{n+1}}(x)\\
&\ge -d_\varphi^{E^h_n}(\Pi_{\partial E^h_{n+1}}(x))-h\mathrm{div}\, n_\varphi^{E^h_{n+1}}(\Pi_{\partial E^h_{n+1}}(x))+O\left(h|d_\varphi^{E^h_{n+1}}(x)|\right)\\
&=G(t+h,\Pi_{\partial E^h_{n+1}}(x))-G(t,\Pi_{\partial E^h_{n+1}}(x))+O\left(h|d_\varphi^{E^h_{n+1}}(x)|\right)\\
&=G(t+h,x)-G(t,x)+O\left(h|d_\varphi^{E^h_{n+1}}(x)|\right).
\end{aligned}$$

Dividing by h and letting $h\to 0^+$, we then get

$$\frac{\partial(d_\varphi^E-G)}{\partial t}(t,x)-\mathrm{div}\,\nabla\varphi^\circ(\nabla d_\varphi^E)(t,x) \ \ge\ O\left(|d_\varphi^E(t,x)|\right) \qquad (t,x)\in U\times[0,T]\cap\{d_\varphi^E(t,x)>0\},$$

which implies

$$\frac{\partial(d_\varphi^E-G)}{\partial t}(t,x)-\mathrm{div}\,\nabla\varphi^\circ(\nabla d_\varphi^E)(t,x) \ \ge\ O\left(|d_\varphi^E(t,x)|\right) \qquad (t,x)\in U\times[0,T].$$

By taking $x\in U$ such that $d_\varphi^{E^h_{n+1}}(x)\le 0$, reasoning as above we get

$$\frac{\partial d_\varphi^E}{\partial t}(t,x)-\mathrm{div}\,\nabla\varphi^\circ(\nabla d_\varphi^E)(t,x)-g(t,x) \ \le\ O\left(|d_\varphi^E(t,x)|\right) \qquad (t,x)\in U\times[0,T],$$

thus obtaining (18). Q.E.D.

Remark 5. When $\varphi(x)=|x|$ and $G_2=0$, existence and uniqueness of φ-regular flows has been proved in [18] in any dimension.

4.2. Uniqueness of φ-regular flows.

We now show uniqueness of the regular evolutions given by Theorem 1.

Theorem 2. *Given an initial set E_0, the flow of Theorem 1 is unique. More precisely, if two flows E, E' are given, starting from initial sets $E_0\subseteq E'_0$, then $E(t)\subseteq E'(t)$ for all $t\in[0,\min\{T,T'\}]$ (where T,T' are respectively the time of existence of regular flows starting from E_0 and E'_0).*

The thesis essentially follows from the results in [15]. Indeed, in [15] it is proved a comparison result for strict C^2 sub- and superflows, based again on a consistency result for the scheme defined in Section 3. A strict C^2 subflow is defined a in Theorem 1, except that $d_\varphi^E(t,x)$ is required to be in $C^0([0,T];C^2(U))$, and (18) is replaced with (for $0\le t<s\le T$, $x\in U$)

$$d_\varphi^E(s,x)-d_\varphi^E(t,x)-\int_t^s \mathrm{div}\,\nabla\varphi^\circ(\nabla d_\varphi^E)(\tau,x)\,d\tau-G(s,x)+G(t,x) \ \le\ -\delta(s-t) \tag{24}$$

for some $\delta > 0$. A superflow will satisfy the reverse inequality, with $-\delta(s-t)$ replaced with $\delta(s-t)$. For technical reasons (in order to make sure, in fact, that the duration time of these flows is independent on δ), we will ask that these flows are defined, in fact, in a tubular neighborhood W of $\bigcup_{0\le t\le T}\partial E(t)$, not necessarily of the form $[0,T]\times U$.

The thesis then follows from the consistency result in [15, Thm. 3.3], once we show the following approximation result.

Lemma 5. *Let $E(t)$ be an evolution as in Theorem 1, starting from a compact set E_0 satisfying the RW_φ-conditions for some $R > 0$. Then, there exists $T' > 0$ such that for any $\varepsilon > 0$, there exist a set E_0' and a strict C^2 subflow $E'(t)$ starting from E_0' such that for all $t\in[0,T']$, $E(t)\subset\subset E'(t)\subset\{d_\varphi^E(t,\cdot)\le\varepsilon\}$.*

Proof. We sketch the proof and refer to [3] for more details.

The idea is to let first $d^\alpha = d_\varphi^E - \alpha t - \alpha/(4\lambda)$, for some small $\alpha > 0$, with $\alpha(T + 1/(4\lambda)) < \varepsilon$. One can then deduce from (18) that, for all $s > t$,

$$d^\alpha(s,x) - d^\alpha(t,x) - \int_t^s \operatorname{div}\nabla\varphi^\circ(\nabla d^\alpha)(\tau,x)\,d\tau - G(s,x) + G(t,x)$$
$$\le\ (s-t)(\lambda\max_{t\le\tau\le s}|d_\varphi^E(\tau,x)| - \alpha)\ \le\ \lambda(s-t)\left(\max_{t\le\tau\le s}|d^\alpha(\tau,x)| + \alpha(s-\tfrac{3}{4}\lambda^{-1})\right).$$

Let $T' := \min\{T, 1/(2\lambda)\}$, and let $\beta = \alpha/(8\lambda)$: then if we let $W = \{(t,x)\,:\,0\le t\le T\,,|d^\alpha(t,x)| < \beta\}$, we deduce that for x,t,s with $[t,s]\times\{x\}\subset W$,

$$d^\alpha(s,x) - d^\alpha(t,x) - \int_t^s \operatorname{div}\nabla\varphi^\circ(\nabla d^\alpha)(\tau,x)\,d\tau - G(s,x) + G(t,x)\ \le\ -\beta\lambda(s-t)\,.$$

Hence $\{d^\alpha\le 0\}$ is almost a C^2 subflow, except for the fact that it is not C^2. However, this is not really an issue, as we will now check. Consider indeed a spatial mollifier

$$\eta_r(x) = \frac{1}{r^d}\eta\left(\frac{x}{r}\right) \tag{25}$$

where as usual $\eta\in C_c^\infty(B(0,1);\mathbb{R}_+)$, $\int_{\mathbb{R}^d}\eta(x)\,dx = 1$. Let $d_r^\alpha = \eta_r * d^\alpha$ (for r small). Observing, as before, that $G(s,x)-G(t,x)$ is $(s-t)L$-Lipschitz, one has $|\eta_r*(G(s,\cdot)-G(t,\cdot))(x) - (G(s,x)-G(t,x))|\le(s-t)Lr$. Hence, the level set 0 of d_r^α will be a strict C^2 subflow, for r small enough, if we can check that the difference

$$\eta_r*(\operatorname{div}\nabla\varphi^\circ(\nabla d^\alpha)(\tau,\cdot))(x)\ -\ \operatorname{div}\nabla\varphi^\circ(\eta_r*\nabla d^\alpha)(\tau,x) \tag{26}$$

can be made arbitrarily small for r small enough and any $(\tau,x)\in W$ (possibly reducing slightly the width of W). Now, for $(\tau,x)\in W$,

$$\eta_r*(\operatorname{div}\nabla\varphi^\circ(\nabla d^\alpha)(\tau,\cdot))(x)\ =\ \int_{B(0,r)}\eta_r(z)D^2\varphi^\circ(\nabla d^\alpha(\tau,x-z)) : D^2 d^\alpha(\tau,x-z)\,dz$$

while

$$\operatorname{div}\nabla\varphi^\circ(\eta_r*\nabla d^\alpha)(\tau,x))\ =\ \int_{B(0,r)}\eta_r(z)D^2\varphi^\circ((\eta_r*\nabla d^\alpha(\tau,\cdot))(x)) : D^2 d^\alpha(\tau,x-z)\,dz\,.$$

The difference in (26) is therefore

$$\int_{B(0,r)}\eta_r(z)(D^2\varphi^\circ(\nabla d^\alpha(\tau,x-z)) - D^2\varphi^\circ((\eta_r*\nabla d^\alpha(\tau,\cdot))(x))) : D^2 d^\alpha(\tau,x-z)\,.$$

Now, since $D^2\varphi^\circ$ is at least continuous (uniformly in $\{\varphi^\circ(\xi)\ge 1/2\}$), $\varphi^\circ(\nabla d^\alpha) = 1$ a.e. in W, while $D^2 d^\alpha$ is globally bounded (and ∇d^α uniformly Lipschitz), this difference can be made arbitrarily small as $r\to 0$, and we actually deduce that, in such a case, $E'(t) = \{d_r^\alpha\le 0\}$ is a strict C^2-superflow starting from $E_0' = \{d_\varphi^E\le\beta\}$, which satisfies the thesis of the Lemma. Q.E.D.

Remark 6. The uniqueness result holds in any dimension $d \geq 2$, with exactly the same proof. We point out that it is not necessary to assume that G_1 or G_2 vanishes.

§5. General anisotropies

An important feature of Theorem 1 is that the existence time, as well as the neighborhood where d^E_φ is $C^{1,1}$, are both independent on the anisotropy, and only depend on the radius R for which E_0 satisfies the RW_φ-condition. This allows us to extend the existence result to general anisotropies, by the approximation argument given in Lemma 1.

Theorem 3. *Assume $G_1 = 0$ or $G_2 = 0$, and let (φ, φ°) be an arbitrary anisotropy. Let $E_0 \subset \mathbb{R}^2$ an initial set with compact boundary, satisfying the RW_φ-condition for some $R > 0$. Then, there exist $T > 0$, and a φ-regular flow $E(t)$ defined on $[0, T]$ and starting from E_0.*

More precisely, there exist $R' > 0$ and a neighborhood U of $\bigcup_{0\leq t\leq T} \partial E(t)$ in $\mathbb{R}^2$ such that the sets $E(t)$ satisfy the $R'W_\varphi$-condition for all $t \in [0, T]$, the φ-signed distance function $d^E_\varphi(t, x)$ from $\partial E(t)$ belongs to $C^0([0, T]; \mathrm{Lip}(U))$, $(d^E_\varphi - G) \in \mathrm{Lip}([0, T] \times U)$ and

$$\left|\frac{\partial(d^E_\varphi - G)}{\partial t}(t, x) - \operatorname{div} z(t, x)\right| \;\leq\; \lambda |d^E_\varphi(t, x)| \tag{27}$$

for a.e. $(t, x) \in [0, T] \times U$, where λ is a positive constant and $z \in L^\infty([0, T] \times U; \mathbb{R}^2)$ is such that $z \in \partial\varphi^\circ(\nabla d^E_\varphi)$ a.e. in $[0, T] \times U$. The time T, the radius R', and the constant λ, only depend on R and G.

Remark 7. Comparison and uniqueness for such flows has been shown in [11, 30, 14, 7], although the most general result in these references only covers the case of a time-dependent, Lipschitz continuous forcing term $G(t) = G(0) + \int_0^t c(s)\, ds$, with $c \in L^\infty(0, +\infty)$.

Proof. Let $\varepsilon > 0$ and consider smooth and elliptic anisotropies $(\varphi_\varepsilon, \varphi^\circ_\varepsilon)$, with $\varphi_\varepsilon \geq \varphi$, converging to (φ, φ°) locally uniformly as $\varepsilon \to 0$. By the approximation result in Lemma 1, we can find a sequence of sets E_ε which satisfy the RW_{φ_ε}-condition, and such that $\partial E_\varepsilon \to \partial E$ in the Hausdorff sense. For each ε we consider the evolution $E^\varepsilon(t)$ given by Theorem 1, with $0 \leq t \leq T^\varepsilon$. Since the times T^ε and the width of the neighborhoods U^ε depend only on R and G, up to extracting a subsequence we can assume that $\lim_\varepsilon T^\varepsilon = T$ for some $T > 0$, and there exists a neighborhood U of ∂E_0 such that $\mathbb{R}^d \setminus U^\varepsilon$ converges to $\mathbb{R}^d \setminus U$ in the Hausdorff sense, as $\varepsilon \to 0$. Possibly reducing T and the width of U we can then assume that $T^\varepsilon = T$ and $U^\varepsilon = U$ for all $\varepsilon > 0$.

Letting $W := [0, T] \times U$, and $z_\varepsilon(t, x) = \nabla\varphi^\circ_\varepsilon(\nabla d^{E_\varepsilon}_{\varphi_\varepsilon}(t, x))$, from (18) we get

$$\left|\frac{\partial(d^{E_\varepsilon}_{\varphi_\varepsilon} - G)}{\partial t}(t, x) - \operatorname{div} z_\varepsilon(t, x)\right| \;\leq\; \lambda |d^{E_\varepsilon}_{\varphi_\varepsilon}(t, x)| \tag{28}$$

for a.e. $(t, x) \in W$, where the constant λ depends only on R and G.

As $d^{E_\varepsilon}_{\varphi_\varepsilon} - G$ are uniformly Lipschitz in (t, x), up to a subsequence we can assume that the functions $d^{E_\varepsilon}_{\varphi_\varepsilon}$ converge uniformly in any compact subset of W to a function d^E_φ, such that for all $t \in [0, T]$ $d^E_\varphi(t, \cdot)$ is the signed φ-distance function to the boundary of $E(t) := \{x : d^E_\varphi(t, x) \leq 0\}$. Moreover, $E(0) = E_0$, $E(t)$ is the Hausdorff limit of $E_\varepsilon(t)$ for each $t \in [0, T]$, and satisfies the $R'W_\varphi$-condition, with $R' = \lim_\varepsilon R'_\varepsilon$.

Up to a subsequence we can also assume that there exists $z \in L^\infty(W)$ with $z_\varepsilon(t, x) \stackrel{*}{\rightharpoonup} z(t, x)$, $\operatorname{div} z_\varepsilon \stackrel{*}{\rightharpoonup} \operatorname{div} z$ and $\partial_t(d^{\varphi_\varepsilon}_{E_\varepsilon} - G) \stackrel{*}{\rightharpoonup} \partial_t(d^E_\varphi - G)$ in $L^\infty(W)$, so that (27) holds a.e. in W.

It remains to check that $z(t, x) \in \partial\varphi^\circ(\nabla d^E_\varphi(t, x))$. Since by construction $z(t, x) \in W_\varphi$ for a.e. $(t, x) \in W$, it is enough to show that

$$z \cdot \nabla d^E_\varphi \;=\; \varphi^\circ(\nabla d^E_\varphi) \;=\; 1 \tag{29}$$

a.e. in W. Recalling that $z_\varepsilon \cdot \nabla d^{E_\varepsilon}_{\varphi_\varepsilon} = \varphi^\circ_\varepsilon(\nabla d^{E_\varepsilon}_{\varphi_\varepsilon}) = 1$ a.e. in W and letting $\psi \in C^\infty_c(W)$, we have

$$\int_W \psi \, dxdt \,=\, \int_W \psi \left(z_\varepsilon \cdot \nabla d^{E_\varepsilon}_{\varphi_\varepsilon}\right) \, dxdt \,=\, -\int_W d^{E_\varepsilon}_{\varphi_\varepsilon} \left(z_\varepsilon \cdot \nabla \psi + \psi \mathrm{div}\, z_\varepsilon\right) \, dxdt.$$

Passing to the limit in the righ-hand side we then get

$$\int_W \psi \, dxdt \,=\, -\int_W d^E_\varphi \left(z \cdot \nabla \psi + \psi \mathrm{div}\, z\right) \, dxdt \,=\, \int_W \psi \left(z \cdot \nabla d^E_\varphi\right) \, dxdt$$

which gives (29). Q.E.D.

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