NONLOCAL QUANTITATIVE ISOPERIMETRIC INEQUALITIES

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Abstract. We show a quantitative-type isoperimetric inequality for fractional perimeters where the deficit of the t-perimeter, up to moltiplicative constants, controls from above that of the s-perimeter, with s smaller than t. To do this we consider a problem of independent interest: we characterize the volume-constrained minimizers of a nonlocal free energy given by the difference of the t-perimeter and the s-perimeter. In particular, we show that balls are the unique minimizers if the volume is sufficiently small, depending on $t-s$, while the existence vs. nonexistence of minimizers for large volumes remains open. We also consider the corresponding isoperimetric problem and prove existence and regularity of minimizers for all s, t. When $s=0$ this problem reduces to the fractional isoperimetric problem, for which it is well known that balls are the only minimizers.

Contents

1. Introduction

In this paper we deal with two nonlocal isoperimetric problems, which are closely related one with the other. To introduce them, we recall the definition and some properties of the fractional perimeter. Given a number $\alpha \in (0,1)$, for a measurable set $E \subset \mathbb{R}^N$, the fractional perimeter $P_\alpha(E)$ is defined as the (squared) $H^{\alpha/2}$-seminorm of the characteristic function of E, that is,

$$P_\alpha(E) := \frac{1}{2}\,[\chi_E]^2_{H^{\alpha/2}} = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\chi_E(x)-\chi_E(y)|^2}{|x-y|^{N+\alpha}}\,dx\,dy = \int_E\int_{E^c} \frac{dx\,dy}{|x-y|^{N+\alpha}}.$$

The notion of fractional perimeter has been introduced in [36, 9] and it has been extensively studied in several recent papers (see for instance [24, 33, 34, 11, 18, 15] and references therein). In particular, according [10, Theorem 1] (see also [7, 14, 3]), we have that the fractional perimeter P_α, if suitably renormalized, approaches the classical perimeter P as $\alpha \nearrow 1$. More precisely, if ∂E is of class $C^{1,\gamma}$ for some $\gamma > 0$, we have

$$\lim_{\alpha\to 1^-}(1-\alpha)P_\alpha(E) = N\omega_N P(E), \tag{1.1}$$

where ω_N denotes the volume of the N-dimensional ball of radius 1. The convergence of the variational problems (and of the minimizers in particular), can also be addressed in terms of Γ-convergence, see [3].

On the other hand, the fractional perimeter P_α approaches the Lebesgue measure $|\cdot|$ as $\alpha \searrow 0$, that is,

$$\lim_{\alpha\to 0^+} \alpha P_\alpha(E) = N\omega_N |E|, \tag{1.2}$$

if $P_{\bar\alpha}(E) < +\infty$ for some $\bar\alpha > 0$ (see [31] and [17, Corollary 2.6]).

To introduce the first problem we consider we define, for $t \in (0,1)$, the *isoperimetric deficit* of the t-perimeter by

$$\delta P_t(E) := \frac{P_t(E) - P_t(B_E)}{P_t(B_E)} \tag{1.3}$$

where B_E is a ball of measure $|E|$. The fractional isoperimetric inequality (see e.g. [20, 10] and references therein), stating that among sets of fixed measure the ball minimizes the fractional perimeter, reads in term of the isoperimetric deficit as

$$\delta P_t(E) \geqslant 0.$$

Notice that for any $t \in (0,1)$ the isoperimetric deficit is a 0 homogeneous quantity. Moreover thanks to (1.1) and (1.2), for $t \to 1$ and $s \to 0$ it converges to the classical deficit (see for instance [21])

$$\delta P(E) = \frac{P(E) - P(B_E)}{P(B_E)},$$

and to 0 respectively. In the last years there has been a renewed interest into the study of quantitative stability isoperimetric inequalities, which is a stronger versions of the isoperimetric inequality of the form

$$\delta P(E) \geqslant \phi(E),$$

where $\phi(E)$ is a non-negative quantity which measures the distance between the set E and the set of the balls contained in $\mathbb{R}^N$. A cornerstone example has been given in the paper [23] where the authors show an inequality of the form

$$\delta P(E) \geqslant C_N \alpha(E) := C_N \min_{x\in\mathbb{R}^N} \frac{|E\Delta(B_E + x)|^2}{|E|^2}, \tag{1.4}$$

proving that the exponent 2 is asimptotically optimal, as E approaches B_E. Here C_N is a dimensional constant while $|E\Delta F|$ indicates the Lebesgue measure of the symmetric difference between E and F. The quantity $\alpha(E)$ is usually referred to as *Fraenkel asymmetry.* Recently the (sharp) fractional counterpart of (1.4) has been shown in [18]. Namely it is proved that there exists a constant $C_{N,t}$ such that for any $E \subset \mathbb{R}^N$ it holds

$$\delta P_t(E) \geqslant C_{N,t}\alpha(E)^2. \tag{1.5}$$

Here, again the exponent 2 is optimal.

The first main result of this paper is the following.

Theorem 1.1. *Let $0 < s < t < 1$. Then there exists a constant $C(N,s,t)$ such that for any $E \subset \mathbb{R}^N$ the following inequality holds true*

$$\delta P_t(E) \geqslant C(N,s,t)\delta P_s(E). \tag{1.6}$$

Moreover the constant $C(N,s,t)$ is bounded as $s \to 0$ and $t \to 1$.

Some comments about the proof of Theorem 1.1 are in order. First we notice that in view of (1.5), inequality (1.6) might be seen as a stronger version of the quantitative isoperimetric inequality. To get (1.6) we investigate another variational problem:

$$\min_{|E|=m} \mathcal{F}_{s,t}(E) \qquad m \in (0,+\infty)\,, \tag{1.7}$$

where

$$\mathcal{F}_{s,t}(E) := \begin{cases} (1-t)P_t(E) - sP_s(E) & \text{if } 0<s<t<1 \\[2ex] N\omega_N P(E) - sP_s(E) & \text{if } 0<s<t=1 \\[2ex] (1-t)P_t(E) - N\omega_N|E| & \text{if } 0=s<t<1 \\[2ex] N\omega_N P(E) - N\omega_N|E| & \text{if } s=0 \text{ and } t=1. \end{cases} \tag{1.8}$$

Notice that thanks to (1.2) and (1.1), for all $s,t \in (0,1)$ we have

$$\mathcal{F}_{s,t}(E) \underset{t\to 1}{\to} \mathcal{F}_{s,1}(E) \underset{s\to 0}{\to} \mathcal{F}_{0,1}(E) \quad \text{and} \quad \mathcal{F}_{s,t}(E) \underset{s\to 0}{\to} \mathcal{F}_{0,t}(E) \underset{t\to 1}{\to} \mathcal{F}_{0,1}(E)\,, \tag{1.9}$$

that is, $\mathcal{F}_{s,t}$ depends continuously on $s,t \in [0,1]$, with $s<t$.

Problem (1.7) is, in our opinion, of independent interest as it is reminiscent of recent results about isoperimetric problems with nonlocal competing term arising in mathematical physics, where the functionals take the form

$$\mathcal{F} = P + \mathcal{NL}$$

being P the perimeter and $\mathcal{NL}$ the nonlocal term, see for instance [28, 29, 13, 26, 22, 18, 5, 27]. We mention in particular the works by Knüpfer and Muratov [28, 29] where the authors consider the case where the nonlocal term is given by a Coulombic potential. In our framework, the energy in (1.9) presents a competing effect between the term P_t which has the tendency to "aggregate" the sets into balls, and P_s, which acts in the opposite way. We will see that, at small scales, the aggregating effect is predominant, but this does not occur at large scales. More precisely, as a first result we show that minimizers exist and are regular at least for small volumes.

Theorem 1.2. *For any* $0 \leqslant s < t \leqslant 1$*, there exists* $\bar m_0 = \bar m_0(N, t-s) > 0$ *such that for all* $m \in (0, \bar m_0)$*, problem* (1.7) *has a minimizer* $F \subset \mathbb{R}^N$*. Moreover* F *is bounded with boundary of class* $C^{1,\beta}$*, for some* $\beta = \beta(N, t-s) \in (0,1)$*, outside a closed singular set of Hausdorff dimension at most* $N-2$ *(respectively* $N-8$ *if* $t=1$*).*

Exploiting the fractional isoperimetric inequality in a quantitative form proved in [18], we then show that the the minimizer found in Theorem 1.2 is necessarily a ball, if the volume m is sufficiently small.

Theorem 1.3. *For any* $0 \leqslant s < t \leqslant 1$ *and* $\bar m_0$ *as in Theorem 1.2, there exists* $\bar m_1 = \bar m_1(N, t-s) \in (0, \bar m_0]$ *such that for all* $m \in (0, \bar m_1)$*, the only minimizer of problem* (1.7) *is given by the ball of measure* m*.*

Once Theorem 1.3 is settled, the proof of Theorem 1.1 easily follows. We stress that our estimates, similarly to those in [18], depend only on a lower bound on the difference $t-s$, and pass to the limit as $s \to 0$ and $t \to 1$ (as a matter of fact, the normalizing constants appearing in (1.8) has exactly the purpose of making our estimates stable as $s \to 0$ and $t \to 1$). Moreover, as far as we know, our results are new even in the case $t = 1$. We also point out that we do not know if a minimizer exists for any volume m. However, we show that *a minimizer cannot be a ball if* m *is large enough* (see Theorem 6.3), so the minimization problem can be in general quite rich.

The second problem we consider is the following generalized isoperimetric problem:

$$\min_{E\subset\mathbb{R}^N} \widetilde{\mathcal{F}_{s,t}}(E) \qquad 0 \leqslant s < t \leqslant 1\,, \tag{1.10}$$

where

$$\widetilde{\mathcal{F}_{s,t}}(E) := \begin{cases} \dfrac{\left((1-t)P_t(E)\right)^{N-s}}{\left(sP_s(E)\right)^{N-t}} & \text{if } 0 < s < t < 1 \\[2ex] \dfrac{\left(N\omega_N P(E)\right)^{N-s}}{\left(sP_s(E)\right)^{N-1}} & \text{if } 0 < s < t = 1 \\[2ex] \dfrac{(1-t)P_t(E)^N}{(N\omega_N|E|)^{N-t}} & \text{if } 0 = s < t < 1 \\[2ex] N\omega_N \frac{P(E)^N}{|E|^{N-1}} & \text{if } s = 0 \text{ and } t = 1. \end{cases}$$

Again, thanks to (1.2) and (1.1) we see that

$$\widetilde{\mathcal{F}_{s,t}}(E) \underset{t\to 1}{\to} \widetilde{\mathcal{F}_{s,1}}(E) \underset{s\to 0}{\to} \widetilde{\mathcal{F}_{0,1}}(E) \quad \text{ and } \quad \widetilde{\mathcal{F}_{s,t}}(E) \underset{s\to 0}{\to} \widetilde{\mathcal{F}_{0,t}}(E) \underset{t\to 1}{\to} \widetilde{\mathcal{F}_{0,1}}(E).$$

Since, for $s = 0$ and $t = 1$, problem (1.10) reduces to the classical isoperimetric one, while for $t < 1$ it reduces to the fractional isoperimetric one, we can think to it as a generalized isoperimetric problem for fractional perimeters. Moreover in the cases $s = 0 < t \leqslant 1$ it is well known that the ball is the unique minimizer of $\widetilde{\mathcal{F}_{s,t}}$. Nevertheless we don't know if the ball minimizes $\widetilde{\mathcal{F}_{s,t}}$ for any $0 < s < t \leqslant 1$. Our main result about problem (1.10) is the following.

Theorem 1.4. *For any* $0 \leqslant s < t \leqslant 1$*, there exists a minimizer* $E_{s,t}$ *of problem* (1.10)*. Moreover* $E_{s,t}$ *is bounded and has boundary of class* $C^{1,\beta}$*, for some* $\beta = \beta(N, t-s) \in (0,1)$*, outside a closed singular set of Hausdorff dimension at most* $N-3$ *(respectively* $N-8$ *if* $t = 1$[1]*).*

Remark 1.5. An observation which may support the conjecture that the ball is a minimizer of $\widetilde{\mathcal{F}_{s,t}}$ is the following link between inequality (1.6) and problem (1.10). By the concavity of the map $x \to (1+x)^{(N-t)/(N-s)}$, if the quantity $\delta P_s(E)$ is not too large, then

$$(1+\delta P_s(E))^{(N-t)/(N-s)} \leqslant 1 + \frac{N-t}{N-s}\delta P_s(E). \tag{1.11}$$

Moreover a straightforward computation shows that the inequality

$$\widetilde{\mathcal{F}_{s,t}}(E) \geqslant \widetilde{\mathcal{F}_{s,t}}(B)$$

is equivalent to the following one

$$1 + \delta P_t(E) \geqslant (1+\delta P_s(E))^{\frac{N-t}{N-s}}.$$

Thus, the mixed isoperimetric problem (1.10) has as solution the ball if the mixed quantitative fractional isoperimetric inequality (1.6) holds true with a constant $C(N,s,t)$ greater or equal than $(N-t)/(N-s)$.

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[1] Using [11], one can actually show that the dimension is at most $N-8$ even for t sufficiently close to 1.

We remark that the problems considered in this paper and some techniques exploited to get the proof of Theorem 1.1 are related to the very recent paper [18], since both here and there some nonlocal functionals built by the combination of aggregating and disaggregating terms are studied via variational methods and geometric measure theory techniques. Nevertheless there are several technical and conceptual differences between our case and the one of [18]. For instance, the disaggregating term in [18] comes from a Riesz potential (i.e. it has a locally integrable kernel), and the minimizers of such functional are Λ-minimizers for the aggregating term (i.e. their energy surplus reduces to a volume perturbation). On the contrary, our disaggregating terms have somehow the same type of nonlocal structure as the aggregating ones (for instance they do not come from a locally integrable kernel), and our minimizers are only ω-minimizers of the fractional perimeter. Also, our techniques are different than the ones in [18]; for instance, we highly rely on a relative isoperimetric inequality (see Lemma 2.5) and on conceptually different regularity results (see e.g. the second inequality in (6.2)).

About the proof of Theorem 1.4, if the regularity results presented are basically a straightforward application of already developed tools in Geometric Measure Theory, the existence issue, mainly because of the competition between the numerator and the denominator in the definition of $\widetilde{\mathcal{F}_{s,t}}$, is less straightforward. Indeed, to get the existence part of the proof of Theorem 1.4, it is necessary a quite more original and non-trivial approach. For the reader convenience we added a formal description of the strategy of the proof at the beginning of Section 7.

The paper is organized as follows: in Section 2 we recall and prove some general properties of the fractional perimeters and, more generally, of the fractional Sobolev seminorms. In Sections 3–6 we deal with problem (1.7). Section 3 contains the main tools exploited later to prove Theorems 1.2 and 1.3. The cornerstone of the section is an optimality criterion (see Proposition 3.9) which entails density estimates for minimizers (see Proposition 3.11) and the fact that minimizers must be close to a ball, if the volume is small enough (see Lemma 3.13). An elementary, but important result is then provided by Proposition 3.12, stating that any minimum must be necessary bounded and, if $t = 1$ (that is, $\mathcal{F}_{s,1} = N\omega_N P - sP_s$), also essentially connected. Section 4 contains Theorem 4.2, which solves the existence part of Theorem 1.2, while in Section 5 we prove that any minimizer has smooth boundary, out of a closed singular set. Then, in Section 6 we show that, if the volume m is below a certain threshold $\bar{m}_1 > 0$, then the ball is the unique minimizer for problem (1.7) and we give the proof of Theorem 1.1. Eventually, in Section 7, we deal with problem (1.10). The main result here is given by Theorem 1.4, where we show the existence and regularity of minimizers.

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2. General properties of fractional perimeters

Before starting to prove some properties of fractional perimeters it is convenient to fix some notation which will be used throughout the rest of the paper. We shall denote by c_N a general positive constant depending only on the dimension N which may change from line to line, by δ_0 a fixed constant such that $0 < \delta_0 \leqslant t - s$, and by c_0 a positive constant depending both on N and on δ_0; special constants will be denoted by c_1, c_2,... Relevant dependences on parameters will be emphasized by using parentheses.

As customary, we denote by $B(x_0, R) := \{x \in \mathbb{R}^N : |x - x_0| < R\}$ the open ball centered in $x_0 \in \mathbb{R}^N$ with radius $R > 0$. We shall use the shorter notation $B = B(0, 1)$, with $|B(0, 1)| = \omega_N$. Moreover, when not important and clear from the context, we shall denote by B_m the ball of volume m, that is of radius $R = (m/|B(0, 1)|)^{1/N}$.

Finally, as usual, given two sets E and F of $\mathbb{R}^N$, we denote the symmetric difference between E and F as $E\Delta F = (E \setminus F) \cup (F \setminus E)$.

We begin by a simple result.

Lemma 2.1. *Let $E = E_1 \cup E_2$ a subset of $\mathbb{R}^N$ with $|E_1 \cap E_2| = 0$. Then*

$$P_\alpha(E) = P_\alpha(E_1) + P_\alpha(E_2) - 2\int_{E_1}\int_{E_2} \frac{dx\, dy}{|x - y|^{N+\alpha}}. \tag{2.1}$$

In particular

$$P_\alpha(E) \leqslant P_\alpha(E_1) + P_\alpha(E_2). \tag{2.2}$$

Proof. Let us denote by χ_E the characteristic function of the set E. We have

$$\begin{aligned} P_\alpha(E) &= \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\chi_E(x) - \chi_E(y))^2}{|x - y|^{N+\alpha}} dx\, dy \\ &= \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\chi_{E_1}(x) + \chi_{E_2}(x) - \chi_{E_1}(y) - \chi_{E_2}(y))^2}{|x - y|^{N+\alpha}} dx\, dy \\ &= \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\chi_{E_1}(x) - \chi_{E_1}(y))^2 + (\chi_{E_2}(x) - \chi_{E_2}(y))^2}{|x - y|^{N+\alpha}} \\ &+ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\chi_{E_1}(x) - \chi_{E_1}(y))(\chi_{E_2}(x) - \chi_{E_2}(y))}{|x - y|^{N+\alpha}} dx\, dy \\ &= P_\alpha(E_1) + P_\alpha(E_2) - 2\int_{E_1}\int_{E_2} \frac{dx\, dy}{|x - y|^{N+\alpha}}. \end{aligned}$$

□

For further use, we also prove the following interpolation estimate (by reasoning as in [8, Proposition 4.2 and Corollary 4.4]):

Lemma 2.2. *For any $E \subset \mathbb{R}^N$ and $0 < s < t < 1$ there holds*

$$P_s(E) \leqslant c_N \frac{1}{s}\left(1 - \frac{s}{t}\right)^{-1} |E|^{1-\frac{s}{t}} (1 - t)^{\frac{s}{t}} P_t(E)^{\frac{s}{t}}. \tag{2.3}$$

Proof. We reason as in [8, Prop. 4.2]. Letting $u = \chi_E$, we can write

$$\begin{aligned} P_s(E) &= \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x + h) - u(x)|}{|h|^{N+s}}\, dxdh \\ &= \frac{1}{2}\int_{|h|<1}\int_{\mathbb{R}^N} \frac{|u(x + h) - u(x)|}{|h|^{N+s}}\, dxdh \\ &\quad + \frac{1}{2}\int_{|h|\geqslant 1}\int_{\mathbb{R}^N} \frac{|u(x + h) - u(x)|}{|h|^{N+s}}\, dxdh =: I_1 + I_2. \end{aligned}$$

We recall that, by [8, Lemma A.1] (see also [31]), there exists a constant c_N such that

$$\int_{\mathbb{R}^N} \frac{|u(x + h) - u(x)|}{|h|^t}\, dx \leqslant c_N\, (1 - t) P_t(E)\,, \tag{2.4}$$

for all $|h| > 0$. We then estimate

$$\begin{aligned} I_1 &= \int_{|h|<1}\int_{\mathbb{R}^N} \frac{|u(x+h)-u(x)|}{|h|^{N+s}}\,dxdh \\ &\leqslant c_N(1-t)P_t(E)\int_{|h|<1}\frac{1}{|h|^{N-(t-s)}}\,dh \\ &= c_N\frac{1-t}{t-s}P_t(E), \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} I_2 &= \int_{|h|\geqslant 1}\int_{\mathbb{R}^N} \frac{|u(x+h)-u(x)|}{|h|^{N+s}}\,dxdh \\ &\leqslant 2|E|\int_{|h|\geqslant 1}\frac{1}{|h|^{N+s}}\,dh \\ &= \frac{2N\omega_N}{s}|E|. \end{aligned} \tag{2.6}$$

Putting together (2.5) and (2.6) we then get, up to rename c_N,

$$P_s(E) \leqslant c_N\frac{1-t}{t-s}P_t(E) + \frac{N\omega_N}{s}|E|. \tag{2.7}$$

If we evaluate (2.7) on the set λE, with $\lambda > 0$, we obtain

$$\lambda^{N-s}P_s(E) \leqslant c_N\frac{1-t}{t-s}\lambda^{N-t}P_t(E) + \lambda^N\frac{N\omega_N}{s}|E|,$$

that is,

$$\lambda^{t-s}P_s(E) - \lambda^t\frac{N\omega_N|E|}{s} \leqslant \frac{c_N(1-t)}{t-s}P_t(E). \tag{2.8}$$

The expression at the left-hand side of (2.8) reaches its maximum at

$$\lambda = \left(\frac{s(t-s)P_s(E)}{2N\omega_N t|E|}\right)^{\frac{1}{s}}.$$

Substituting this value of λ into (2.8) we get (2.3). □

Remark 2.3. If we let $t \to 1^-$ in (2.3), we recover the estimate in [8, Cor. 4.4]:

$$P_s(E) \leqslant \frac{c_N}{s(1-s)}|E|^{1-s}P(E)^s. \tag{2.9}$$

Indeed the proof of Lemma 2.2 extends to the case $t = 1$, by substituting $(1-t)P_t(E)$ with $P(E)$ in the right hand side of (2.4).

We show now a version of the local fractional isoperimetric inequality. For this, we recall that the fractional perimeter of a set E in a bounded set Ω is defined by

$$P_\alpha(E,\Omega) := \int_{E\cap\Omega}\int_{\mathbb{R}^N\setminus E}\frac{dx\,dy}{|x-y|^{N+\alpha}} + \int_{\Omega\setminus E}\int_{E\setminus\Omega}\frac{dx\,dy}{|x-y|^{N+\alpha}}. \tag{2.10}$$

With this setting, we have a variant of Lemma 2.1 as follows:

Lemma 2.4. *Let Ω_1 and Ω_2 be disjoint bounded sets. Then*

$$P_\alpha(E,\Omega_1) + P_\alpha(E,\Omega_2) \leqslant P_\alpha(E,\Omega_1\cup\Omega_2) + 2\int_{\Omega_1}\int_{\Omega_2}\frac{dx\,dy}{|x-y|^{N+\alpha}}.. \tag{2.11}$$

Proof. We use (2.10) (omitting the integrands for simplicity) to compute

$$\begin{aligned}
&P_\alpha(E,\Omega_1\cup\Omega_2)-P_\alpha(E,\Omega_1)-P_\alpha(E,\Omega_2)\\
=\ &\int_{E\cap(\Omega_1\cup\Omega_2)}\int_{\mathbb{R}^N\setminus E}+\int_{(\Omega_1\cup\Omega_2)\setminus E}\int_{E\setminus(\Omega_1\cup\Omega_2)}\\
&-\int_{E\cap\Omega_1}\int_{\mathbb{R}^N\setminus E}-\int_{\Omega_1\setminus E}\int_{E\setminus\Omega_1}-\int_{E\cap\Omega_2}\int_{\mathbb{R}^N\setminus E}-\int_{\Omega_2\setminus E}\int_{E\setminus\Omega_1}\\
=\ &\int_{E\cap\Omega_1}\int_{\mathbb{R}^N\setminus E}+\int_{E\cap\Omega_2}\int_{\mathbb{R}^N\setminus E}+\int_{\Omega_1\setminus E}\int_{E\setminus(\Omega_1\cup\Omega_2)}+\int_{\Omega_2\setminus E}\int_{E\setminus(\Omega_1\cup\Omega_2)}\\
&-\int_{E\cap\Omega_1}\int_{\mathbb{R}^N\setminus E}-\int_{\Omega_1\setminus E}\int_{E\setminus\Omega_1}-\int_{E\cap\Omega_2}\int_{\mathbb{R}^N\setminus E}-\int_{\Omega_2\setminus E}\int_{E\setminus\Omega_1}\\
=\ &\int_{\Omega_1\setminus E}\int_{E\setminus(\Omega_1\cup\Omega_2)}+\int_{\Omega_2\setminus E}\int_{E\setminus(\Omega_1\cup\Omega_2)}-\int_{\Omega_1\setminus E}\int_{E\setminus\Omega_1}-\int_{\Omega_2\setminus E}\int_{E\setminus\Omega_1}\\
=\ &-\int_{\Omega_1\setminus E}\int_{(E\setminus\Omega_1)\cap\Omega_2}-\int_{\Omega_2\setminus E}\int_{(E\setminus\Omega_2)\cap\Omega_1}.
\end{aligned}$$

This implies (2.11). □

Then, we have the following local fractional isoperimetric inequality:

Lemma 2.5. *Let Ω be a open bounded set with Lipschitz boundary and let $E\subseteq\mathbb{R}^N$ such that $|E\cap\Omega|<|\Omega|/2$. Then there exists a constant $C=C(\Omega,N,\alpha)$ such that*

$$P_\alpha(E,\Omega)\geqslant C|E\cap\Omega|^{\frac{N-\alpha}{N}}. \tag{2.12}$$

Proof. The case $t=1$ is classical and we refer to [30, Section $II.1.6$] for its proof. Notice that (2.12) does not pass to the limit as $\alpha\to1$, as we do not control the constant C. However, we don't need such uniformity when we apply this estimate in the proof of Theorem 7.2.

We begin by recalling the Poincaré-type inequality for fractional Sobolev spaces (see for instance [7, Equations (2) and (3)]): for any $p\geqslant1$ and $\alpha\in(0,1)$, given a function $f\in L^p(\Omega)$ we have that

$$\int_\Omega\int_\Omega\frac{|f(x)-f(y)|^p}{|x-y|^{N+\alpha p}}\geqslant C(N,\alpha,p,\Omega)\,\|f-f_\Omega\|^p_{L^q(\Omega)}, \tag{2.13}$$

where

$$f_\Omega=\frac{1}{|\Omega|}\int_\Omega|f|\,dx$$

and

$$\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{N}. \tag{2.14}$$

By applying (2.13) with $p = 1$, $\alpha \in (0, 1)$ and $f = \chi_E$, and by the very definition of $P_\alpha(E)$ we get that

$$\begin{aligned} 2P_\alpha(E,\Omega) &\geqslant \int_\Omega\int_\Omega \frac{|\chi_E(x)-\chi_E(y)|}{|x-y|^{N+\alpha}} \\ &\geqslant C(N,\alpha,\Omega)\left(\int_\Omega \left|\chi_E(x) - \frac{|E\cap\Omega|}{|\Omega|}\right|^q dx\right)^{1/q} \\ &= C(N,\alpha,\Omega)\left[|E\cap\Omega|\left(1-\frac{|E\cap\Omega|}{|\Omega|}\right)^q + |\Omega\setminus E|\left(\frac{|E\cap\Omega|}{|\Omega|}\right)^q\right]^{1/q} \\ &\geqslant C(N,\alpha,\Omega)|E\cap\Omega|^{1/q}\left(1-\frac{|E\cap\Omega|}{|\Omega|}\right) \\ &\geqslant \frac{C(N,\alpha,\Omega)}{2}|E\cap\Omega|^{1/q}. \end{aligned}$$

Since, by (2.14), $q = N/(N-\alpha)$, the proof is concluded. □

Beside the local fractional isoperimetric inequality (2.12), we recall from [19] the standard (fractional) one: if $0 < t_0 \leqslant \alpha \leqslant 1$ then it holds (if $|E| < +\infty$)

$$(1-\alpha)P_\alpha(E) \ \geqslant\ c(N,t_0)|E|^{\frac{N-\alpha}{N}}, \qquad \text{where } c(N,t_0) = \frac{c_N}{t_0}, \tag{2.15}$$

We now recall some basic facts on hypersingular Riesz operators on the sphere, following [32, pp. 159-160] (see also [18, pp. 4-5]). We denote by $\mathcal{S}_k$ the space of spherical harmonics of degree k, and by $d(k)$ the dimension of $\mathcal{S}_k$. For $\alpha \in (0, 1)$ we also let $\mathcal{J}_\alpha$ be the operator defined as

$$\mathcal{J}_\alpha u(x) = 2\,\text{p.v.}\int_{\partial B}\frac{u(x)-u(y)}{|x-y|^{N+\alpha}}\,d\mathcal{H}^{N-1}(y) \qquad \text{for } u \in C^2(\partial B),$$

(with the symbol p.v. we mean that the integral is considered in the Cauchy principal value sense) and we let λ_k^α be the k^{th} eigenvalue of $\mathcal{J}_\alpha$, that is,

$$\mathcal{J}_\alpha Y = \lambda_k^\alpha Y \qquad \text{for any } Y \in \mathcal{S}_k.$$

We stress that, in the notation λ_k^α, the superscript α denotes an index and not an exponent. We then have $\lambda_k^\alpha \to +\infty$ as $k \to +\infty$, and

$$\lambda_0^\alpha = 0 \qquad \lambda_{k+1}^\alpha > \lambda_k^\alpha \qquad \forall k \in \mathbb{N}\cup\{0\}.$$

The precise formulation of λ_k^α is the following

$$\lambda_k^\alpha = \frac{2^{1+\alpha}\pi^{\frac{N-1}{2}}}{1+\alpha}\frac{\Gamma\left(\frac{1-\alpha}{2}\right)}{\Gamma\left(\frac{N+\alpha}{2}\right)}\left(\frac{\Gamma\left(k+\frac{N+\alpha}{2}\right)}{\Gamma\left(k+\frac{N-2-\alpha}{2}\right)} - \frac{\Gamma\left(\frac{N+\alpha}{2}\right)}{\Gamma\left(\frac{N-2-\alpha}{2}\right)}\right). \tag{2.16}$$

By standard properties of the function Γ it is easy to show that there is a constant c_N such that

$$\frac{1}{c_N} \leqslant (1-\alpha)\frac{2^{1+\alpha}\pi^{\frac{N-1}{2}}}{1+\alpha}\frac{\Gamma\left(\frac{1-\alpha}{2}\right)}{\Gamma\left(\frac{N+\alpha}{2}\right)} \leqslant c_N, \tag{2.17}$$

and

$$\lambda_k^*(s) \leqslant c_N\lambda_k^*(t) \qquad \text{for all } 0 < s \leqslant t < 1, \tag{2.18}$$

where

$$\lambda_k^*(s) = \left(\frac{\Gamma\left(k+\frac{N+s}{2}\right)}{\Gamma\left(k+\frac{N-2-s}{2}\right)} - \frac{\Gamma\left(\frac{N+s}{2}\right)}{\Gamma\left(\frac{N-2-s}{2}\right)}\right).$$

From (2.17) and (2.18) it follows

$$(1-s)\lambda_k^s \leqslant c_N(1-t)\lambda_k^t \qquad \text{for all } 0<s\leqslant t<1. \tag{2.19}$$

Notice that if we let $\{Y_k^i\}_{i=1}^{d(k)}$ be an orthonormal basis of $\mathcal{S}_k$ in $L^2(\partial B)$, and denote by

$$a_k^i(u) := \int_{\partial B} u\, Y_k^i \, d\mathcal{H}^{N-1},$$

the Fourier coefficients of $u \in L^2(\partial B)$ corresponding to Y_k^i, we have

$$\begin{aligned} [u]^2_{H^{\frac{1+\alpha}{2}}(\partial B)} &:= \int_{\partial B}\int_{\partial B} \frac{|u(x)-u(y)|^2}{|x-y|^{N+\alpha}}\, d\mathcal{H}^{N-1}(x)\, d\mathcal{H}^{N-1}(y) \\ &= \int_{\partial B} u\, \mathcal{J}_\alpha u\, d\mathcal{H}^{N-1} \\ &= \sum_{k=0}^{\infty}\sum_{i=0}^{d(k)} \lambda_k^\alpha\, a_k^i(u)^2. \end{aligned} \tag{2.20}$$

The following estimate will be crucial in the proof of Theorem 1.3.

Proposition 2.6. *Let $u \in H^{\frac{1+t}{2}}(\partial B)$ and $0<s\leqslant t<1$ then the following estimate holds*

$$(1-s)[u]^2_{H^{\frac{1+s}{2}}(\partial B)} \leqslant c_N(1-t)[u]^2_{H^{\frac{1+t}{2}}(\partial B)}. \tag{2.21}$$

Proof. Using the estimate (2.19) we get

$$\begin{aligned} (1-s)[u]^2_{H^{\frac{1+s}{2}}(\partial B)} &= (1-s)\sum_{k=0}^{\infty}\sum_{i=0}^{d(k)} \lambda_k^s a_k^i(u)^2 \\ &\leqslant c_N(1-t)\sum_{k=0}^{\infty}\sum_{i=0}^{d(k)} \lambda_k^t a_k^i(u)^2 \\ &= c_N(1-t)[u]^2_{H^{\frac{1+t}{2}}(\partial B)}. \end{aligned}$$

□

Remark 2.7. We note that the result established in the previous proposition remains true also in the case $t=1$. Indeed, since

$$\lim_{t\to 1^-}(1-t)[u]^2_{H^{\frac{1+t}{2}}(\partial B)} = c_N[u]^2_{H^1(\partial B)}$$

as established in [6, Cor. 2], we can pass to the limit $t \to 1^-$ in (2.21).

3. PRELIMINARY ESTIMATES ON THE ENERGY FUNCTIONAL

In the following we shall consider parameters $s,t \in (0,1)$ satisfying

$$t-s \geqslant \delta_0 > 0\,. \tag{3.1}$$

All the constants in this work, unless differently specified, will depend only on N and δ_0, so that it will be possible to pass to the limits in a straightforward way as $s \to 0^+$ or $t \to 1^-$.

Proposition 3.1. *There exists $c_0 = c_0(N,\delta_0)$ such that, for any $E \subset \mathbb{R}^N$ and $0<s<t<1$ satisfying* (3.1)*, it holds*

$$\mathcal{F}_{s,t}(E) \geqslant \frac{(1-t)P_t(E)}{2} - c_0|E|. \tag{3.2}$$

Proof. Set $m := |E|$. We apply Young inequality with exponents $\frac{t}{t-s}$ and $\frac{t}{s}$ to the right hand side of (2.3) getting

$$\begin{aligned} c_N \frac{1}{s}\left(1-\frac{s}{t}\right)^{-1} |E|^{1-\frac{s}{t}}(1-t)^{\frac{s}{t}} P_t(E)^{\frac{s}{t}} &= \left[c_N \frac{2^{\frac{s}{t}}}{s}\left(1-\frac{s}{t}\right)^{-1} m^{1-\frac{s}{t}}\right]\left[2^{-1}(1-t)P_t(E)\right]^{\frac{s}{t}} \\ &\leqslant \left[c_N \frac{2^{\frac{s}{t}}}{s}\left(1-\frac{s}{t}\right)^{-1} m^{1-\frac{s}{t}}\right]^{\frac{t}{t-s}} + \frac{(1-t)P_t(E)}{2}. \end{aligned}$$

Thus (2.3) gives that

$$\begin{aligned} \mathcal{F}_{s,t}(E) &= (1-t)P_t(E) - sP_s(E) \geqslant (1-t)P_t(E) - c_N\left(1-\frac{s}{t}\right)^{-1} |E|^{1-\frac{s}{t}}(1-t)^{\frac{s}{t}} P_t(E)^{\frac{s}{t}} \\ &\geqslant (1-t)P_t(E) - \left[2^{\frac{s}{t}} c_N\left(1-\frac{s}{t}\right)^{-1} m^{1-\frac{s}{t}}\right]^{\frac{t}{t-s}} - \frac{(1-t)P_t(E)}{2} \\ &= \frac{(1-t)P_t(E)}{2} - \left[2^{\frac{s}{t}} c_N \frac{t}{t-s}\right]^{\frac{t}{t-s}} m \end{aligned}$$

and this concludes the proof. □

Corollary 3.2. *Let* $|E| = m$. *Then both* $P_t(E)$ *and* $P_s(E)$ *are bounded above by quantities only depending on* m *and* $\mathcal{F}_{s,t}(E)$. *More explicitly*

$$(1-t)P_t(E) \leqslant 2(\mathcal{F}_{s,t}(E) + c_0 m) \tag{3.3}$$

$$\text{and} \qquad sP_s(E) \leqslant c_0^{1-\frac{s}{t}} m^{1-\frac{s}{t}} (\mathcal{F}_{s,t}(E) + c_0 m)^{\frac{s}{t}}, \tag{3.4}$$

with c_0 *as in Proposition 3.1.*

Proof. We obtain (3.3) easily from Proposition 3.1. Then (3.4) follows from (2.3) and (3.3). □

Now we define the isovolumetric function $\phi : (0,+\infty) \to \mathbb{R}$ as

$$\phi(m) = \inf_{|E|=m} \mathcal{F}_{s,t}(E) \qquad m \in (0,+\infty).$$

A general estimate on $\phi(m)$ goes as follows:

Lemma 3.3. *We have*

$$-c_0 m \leqslant \phi(m) \leqslant c_1 m^{\frac{N-t}{N}}\left(1 - \frac{c_2}{c_1} m^{\frac{t-s}{N}}\right), \tag{3.5}$$

with c_0 *as in Proposition 3.1 and*

$$c_1 := \frac{(1-t)P_t(B)}{|B|^{\frac{N-t}{N}}} \quad \text{and} \quad c_2 := \frac{sP_s(B)}{|B|^{\frac{N-s}{N}}}. \tag{3.6}$$

Proof. Let us begin by proving the estimate from above of $\phi(m)$. For this, we take the unit ball B and we follow the notation introduced in Section 2 to denote by B_m the ball of volume m. Then,

$$P_t(B_m) = \frac{P_t(B)}{|B|^{\frac{N-t}{N}}} m^{\frac{N-t}{N}}$$

and

$$P_s(B_m) = \frac{P_s(B)}{|B|^{\frac{N-s}{N}}} m^{\frac{N-s}{N}}.$$

By minimality, we get, with c_1 and c_2 as in (3.6),

$$\phi(m) \leqslant \mathcal{F}_{s,t}(B_m) = (1-t)P_t(B_m) - sP_s(B_m) = c_1 m^{\frac{N-t}{N}} \left(1 - \frac{c_2}{c_1} m^{\frac{t-s}{N}}\right),$$

that proves (3.5).

The first inequality in (3.5) follows from Proposition 3.1. □

Remark 3.4. We recall the fractional isoperimetric inequality, which holds true for any measurable set E such that $|E| < +\infty$:

$$|E|^{\frac{N-t}{N}} \leqslant c_N t(1-t)P_t(E). \tag{3.7}$$

For the optimal constant c_N we refer to [20] (see in particular Equations (1.10) and (4.2) there).

Lemma 3.5. *There exist $m_0 = m_0(N, \delta_0)$ and $m_1 = m_1(N, \delta_0)$ such that:*

$$\text{if } m > m_1, \text{ then } \phi(m) < 0; \tag{3.8}$$

$$\text{if } m \in (0, m_0), \text{ then } \phi(m) \geqslant \tfrac{c_N}{t} m^{\frac{N-t}{N}} > 0. \tag{3.9}$$

Moreover

$$\lim_{m \to 0^+} \phi(m) = 0. \tag{3.10}$$

Proof. We have that (3.8) and (3.10) plainly follow from (3.5).

Now we prove (3.9). For this, we use Proposition 3.1 and the fractional isoperimetric inequality in the form (3.7) to obtain that, if $|E| = m$,

$$\mathcal{F}_{s,t}(E) \geqslant \frac{(1-t)P_t(E)}{2} - c_0 m \geqslant \frac{m^{\frac{N-t}{N}}}{2\, c_N\, t} - c_0 m = \frac{m^{\frac{N-t}{N}}}{2\, c_N\, t}\left(1 - 2c_0\, c_N\, t\, m^{\frac{t}{N}}\right).$$

In particular, if m is small enough, we have that

$$\mathcal{F}_{s,t}(E) \geqslant \frac{m^{\frac{N-t}{N}}}{4\, c_N\, t}$$

and this implies (3.9). □

Lemma 3.6. *Let m_1 be as in Lemma 3.5, and let F be a minimizer of $\mathcal{F}_{s,t}$ among sets of measure $m > m_1$. We have*

$$\frac{c_N}{t}\, m^{\frac{N-t}{N}} \leqslant (1-t)P_t(F) < c_0\, m \quad \text{and} \quad \frac{c_N}{t}\, m^{\frac{N-t}{N}} < sP_s(F) \leqslant c_0\, m, \tag{3.11}$$

for some $c_0 > 0$.

Proof. By Lemma 3.5 we know that $(1-t)P_t(F) < sP_s(F)$, hence from (2.3) and from the fractional isoperimetric inequality (3.7) we get

$$\frac{m^{\frac{N-t}{N}}}{c_N\, t} \leqslant (1-t)P_t(F) < sP_s(F) \leqslant c_0^{\frac{t-s}{t}}\, 2^{-\frac{s}{t}} m^{1-\frac{s}{t}} [(1-t)P_t(F)]^{\frac{s}{t}}$$

with c_0 given in Proposition 3.1. Then $(1-t)P_t(F) < c_0\, 2^{-\frac{s}{t-s}}\, m$. This and (2.3) also implies the desired bound on $sP_s(F)$. □

Remark 3.7. By inspecting the proof of the Lemma 3.5 we obtain explicit estimates for m_0 and m_1:

$$\begin{aligned} m_0 &\geqslant [4c_0\, c_N\, t]^{-\frac{N}{t}} = \left[4\left(\frac{c_N\, t\, 2^{s/t}}{t-s}\right)^{\frac{t}{t-s}} c_N\, t\right]^{-\frac{N}{t}} \\ m_1 &\leqslant \left(\frac{c_1}{c_2}\right)^{\frac{N}{t-s}} = \left[\frac{(1-t)P_t(B)}{sP_s(B)}\right]^{\frac{N}{t-s}} |B|. \end{aligned}$$

Moreover, the first inequality in the second formula in (3.11)

$$\frac{c_N}{t}|F|^{\frac{N-t}{N}} < sP_s(F)$$

entails that $|F| \to \infty$ as $t \to 0$ (and thus $\delta_0 \to 0$). Indeed, letting $t = s + \delta_0$, and using the fact that $sP_s(F) \to N\omega_N|F|$ as $s \to 0$, after an elementary computation we get that

$$m_1 \geqslant |F| \geqslant \left(\frac{c_N}{s+\delta_0}\right)^{\frac{N}{s+\delta_0}}.$$

which gives also a lower bound on m_1 in terms of s and δ_0. Notice that if $t \to 0$, then also s and δ_0 converge to 0 and so $m_1 \to \infty$. Also it is not a direct consequence of our investigation, we stress that it is natural to expect that also if only s converges to 0, then m_1 diverges to $+\infty$.

We state an elementary numerical inequality which will be useful in the proof of the forthcoming Proposition 3.9.

Lemma 3.8. *Let $\gamma > 0$ and $\lambda = (1+\gamma)^{1/N}$. Then, there exists γ_0, possibly depending on N, s and t, such that for any $\gamma \in (0, \gamma_0)$ and for any a, $b \geqslant 0$, it holds that*

$$(\lambda^{N-t} - 1)a - (\lambda^{N-s} - 1)b \leqslant \gamma\,(a-b). \tag{3.12}$$

Proof. To prove (3.12), we notice that

$$\lim_{\gamma\to 0}(N-s)(1+\gamma)^{\frac{t-s}{N}} - (N-t) = t-s > 0,$$

hence we may take γ small enough, such that

$$(N-s)(1+\gamma)^{\frac{t-s}{N}} - (N-t) \geqslant \frac{t-s}{2}. \tag{3.13}$$

So we write

$$f(\gamma) := \Big((1+\gamma)^{\frac{N-t}{N}} - 1\Big)a - \Big((1+\gamma)^{\frac{N-s}{N}} - 1\Big)b = (\lambda^{N-t} - 1)a - (\lambda^{N-s} - 1)b,$$

and we notice that $f(0) = 0$ and

$$\begin{aligned} f'(\gamma) &= \frac{N-t}{N}(1+\gamma)^{-\frac{t}{N}}a - \frac{N-s}{N}(1+\gamma)^{-\frac{s}{N}}b \\ &= \frac{N-t}{N}(1+\gamma)^{-\frac{t}{N}}(a-b) - \frac{b(1+\gamma)^{-\frac{t}{N}}}{N}\big[(N-s)(1+\gamma)^{\frac{t-s}{N}} - (N-t)\big] \\ &\leqslant (a-b) - \frac{b(1+\gamma)^{-\frac{t}{N}}(t-s)}{2N}, \end{aligned}$$

thanks to (3.13). In particular, $f'(\gamma) \leqslant (a-b)$ and thus $f(\gamma) \leqslant \gamma\,(a-b)$, that establishes (3.12). □

Proposition 3.9 (Non-optimality criterion)**.** *There exists* $\varepsilon = \varepsilon(N, \delta_0)$ *such that, if* $F \subset \mathbb{R}^N$ *is such that* $F = F_1 \cup F_2$*, with* $|F_1 \cap F_2| = 0$*,*

$$|F_2| \leqslant \varepsilon \min(1, |F_1|), \tag{3.14}$$

$$\text{and} \qquad (1-t)[P_t(F_1) + P_t(F_2) - P_t(F)] \leqslant \frac{\mathcal{F}_{s,t}(F_2)}{2}, \tag{3.15}$$

then there exists a set G *with* $|G| = |F|$ *and* $\mathcal{F}_{s,t}(G) < \mathcal{F}_{s,t}(F)$ *(i.e.,* F *is not a minimizer).*

In addition, we have that the set G *is either a ball of volume* m*, or a dilation of the set* F_1*, according to the following formula:*

$$G = \sqrt[N]{1 + \frac{|F_2|}{|F_1|}}\, F_1 \tag{3.16}$$

Proof. Let $m := |F|$, $m_1 := |F_1|$ and $m_2 := |F_2|$. We may suppose that $\mathcal{F}_{s,t}(F)$ is less than or equal than $\mathcal{F}_{s,t}$ of the ball of volume m, B_m, otherwise we can take G equal to such a ball, decrease the energy and finish our proof. That is, we may suppose that

$$\mathcal{F}_{s,t}(F) \leqslant \mathcal{F}_{s,t}(B_m) \leqslant (1-t)P_t(B_m) \leqslant \frac{(1-t)P_t(B)}{|B|^{\frac{N-t}{N}}} m^{\frac{N-t}{N}}. \tag{3.17}$$

Let $G = \lambda F_1$, with $\lambda := \sqrt[N]{1+\gamma}$ and $\gamma = m_2/m_1$. Notice that this is in agreement with (3.16), and also $|G| = m$. Moreover, by (3.14) we have that

$$\gamma \leqslant \frac{\varepsilon \min(1, m_1)}{m_1} \leqslant \varepsilon,$$

so that $\gamma \in (0,1)$ can be taken as small as we like.

By applying inequality (3.12) with $a = (1-t)P_t(F_1)$ and $b = sP_s(F_1)$, we obtain that

$$(\lambda^{N-t} - 1)\,(1-t)P_t(F_1) - (\lambda^{N-s} - 1)\, sP_s(F_1) \leqslant \gamma\, [(1-t)P_t(F_1) - sP_s(F_1)].$$

As a consequence we get

$$\begin{aligned}
\mathcal{F}_{s,t}(G) &= (1-t)P_t(G) - sP_s(G) \\
&= \lambda^{N-t}(1-t)P_t(F_1) - \lambda^{N-s} sP_s(F_1) \\
&= \mathcal{F}_{s,t}(F_1) + \left[(\lambda^{N-t} - 1)(1-t)P_t(F_1) - (\lambda^{N-s} - 1)sP_s(F_1)\right] \\
&\leqslant (1+\gamma)\mathcal{F}_{s,t}(F_1)\,.
\end{aligned}$$

Thus we have, by (2.2) and (3.15),

$$\begin{aligned}
\mathcal{F}_{s,t}(G) - \mathcal{F}_{s,t}(F) &\leqslant (1+\gamma)\mathcal{F}_{s,t}(F_1) - (1-t)P_t(F) + sP_s(F) \\
&\leqslant (1+\gamma)\mathcal{F}_{s,t}(F_1) - (1-t)P_t(F) + sP_s(F_1) + sP_s(F_2) \\
&\leqslant (1+\gamma)\mathcal{F}_{s,t}(F_1) + sP_s(F_1) + sP_s(F_2) \\
&\quad + \frac{1}{2}\mathcal{F}_{s,t}(F_2) - (1-t)P_t(F_1) - (1-t)P_t(F_2) \\
&= \gamma \mathcal{F}_{s,t}(F_1) - \frac{1}{2}\mathcal{F}_{s,t}(F_2). \qquad (3.18)
\end{aligned}$$

Furthermore by (3.9), since m_2 can be chosen in $(0, m_0)$, m_0 as in Lemma 3.5 (up to decreasing the value of ε), we have

$$\mathcal{F}_{s,t}(F_2) \geqslant \phi(m_2) \geqslant \frac{c_N}{t}\, m_2^{\frac{N-t}{N}}. \tag{3.19}$$

Also, using again (2.2) and (3.15), we have that

$$\begin{aligned}\mathcal{F}_{s,t}(F_1) &= [(1-t)P_t(F_1)+(1-t)P_t(F_2)-sP_s(F_1)-sP_s(F_2)]-\mathcal{F}_{s,t}(F_2)\\ &\leqslant \mathcal{F}_{s,t}(F)+[(1-t)P_t(F_1)+(1-t)P_t(F_2)-(1-t)P_t(F)-\mathcal{F}_{s,t}(F_2)]\\ &\leqslant \mathcal{F}_{s,t}(F)-\frac{1}{2}\mathcal{F}_{s,t}(F_2)<\mathcal{F}_{s,t}(F).\end{aligned}$$

This, (3.18) and (3.19) give that

$$\mathcal{F}_{s,t}(G)-\mathcal{F}_{s,t}(F)\leqslant \gamma\mathcal{F}_{s,t}(F)-\frac{1}{2}\mathcal{F}_{s,t}(F_2)\leqslant \gamma\mathcal{F}_{s,t}(F)-\frac{c_N}{2t}\,m_2^{\frac{N-t}{N}}.$$

Accordingly, recalling (3.17) we conclude that

$$\begin{aligned}\mathcal{F}_{s,t}(G)-\mathcal{F}_{s,t}(F) &\leqslant \frac{(1-t)P_t(B)}{|B|^{\frac{N-t}{N}}}\,\gamma(m_1+m_2)^{\frac{N-t}{N}}-\frac{c_N}{2t}\,m_2^{\frac{N-t}{N}}\\ &= m_2^{\frac{N-t}{N}}\left[\frac{(1-t)P_t(B)}{|B|^{\frac{N-t}{N}}}\,\gamma\Big(\gamma^{-1}+1\Big)^{\frac{N-t}{N}}-\frac{c_N}{2t}\right]\\ &\leqslant m_2^{\frac{N-t}{N}}\left[\frac{(1-t)P_t(B)}{|B|^{\frac{N-t}{N}}}\,\gamma\left(2\gamma^{-1}\right)^{\frac{N-t}{N}}-\frac{c_N}{2t}\right]\\ &= m_2^{\frac{N-t}{N}}\left[\frac{2^{\frac{N-t}{N}}(1-t)P_t(B)}{|B|^{\frac{N-t}{N}}}\,\gamma^{\frac{t}{N}}-\frac{c_N}{2t}\right]\end{aligned}$$

which is negative if γ is small enough, i. e.

$$\gamma<\left[\frac{c_N}{2t}\,\frac{|B|^{\frac{N-t}{N}}}{2^{\frac{N-t}{N}}(1-t)P_t(B)}\right]^{\frac{N}{t}}.$$

The proof is concluded. □

When (3.15) does not hold, one obtains for free some interesting density bounds.

Given a measurable set E we denote by $\partial^m E$ the measure theoretic boundary of E defined as

$$\partial^m E=\{x\in\mathbb{R}^N:\ |E\cap B_r(x)|>0 \text{ and } |E\setminus B_r(x)|>0 \text{ for all } r>0\}.$$

Lemma 3.10. *Let F be a set of finite t-perimeter and volume m, and let $x_0\in\mathbb{R}^N$. Assume*

$$\text{either } F_1:=F\setminus B(x_0,r) \text{ and } F_2:=F\cap B(x_0,r), \tag{3.20}$$

$$\text{or } F_2:=F\setminus B(x_0,r) \text{ and } F_1:=F\cap B(x_0,r), \tag{3.21}$$

and suppose that $|F_2|<m_0$, with m_0 be as in Lemma 3.5, and

$$(1-t)[P_t(F_1)+P_t(F_2)-P_t(F)]\geqslant\frac{\mathcal{F}_{s,t}(F_2)}{2}. \tag{3.22}$$

Then

$$\int_{F_1}\int_{F_2}\frac{dx\,dy}{|x-y|^{N+t}}\geqslant\frac{c_N}{t(1-t)}|F_2|^{\frac{N-t}{N}}. \tag{3.23}$$

If $x_0\in\partial^m F$ and (3.22) holds for any $r\leqslant r_0$, we also have the estimate

$$|F\cap B(x_0,r)|\geqslant c_0\,r^N\qquad \text{for all } r\in(0,r_0], \tag{3.24}$$

where the constant $c_0>0$ depend only on N and δ_0.

Proof. Without loss of generality we can assume $x_0 = 0$. Also, using either (3.20) or (3.21), (3.22) and (2.1), we have that

$$\int_{F_1}\int_{F_2} \frac{dx\,dy}{|x-y|^{N+t}} = \frac{1-t}{2(1-t)}(P_t(F_1)+P_t(F_2)-P_t(F)) \geqslant \frac{\mathcal{F}_{s,t}(F_2)}{4(1-t)} \geqslant \frac{\phi(|F_2|)}{4(1-t)}.$$

This and (3.9) (which can be used here thanks to the fact that we are assuming $|F_2| < m_0$) imply (3.23).

Now we prove (3.24). For this, we take F_1 and F_2 as in (3.20) and we define $\mu(r) := |B(0,r)\cap F| = |F_2|$. Note that by the co-area formula

$$\mu'(r) = \mathcal{H}^{N-1}(\partial B(0,r)\cap F), \quad \text{for a. e. } r.$$

Then, by (3.23) and the fact that $F_1 := F\setminus B(0,r) \subset (B(0,r))^c$,

$$\frac{c_N}{t(1-t)}\,\mu(r)^{\frac{N-t}{N}} \leqslant \int_{F_2}\int_{F_1} \frac{dx\,dy}{|x-y|^{N+t}} \leqslant \int_{F_2}\int_{(B(0,r))^c} \frac{dx\,dy}{|x-y|^{N+t}}.$$

For any $x \in F\cap B(0,r)$, we have

$$\int_{(B(0,r))^c} \frac{dy}{|x-y|^{N+t}} \leqslant \int_{(B(x,r-|x|))^c} \frac{dy}{|x-y|^{N+t}} = \frac{N\omega_N}{t}(r-|x|)^{-t}$$

that leads to

$$\int_{F_2}\int_{(B(0,r))^c} \frac{dx\,dy}{|x-y|^{N+t}} \leqslant \frac{c_N}{t}\int_0^r \mu'(z)(r-z)^{-t}\,dz.$$

Finally we arrive at the following integro-differential inequality

$$\mu(r)^{\frac{N-t}{N}} \leqslant c_N(1-t)\int_0^r \mu'(z)(r-z)^{-t}\,dz.$$

We may integrate the last inequality in the r variable on the interval $(0,\rho)$ and get

$$\int_0^\rho \mu(r)^{\frac{N-t}{N}}\,dr \leqslant c_N(1-t)\int_0^\rho\int_0^r \mu'(z)(r-z)^{-t}\,dz\,dr,$$

interchanging the order of integration,

$$\int_0^\rho\int_0^r \mu'(z)(r-z)^{-t}\,dz\,dr = \int_0^\rho \mu'(z)\int_z^\rho (r-z)^{-t}\,dr\,dz,$$

we get

$$\int_0^\rho \mu(r)^{\frac{N-t}{N}}\,dr \leqslant c_N\,\rho^{1-t}\mu(\rho).$$

Now we arrive at the desired result, indeed, following [12] (see the end of p. 9), it is possible to prove that

$$\mu(r) \geqslant g(r) := \left[\frac{1}{2c_N(N+1-t)}\right]^{\frac{N}{t}} r^N \tag{3.25}$$

for any $r < r_0 = (m_0/\omega_N)^{1/N}$, where g satisfies

$$\int_0^\rho g(r)^{\frac{N-t}{N}}\,dr \geqslant 2c_N\,\rho^{1-t}g(\rho),$$

with the same constant c_N as in (3.25). □

The combination of Proposition 3.9 and Lemma 3.10 yield the following density estimate:

Proposition 3.11. *There exist* $r_0 = r_0(m, N, \delta_0) > 0$ *such that, if* F *is a minimizer for* $\phi(m)$ *and* $x_0 \in \partial^m F$*, there holds*

$$|B(x_0, r) \cap F| \geqslant c_0\, r^N$$

for any $r < r_0$*, where* c_0 *is as in* (3.24)*.*

Proof. Let F_1 and F_2 be as in (3.20). Up to choosing r_0 small enough, that is,

$$\omega_N r_0^N \leqslant \varepsilon(N, \delta_0) \min(1, m)\,,$$

we can suppose that F_1 and F_2 satisfy the hypotheses of Proposition 3.9. Thus, since F is a minimum, we obtain that (3.15) cannot hold true. Hence (3.22) is satisfied, and so we can apply (3.24) in Lemma 3.10 and obtain the desired result. ☐

Proposition 3.12. *Let* F *be a minimum for* $\phi(m)$*. Then* F *is essentially bounded. Moreover, if* $t = 1$*, for any* $s < t$*,* $s \in (0, 1)$*,* F *is also essentially connected in the sense of* [2]*, that is, it cannot be decomposed into two disjoint sets* F_1 *and* F_2 *of positive measure such that* $P(F) = P(F_1) + P(F_2)$*.*

Proof. Let F be a minimum. First we prove that it is bounded. By contradiction, if not, there exists a sequence $x_k \in \partial^m F$ such that $|x_k| \to \infty$ as $k \to \infty$. In particular, up to a subsequence, we may suppose that all the balls $B(x_k, 1)$ are disjoint, hence so are the balls $B(x_k, r)$ when $r \in (0, 1)$. Hence

$$m = |F| \geqslant \sum_k |B(x_k, r) \cap F|.$$

On the other hand, by Proposition 3.11, we know that $|B(x_k, r) \cap F| \geqslant c_0 r^N$ if r is small enough, hence we obtain that

$$m \geqslant \sum_k c_0 r^N = +\infty,$$

which is clearly not possible.

This proves that F is bounded. Now we show that, if $t = 1$, F is also essentially connected. Suppose, by contradiction, that F can be decomposed into two disjoint sets F_1 and F_2 of positive measure such that

$$P(F) = P(F_1) + P(F_2). \tag{3.26}$$

Since F is bounded, so are F_1 and F_2, say F_1, $F_2 \subseteq B(0, R)$, for some $R > 0$. Hence, we consider the translation $F_{2,k} := F_2 + (k, 0, \ldots, 0)$ and we observe that if $x \in F_1$ and $y \in F_{2,k}$ we have that

$$|x - y| \geqslant |y| - |x| \geqslant k - 2R \geqslant \frac{k}{2}$$

if k is large enough. Accordingly, we have that

$$\int_{F_1} \int_{F_{2,k}} \frac{dx\, dy}{|x - y|^{N+s}} \leqslant \int_{B(0,R)} \int_{B(0,R)+(k,0,\ldots,0)} \frac{dx\, dy}{(k/2)^{N+s}} = \frac{c_N R^{2N}}{k^{N+s}}$$

and so

$$\lim_{k \to +\infty} \int_{F_1} \int_{F_{2,k}} \frac{dx\, dy}{|x - y|^{N+s}} = 0. \tag{3.27}$$

Notice also that, if $G_k := F_1 \cup F_{2,k}$ we have that $|G_k| = |F_1| + |F_{2,k}| = |F_1| + |F_2| = |F|$, for k large, and so, by the minimality of F, (2.1), (2.2) and (3.26) we have that

$$\begin{aligned} & N\omega_N P(F_1) + N\omega_N P(F_2) - sP_s(F_1) - sP_s(F_2) + 2s \int_{F_1}\int_{F_2} \frac{dx\,dy}{|x-y|^{N+s}} \\ = \ & N\omega_N P(F) - sP_s(F) \\ = \ & \mathcal{F}_{s,1}(F) \\ \leqslant \ & \mathcal{F}_{s,1}(G_k) \\ = \ & N\omega_N P(G_k) - sP_s(G_k) \\ \leqslant \ & N\omega_N P(F_1) + N\omega_N P(F_{2,k}) - sP_s(F_1) - sP_s(F_{2,k}) + 2s \int_{F_1}\int_{F_{2,k}} \frac{dx\,dy}{|x-y|^{N+s}} \\ = \ & N\omega_N P(F_1) + N\omega_N P(F_2) - sP_s(F_1) - sP_s(F_2) + 2s \int_{F_1}\int_{F_{2,k}} \frac{dx\,dy}{|x-y|^{N+s}}. \end{aligned}$$

Therefore, taking the limit as $k \to +\infty$ and using (3.27), we obtain that

$$2s \int_{F_1}\int_{F_2} \frac{dx\,dy}{|x-y|^{N+s}} \leqslant 0.$$

This says that either F_1 or F_2 must have zero measure, against our assumptions. □

We conclude the section with the following estimate on the fractional isoperimetric deficit, which will be important to localize minimizing sequences.

Lemma 3.13. *There exists $m_2 = m_2(N, \delta_0)$ such that for any $m \in (0, m_2)$ the following statement holds true.*

Let $F \subset \mathbb{R}^N$ be a set of finite perimeter. Assume that $\mathcal{F}_{s,t}(F) \leqslant \mathcal{F}_{s,t}(B_m)$. Then there exists $c_0 > 0$ such that

$$\delta P_t(F) = \frac{P_t(F) - P_t(B_m)}{P_t(B_m)} \leqslant c_0\, m^{\frac{t-s}{N}}. \tag{3.28}$$

In addition, there exists a translation of F (still denoted by F for simplicity) such that

$$|F \,\Delta\, B_m| \leqslant c_0\, m^{1+\frac{t-s}{2N}}. \tag{3.29}$$

Proof. First recall that

$$P_t(B_m) = \frac{P_t(B)}{|B|^{\frac{N-t}{N}}}\, m^{\frac{N-t}{N}}. \tag{3.30}$$

Also, by our assumptions,

$$(1-t)P_t(F) - sP_s(F) = \mathcal{F}_{s,t}(F) \leqslant \mathcal{F}_{s,t}(B_m) \leqslant (1-t)P_t(B_m). \tag{3.31}$$

Using (3.4) we have that

$$\begin{aligned} sP_s(F) \ &\leqslant\ c_0^{1-\frac{s}{t}} m^{1-\frac{s}{t}} [(1-t)P_t(B_m) + c_0\, m]^{\frac{s}{t}} \\ &=\ c_0^{1-\frac{s}{t}} m^{1-\frac{s}{t}} \left[\frac{(1-t)P_t(B)}{|B|^{\frac{N-t}{N}}}\, m^{\frac{N-t}{N}} + c_0\, m \right]^{\frac{s}{t}} \\ &\leqslant\ c_0^{1-\frac{s}{t}} \left[\frac{(1-t)P_t(B)}{|B|^{\frac{N-t}{N}}} + c_0 \right]^{\frac{s}{t}} m^{\frac{N-s}{N}}, \end{aligned}$$

for small m. From this and (3.31), we have that

$$\frac{P_t(F)-P_t(B_m)}{P_t(B_m)} \leqslant c_0^{1-\frac{s}{t}} \left[\frac{(1-t)P_t(B)}{|B|^{\frac{N-t}{N}}}+c_0\right]^{\frac{s}{t}} \frac{|B|^{\frac{N-t}{N}}}{(1-t)P_t(B)}\, m^{\frac{N-s}{N}-\frac{N-t}{N}} \leqslant c_0 m^{\frac{t-s}{N}}\,.$$

This proves (3.28).

To prove (3.29) it is sufficient to use (3.28) and the estimate

$$c_0\,\delta P_t(F) \geqslant \frac{|F\,\Delta\,B_m|^2}{|B_m|^2},$$

which was proved in [18, Theorem 1.1] for any $t \geqslant \delta_0 > 0$. Together with (3.28) and possibly increasing the constant c_0, this implies (3.29). □

4. Existence of minimizers

In order to prove the first statement in Theorem 1.2, and for further use as well, we prove a general result on integro-differential equations:

Lemma 4.1. *Let $m, t \in (0,1)$. Let $c, \bar\rho \geqslant 0$ be such that*

$$c \geqslant (1-t)m^{\frac{t}{N}}\,, \tag{4.1}$$

and let $\mu : [0,+\infty) \to [0,m]$ be a non-increasing function such that

$$-\int_\rho^\infty \mu'(z)(z-\rho)^{-t}\,dz \geqslant \frac{3c}{1-t}\mu(\rho)^{\frac{N-t}{N}} \qquad \textit{for all } \rho \geqslant \bar\rho\,. \tag{4.2}$$

Then, there holds

$$\mu\left(\bar\rho + \frac{(2m)^{\frac{t}{N}}N}{ct}\right) = 0\,. \tag{4.3}$$

Proof. Integrating (4.2) between $R \geqslant \bar\rho$ and $+\infty$, we obtain

$$-\int_R^\infty \left(\int_\rho^\infty \mu'(z)(z-\rho)^{-t}\,dz\right) d\rho \geqslant \frac{3c}{1-t}\int_R^\infty \mu(\rho)^{\frac{N-t}{N}}\,d\rho. \tag{4.4}$$

Also, if $z \in [R, R+1]$ we have that $z - R \leqslant 1$ and so, since $\mu' \leqslant 0$ a. e., we get that

$$-\int_R^{R+1} \mu'(z)(z-R)^{1-t}\,dz \leqslant -\int_R^{R+1} \mu'(z)\,dz = \mu(R)-\mu(R+1).$$

Therefore, interchanging the order of integration in (4.4), integrating by parts and using that $\mu \in [0,m]$ and (4.1), we see that

$$\begin{aligned}
-\int_R^\infty \left(\int_\rho^\infty \mu'(z)(z-\rho)^{-t}\,dz\right) d\rho &= -\int_R^\infty \left(\int_R^z \mu'(z)(z-\rho)^{-t}\,d\rho\right) dz \\
&= -\frac{1}{1-t}\int_R^\infty \mu'(z)(z-R)^{1-t}\,dz \\
&\leqslant \frac{\mu(R)-\mu(R+1)}{1-t} - \frac{1}{1-t}\int_{R+1}^\infty \mu'(z)(z-R)^{1-t}\,dz \\
&= \frac{\mu(R)}{1-t} + \int_{R+1}^\infty \mu(z)(z-R)^{-t}\,dz \\
&\leqslant \frac{\mu(R)}{1-t} + \int_{R+1}^\infty \mu(z)\,dz \\
&\leqslant \frac{\mu(R)}{1-t} + m^{\frac{t}{N}}\int_R^\infty \mu(z)^{\frac{N-t}{N}}\,dz \\
&\leqslant \frac{1}{1-t}\left(\mu(R) + c\int_R^\infty \mu(z)^{\frac{N-t}{N}}\,dz\right).
\end{aligned}$$

Recalling (4.4), this gives the integro-differential inequality

$$\mu(\rho) \geqslant 2c\int_\rho^\infty \mu(z)^{\frac{N-t}{N}}\,dz \qquad \text{for all } \rho \geqslant \bar\rho\,. \tag{4.5}$$

Let now

$$g(\rho) := \begin{cases} \left[(2\mu(\bar\rho))^{\frac{t}{N}} - \dfrac{ct}{N}(\rho-\bar\rho)\right]^{\frac{N}{t}} & \text{if } \rho \in \left[\bar\rho, \bar\rho + \dfrac{(2\mu(\bar\rho))^{\frac{t}{N}}N}{ct}\right] \\[2ex] 0 & \text{if } \rho > \bar\rho + \dfrac{(2\mu(\bar\rho))^{\frac{t}{N}}N}{ct}\,. \end{cases}$$

Notice that g is continuous and it satisfies

$$2c\int_\rho^\infty g(z)^{\frac{N-t}{N}}\,dz = 2g(\rho) \qquad \text{for all } \rho \in \left[\bar\rho, \bar\rho + \frac{(2\mu(\bar\rho))^{\frac{t}{N}}N}{ct}\right]. \tag{4.6}$$

We now claim that

$$g(\rho) \geqslant \mu(\rho) \qquad \text{for all } \rho \in \left[\bar\rho, \bar\rho + \frac{(2\mu(\bar\rho))^{\frac{t}{N}}N}{ct}\right]. \tag{4.7}$$

Indeed, we consider the set $I := \{\rho > \bar\rho \,:\, \mu(z) \geqslant g(z) \text{ for all } z \geqslant \rho\}$. By construction, $I \subseteq [\bar\rho, +\infty)$. Furthermore, if $z \geqslant \bar\rho + [(2\mu(\bar\rho))^{\frac{t}{N}}N]/ct$ then $g(z) = 0 \leqslant \mu(z)$, therefore $\bar\rho + [(2\mu(\bar\rho))^{\frac{t}{N}}N]/ct \in I$. As a consequence, we can define $R_* := \inf I$, and we have that

$$R_* \in [\bar\rho,\ \bar\rho + [(2\mu(\bar\rho))^{\frac{t}{N}}N]/ct]. \tag{4.8}$$

By definition of R_*, there exists a sequence $R_n \to R_*$, with $R_n \leqslant R_*$, such that $g(R_n) > \mu(R_n)$. Then, recalling (4.5) and (4.6), we have

$$\begin{aligned} g(R_n) &> \mu(R_n) \\ &\geqslant 2c \int_{R_n}^{\infty} \mu(z)^{\frac{N-t}{N}} dz \\ &\geqslant 2c \int_{R_n}^{R_*} \mu(z)^{\frac{N-t}{N}} dz + 2c \int_{R_*}^{\infty} g(z)^{\frac{N-t}{N}} dz \\ &= 2c \int_{R_n}^{R_*} \mu(z)^{\frac{N-t}{N}} dz + 2g(R_*). \end{aligned} \tag{4.9}$$

Passing to the limit in (4.9) as $n \to +\infty$ we get $g(R_*) \geqslant 2g(R_*)$, which means $g(R_*) = 0$. This implies that $R_* \geqslant \bar{\rho} + [(2\mu(\bar{\rho}))^{\frac{t}{N}} N]/ct$.

This information, combined with (4.8), gives that $R_* = \bar{\rho} + [(2\mu(\bar{\rho}))^{\frac{t}{N}} N]/ct$, and this in turn implies (4.7).

Then, we evaluate (4.7) at $\rho = \bar{\rho} + [(2\mu(\bar{\rho}))^{\frac{t}{N}} N]/ct$ and we obtain (4.3).

□

With the above result, we are able to prove the first statement in Theorem 1.2, concerning the existence of minimizers for small volumes.

Theorem 4.2. *For any $0 \leqslant s < t \leqslant 1$, $t - s \geqslant \delta_0 > 0$, there exists $\bar{m}_0 = \bar{m}_0(N, \delta_0) > 0$ such that for all $m \in (0, \bar{m}_0)$, problem (1.7) has a minimizer $F \subset \mathbb{R}^N$.*

Proof. Suppose $0 < s < t < 1$. We use the Direct Method of the Calculus of Variations. Let us consider a minimizing sequence $\{F_k\} \subset \mathbb{R}^N$, that is a sequence of sets of finite t-perimeter F_k with $|F_k| = m$ such that

$$\lim_{k \to \infty} \mathcal{F}_{s,t}(F_k) = \phi(m). \tag{4.10}$$

Let also set $r_m := (m/\omega_N)^{1/N} > 0$, so that $|B(0, r_m)| = m$. Our goal is to show that we can reduce ourselves to the case in which F_k lies in a large ball, independent of k. More precisely, we claim that there exist $\rho_* > 0$ and sets G_k, with $|G_k| = m$, such that

$$G_k \subseteq B(0, \rho_*) \text{ and } \mathcal{F}_{s,t}(G_k) \leqslant \mathcal{F}_{s,t}(F_k). \tag{4.11}$$

To prove it, we take $\rho \geqslant r_m$ and we set

$$X_k^\rho := F_k \cap B(0, \rho) \text{ and } Y_k^\rho := F_k \setminus B(0, \rho). \tag{4.12}$$

We distinguish two cases:

$$\text{either for any } \rho \geqslant r_m \text{ we have } \quad (1-t)[P_t(X_k^\rho) + P_t(Y_k^\rho) - P_t(F_k)] \geqslant \frac{\mathcal{F}_{s,t}(Y_k^\rho)}{2} \tag{4.13}$$

$$\text{or there exists } \rho \geqslant r_m \text{ such that } \quad (1-t)[P_t(X_k^\rho) + P_t(Y_k^\rho) - P_t(F_k)] \leqslant \frac{\mathcal{F}_{s,t}(Y_k^\rho)}{2}. \tag{4.14}$$

Let us first deal with (4.13). In this case we can apply Lemma 3.10 using the setting in (3.21): accordingly, from (3.23) we see that

$$\int_{X_k^\rho} \int_{Y_k^\rho} \frac{dx\, dy}{|x-y|^{N+t}} \geqslant \frac{c_N}{t(1-t)} |Y_k^\rho|^{\frac{N-t}{N}}.$$

Let us define the non-increasing function $\eta(\rho) := |F_k \setminus B(0,\rho)| = |Y_k^\rho|$. Note that by the co-area formula

$$\eta'(\rho) = -\mathcal{H}^{N-1}(\partial B(0,\rho) \cap F), \quad \text{for a. e. } \rho > 0.$$

Proceeding as in the proof of Lemma 3.10, we have

$$\begin{aligned}
\int_{Y_k^\rho}\int_{X_k^\rho} \frac{dx\,dy}{|x-y|^{N+t}} &\leqslant \int_{Y_k^\rho}\int_{B(0,\rho)} \frac{dx\,dy}{|x-y|^{N+t}} \\
&\leqslant \int_{Y_k^\rho}\left(\int_{(B(y,|y|-\rho))^c} \frac{dx}{|x-y|^{N+t}}\right) dy \\
&\leqslant -\frac{N\omega_N}{t}\int_\rho^\infty \eta'(z)(z-\rho)^{-t}\,dz,
\end{aligned}$$

whence

$$-\int_\rho^\infty \eta'(z)(z-\rho)^{-t}\,dz \geqslant \frac{c_N}{1-t}\eta(\rho)^{\frac{N-t}{N}},$$

that is, η satisfies inequality (4.2). We now apply Lemma 4.1 with $\mu = \eta$, $c = c_N/3$ and $\bar\rho = r_m$. Notice that, possibly reducing $\bar m_0$, we can ensure that condition (4.1) is satisfied. From (4.3) we conclude that

$$\eta\left(r_m + \frac{3(2m)^{\frac{t}{N}}N}{c_N t}\right) = 0\,,$$

that is,

$$F_k \subseteq B\left(0, r_m + \frac{3(2m)^{\frac{t}{N}}N}{c_N t}\right).$$

This proves (4.11) with ρ_* given by

$$\rho_* := r_m + \frac{3(2m)^{\frac{t}{N}}N}{c_N t}$$

in the case where (4.13) holds (here one can take $G_k := F_k$).

We now deal with case (4.14). In this case, we use (3.29) and we obtain (up to a translation of F_k that is still denoted by F_k) that

$$|F_k \setminus B(0,r_m)| + |B(0,r_m) \setminus F_k| = |F_k \,\Delta\, B(0,r_m)| \leqslant c_0\, m^{1+\frac{t-s}{2N}}\,,$$

c_0 as in Lemma 3.13. In particular, if $\rho \geqslant r_m$ is the one given by (4.14) we have that

$$\begin{aligned}
|F_k \cap B(0,\rho)| &\geqslant |F_k \cap B(0,r_m)| \\
&= |B(0,r_m)| - |B(0,r_m) \setminus F_k| \\
&\geqslant m - c_0\, m^{1+\frac{t-s}{2N}} \\
&\geqslant \frac{m}{2}
\end{aligned}$$

if m is small enough, i. e.

$$m \leqslant \left[\frac{1}{2c_0}\right]^{\frac{2N}{t-s}} \tag{4.15}$$

and moreover

$$|F_k \setminus B(0,\rho)| \leqslant |F_k \setminus B(0,r_m)| \leqslant c_0\, m^{1+\frac{t-s}{2N}}.$$

Therefore, for small m, recalling (4.12) we see that

$$\begin{aligned} 2c_0\, m^{\frac{t-s}{2N}} \min\big(1, |X_k^\rho|\big) &= 2c_0\, m^{\frac{t-s}{2N}} \min\big(1, |F_k \cap B(0,\rho)|\big) \geqslant 2c_0\, m^{\frac{t-s}{2N}} \frac{m}{2} \\ &= c_0\, m^{1+\frac{t-s}{2N}} \geqslant |F_k \setminus B(0,\rho)| = |Y_k^\rho|. \end{aligned}$$

Thanks to this and (4.14), we can apply Proposition 3.9, with $\varepsilon := 2c_0\, m^{\frac{t-s}{2N}}$, $F_1 := X_k^\rho$ and $F_2 := Y_k^\rho$.

Hence, from Proposition 3.9, we find G_k such that $\mathcal{F}_{s,t}(G_k) \leqslant \mathcal{F}_{s,t}(F_k)$; notice also that, in light of (3.16), we know that G_k is either a ball or a dilation of X_k^ρ, which is contained in $B(0, 2\rho)$. Thus also G_k is contained in a ball of universal radius, and this establishes (4.11) also in case (4.14).

Thus, by (4.11), we have constructed a minimizing sequence G_k that is uniformly contained in a fixed ball. By Proposition 3.1, we also obtain that

$$(1-t)P_t(G_k) \leqslant 2[\mathcal{F}_{s,t}(G_k) + c_0 m] \leqslant 2[\mathcal{F}_{s,t}(B(0, r_m)) + c_0 m],$$

hence the t-perimeter of G_k is bounded uniformly in k.

By the compact embedding of $H^{\frac{t}{2}}$ into $H^{\frac{s}{2}}$ (see [16, Section 7]), up to extracting a subsequence, the sets G_k converge in $W^{s,1}$ (hence also in L^1) to a limit set G, and it holds

$$\lim_{k\to+\infty} P_s(G_k) = P_s(G).$$

The lower semicontinuity of the t-perimeter yields that

$$\liminf_{k\to+\infty} P_t(G_k) \geqslant P_t(G)$$

Hence, by (4.10) and (4.11),

$$\begin{aligned} \mathcal{F}_{s,t}(G) &= (1-t)P_t(G) - sP_s(G) \leqslant \liminf_{k\to+\infty} [(1-t)P_t(G_k) - sP_s(G_k)] \\ &= \liminf_{k\to+\infty} \mathcal{F}_{s,t}(G_k) \leqslant \liminf_{k\to+\infty} \mathcal{F}_{s,t}(F_k) \leqslant \phi(m), \end{aligned}$$

hence $\mathcal{F}_{s,t}(G) = \phi(m)$ and so $F := G$ is the desired minimizer.

In the case $0 = s < t \leqslant 1$, our problem reduces to the (fractional) isoperimetric problem, hence it is well known that there exists a minimizer F for (1.7) and it is a ball of volume m, for any $m > 0$.

When $0 < s < t = 1$ the previous arguments can be easily adapted, including the analog of Lemma 4.1 which becomes an ordinary differential inequality, and the only difference is that one needs to use the compact embedding of BV into $H^{\frac{s}{2}}$ for $0 < s < 1$. □

5. Regularity of minimizers

The aim of this section is to prove the regularity and rigidity theory necessary to prove the second statement in Theorem 1.2 and Theorem 1.3. We begin with a simple observation.

Lemma 5.1. *Let ϕ be the function describing problem* (1.7)*. Then F is a minimizer of $\phi(m)$ if and only if $F/m^{1/N}$ is a minimizer of problem*

$$\min\{(1-t)P_t(U) - m^{\frac{t-s}{N}} sP_s(U) : |U| = 1\}.$$

Proof. Let $F \subseteq \mathbb{R}^N$ such that $|F| = m$ and let $U = F/m^{1/N}$. Then

$$\begin{aligned} (1-t)P_t(F) - sP_s(F) &= (1-t)P_t(m^{1/N}U) - sP_s(m^{1/N}U) \\ &= m^{\frac{N-t}{N}}\left[(1-t)P_t(U) - m^{\frac{t-s}{N}} sP_s(U)\right], \end{aligned}$$

which gives the desired result. □

The previous lemma allows us to consider, in what follows, the functional

$$\mathcal{F}^{\varepsilon}_{s,t} = (1-t)P_t - \varepsilon s P_s,$$

where we set $\varepsilon = m^{(t-s)/N}$. Indeed, the behavior of a minimizer of $\phi(m)$ is the same, up to a rescaling, to that of

$$\min\left\{\mathcal{F}^{\varepsilon}_{s,t}(E) : |E| = \omega_N\right\}. \tag{5.1}$$

Indeed

$$\begin{aligned} &F \text{ is a minimizer for problem (5.1) if and only if} \\ &\left(\tfrac{m}{\omega_N}\right)^{\frac{1}{N}} F \text{ is a minimizer for problem (1.7) with } \varepsilon = \left(\tfrac{m}{\omega_N}\right)^{\frac{t-s}{N}}. \end{aligned} \tag{5.2}$$

The next lemma allows us to say that if F is a set of $\mathbb{R}^N$ such that $||F| - \omega_N|$ is small enough than the volume constraint can be dropped. Let us consider the following problem:

$$\min\left\{\mathcal{G}_{\varepsilon,\Lambda}(E) : ||E| - \omega_N| < \Lambda^{-1}\right\}, \tag{5.3}$$

for some $\Lambda > 0$, where

$$\mathcal{G}_{\varepsilon,\Lambda}(E) = (1-t)P_t(E) - \varepsilon s P_s(E) + \Lambda||E| - \omega_N|.$$

Letting

$$\varepsilon_0 := \left(\frac{\bar{m}_0}{\omega_N}\right)^{\frac{t-s}{N}}, \tag{5.4}$$

with $\bar{m}_0$ as in Theorem 4.2, we have the following result:

Lemma 5.2. *There exists $\Lambda_0 = \Lambda_0(N, \delta_0) > 0$ such that F_ε is a volume constrained minimizer of problem (5.1), with $\varepsilon < \varepsilon_0$, if and only if F_ε is a minimizer of problem (5.3), for any $\Lambda \geqslant \Lambda_0(1 + \varepsilon_0)$.*

Proof. First, let F_ε be a minimizer of problem (5.3) with $|F_\varepsilon| = \omega_N$. Then, for any set G with $|G| = \omega_N$, we have that

$$\mathcal{F}^{\varepsilon}_{s,t}(G) = \mathcal{G}_{\varepsilon,\Lambda}(G) \geqslant \mathcal{G}_{\varepsilon,\Lambda}(F_\varepsilon) = \mathcal{F}^{\varepsilon}_{s,t}(F_\varepsilon),$$

which shows that F_ε is a minimizer of problem (5.1).

Viceversa, we prove that a volume constrained minimizer F_ε of problem (5.1), with $\varepsilon < \varepsilon_0$, is also a minimizer of (5.3) for any Λ sufficiently large. For this, we argue by contradiction and we assume that there exist a sequence $\Lambda_n \to +\infty$, and sets $E_n \subset \mathbb{R}^N$ such that, letting $\mathcal{G}_n := \mathcal{G}_{\varepsilon,\Lambda_n}$, we have

$$\mathcal{G}_n(E_n) < \mathcal{G}_n(F_\varepsilon) = \mathcal{F}^{\varepsilon}_{s,t}(F_\varepsilon). \tag{5.5}$$

Notice that for all $n \in \mathbb{N}$ there holds

$$\sigma_n := \big||E_n| - \omega_N\big| > 0. \tag{5.6}$$

Indeed, if by contradiction we suppose that $\sigma_n = 0$ for some $n \in \mathbb{N}$, we would have that $|E_n| = \omega_N$, thus

$$\mathcal{G}_n(E_n) = \mathcal{F}^{\varepsilon}_{s,t}(E_n) \geqslant \mathcal{F}^{\varepsilon}_{s,t}(F_\varepsilon),$$

due to the minimality of F_ε. This would be in contradiction with (5.5), and so (5.6) is proved. We also claim that there exists a constant $c_0 > 0$ independent of n, such that

$$(1-t)P_t(E_n) \leqslant c_0 \qquad \text{and} \qquad sP_s(E_n) \leqslant c_0 \qquad \text{for all } n \in \mathbb{N}. \tag{5.7}$$

To show this, proceeding as in Proposition 3.1 and thanks to (5.5), we see that, for $\Lambda_n \geqslant c_0\, \varepsilon_0^{\frac{t}{t-s}}$, we have

$$\begin{aligned}
(1-t)P_t(E_n) &\leqslant 2\left[\mathcal{F}^{\varepsilon}_{s,t}(E_n) + c_0\varepsilon_0^{\frac{t}{t-s}}|E_n|\right] \\
&\leqslant 2\left[\mathcal{F}^{\varepsilon}_{s,t}(E_n) + c_0\, \varepsilon_0^{\frac{t}{t-s}}\big||E_n| - \omega_N\big| + \omega_N c_0 \varepsilon_0^{\frac{t}{t-s}}\right] \\
&\leqslant 2\left[\mathcal{G}_n(E_n) + \omega_N c_0\, \varepsilon_0^{\frac{t}{t-s}}\right] \\
&\leqslant 2\left[\mathcal{F}^{\varepsilon}_{s,t}(F_\varepsilon) + \omega_N c_0\, \varepsilon_0^{\frac{t}{t-s}}\right] \\
&\leqslant 2\left[\mathcal{F}^{\varepsilon}_{s,t}(B) + \omega_N c_0\, \varepsilon_0^{\frac{t}{t-s}}\right] \\
&\leqslant 2\left[(1-t)P_t(B) + \omega_N c_0\, \varepsilon_0^{\frac{t}{t-s}}\right],
\end{aligned}$$

recalling that B denotes the ball centered in 0 and radius 1, with $|B(0,1)| = \omega_N$. This gives the bound for $(1-t)P_t(E_n)$, and then the bound on $sP_s(E_n)$ follows from (2.3). This proves (5.7).

From (5.5) and (5.7) it follows that $\Lambda_n\sigma_n$ is also uniformly bounded, that is,

$$\sigma_n \leqslant \frac{c_0}{\Lambda_n} \to 0 \qquad \text{as } n \to +\infty.$$

Moreover, for $\sigma_n \leqslant 1/2$ we have, supposing $\sigma_n = |E_n| - \omega_N > 0$ (the other case can be treated in a similar way),

$$\left(\frac{|E_n|}{\omega_N}\right)^{-\frac{N-s}{N}} = \left(1 + \frac{\sigma_n}{\omega_N}\right)^{-\frac{N-s}{N}} \geqslant 1 - \frac{N-s}{N}\frac{\sigma_n}{\omega_N}\,, \tag{5.8}$$

and similarly

$$\left(\frac{|E_n|}{\omega_N}\right)^{-\frac{N-t}{N}} = \left(1 - \frac{\sigma_n}{\omega_N}\right)^{-\frac{N-t}{N}} \leqslant 1 + C\frac{N-t}{N}\frac{\sigma_n}{\omega_N}, \tag{5.9}$$

with $C = C(N,s,t)$. We now define

$$\tilde{E}_n = \left(\frac{|E_n|}{\omega_N}\right)^{-\frac{1}{N}} E_n\,,$$

and we use (5.7), (5.8) and (5.9) to obtain

$$sP_s(\tilde{E}_n) = \left(\frac{|E_n|}{\omega_N}\right)^{-\frac{N-s}{N}} sP_s(E_n) \geqslant \left(1 - \frac{N-s}{N}\frac{\sigma_n}{\omega_N}\right) sP_s(E_n) \geqslant sP_s(E_n) - c_0\sigma_n,$$

$$\text{and} \qquad (1-t)P_t(\tilde{E}_n) = \left(\frac{|E_n|}{\omega_N}\right)^{-\frac{N-t}{N}} (1-t)P_t(E_n) \leqslant (1-t)P_t(E_n) + c_0\sigma_n\,,$$

where the constant c_0 may differ from line to line.

Therefore, since $|\tilde{E}_n| = \omega_N$, the minimality of F_ε gives

$$\begin{aligned} \mathcal{F}^\varepsilon_{s,t}(F_\varepsilon) &\leqslant \mathcal{F}^\varepsilon_{s,t}(\tilde{E}_n) = (1-t)P_t(\tilde{E}_n) - \varepsilon s P_s(\tilde{E}_n) \\ &\leqslant (1-t)P_t(E_n) - \varepsilon s P_s(E_n) + c_0(1+\varepsilon_0)\sigma_n \\ &= \mathcal{F}^\varepsilon_{s,t}(E_n) + c_0(1+\varepsilon_0)\sigma_n. \end{aligned}$$

By plugging this into (5.5) we find that

$$\begin{aligned} \mathcal{G}_\varepsilon(E_n) &< \mathcal{F}^\varepsilon_{s,t}(F_\varepsilon) \leqslant \mathcal{F}^\varepsilon_{s,t}(E_n) + c_0(1+\varepsilon_0)\sigma_n \\ &= \mathcal{G}_\varepsilon(E_n) - \Lambda_n\sigma_n + c_0(1+\varepsilon_0)\sigma_n. \end{aligned}$$

We simplify the term $\mathcal{G}_\varepsilon(E_n)$ and we divide by σ_n, which is possible thanks to (5.6), we conclude that

$$0 < -\Lambda_n + c_0(1+\varepsilon_0).$$

This gives a contradiction for Λ_n large enough, and proves that F_ε is a minimizer for problem (5.3). $\square$

Lemma 5.3. *Let F_ε be a minimizer of problem* (5.3) *with $\varepsilon < \varepsilon_0$ and $\Lambda \geqslant \Lambda_0$, ε_0 and Λ_0 as in Lemma 5.2, and let E_ε be a set of finite perimeter with $\big||E_\varepsilon| - \omega_N\big| < 1/\Lambda$. Then,*

$$\begin{aligned} (1-t)P_t(F_\varepsilon) &\leqslant (1-t)P_t(E_\varepsilon) + \varepsilon\, c_N \left(1 - \frac{s}{t}\right)^{-1} |F_\varepsilon \Delta E_\varepsilon|^{1-\frac{s}{t}} [(1-t)P_t(F_\varepsilon\Delta E_\varepsilon)]^{\frac{s}{t}} \\ &\quad + \Lambda\big||E_\varepsilon| - \omega_N\big|. \end{aligned} \tag{5.10}$$

Proof. Notice that, denoting by $\int_U = \int_U f$ for a non-negative function f, the following computation holds

$$\int_{F_\varepsilon}\int_{F_\varepsilon^c} = \int_{F_\varepsilon\setminus E_\varepsilon}\int_{(F_\varepsilon\cup E_\varepsilon)^c} + \int_{F_\varepsilon\setminus E_\varepsilon}\int_{E_\varepsilon\setminus F_\varepsilon} + \int_{F_\varepsilon\cap E_\varepsilon}\int_{(F_\varepsilon\cup E_\varepsilon)^c} + \int_{F_\varepsilon\cap E_\varepsilon}\int_{E_\varepsilon\setminus F_\varepsilon}.$$

By interchanging the roles of F_ε and E_ε, and setting $f(x,y) = |x-y|^{-(s+N)}$ we get

$$\begin{aligned} P_s(F_\varepsilon) - P_s(E_\varepsilon) &= \int_{F_\varepsilon\setminus E_\varepsilon}\int_{(F_\varepsilon\cup E_\varepsilon)^c} - \int_{E_\varepsilon\setminus F_\varepsilon}\int_{(F_\varepsilon\cup E_\varepsilon)^c} + \int_{F_\varepsilon\cap E_\varepsilon}\int_{E_\varepsilon\setminus F_\varepsilon} - \int_{E_\varepsilon\cap F_\varepsilon}\int_{F_\varepsilon\setminus E_\varepsilon} \\ &\leqslant \int_{F_\varepsilon\setminus E_\varepsilon}\int_{(F_\varepsilon\cup E_\varepsilon)^c} + \int_{E_\varepsilon\setminus F_\varepsilon}\int_{F_\varepsilon\cap E_\varepsilon} \leqslant P_s(F_\varepsilon\Delta E_\varepsilon). \end{aligned} \tag{5.11}$$

Therefore, by the minimality of F_ε we get

$$\begin{aligned} (1-t)P_t(F_\varepsilon) &\leqslant (1-t)P_t(E_\varepsilon) + \varepsilon\,[sP_s(F_\varepsilon) - sP_s(E_\varepsilon)] + \Lambda\Big(\big||E_\varepsilon| - \omega_N\big| - \big||F_\varepsilon| - \omega_N\big|\Big) \\ &\leqslant (1-t)P_t(E_\varepsilon) + \varepsilon s P_s(F_\varepsilon\Delta E_\varepsilon) + \Lambda\big||E_\varepsilon| - \omega_N\big|. \end{aligned}$$

Hence the desired result follows from (2.3). $\square$

We point out that from Lemma 5.3 it follows that F_ε is a multiplicative ω-minimizer for the t-perimeter. In the sequel, as customary, the fractional perimeter of a set E in a ball $B(x,R)$ will be denoted by $P_t(E, B(x,R))$.

Corollary 5.4. *There exist c_0 and $R_0 = R_0(N,\delta_0) > 0$ such that the following statement holds true.*

Let ε_0 and Λ_0 be as in Lemma 5.2. Let F_ε be a minimizer of (5.1) *with $\varepsilon < \varepsilon_0$, let $x \in \partial^m F_\varepsilon$, and let E_ε be a set of finite t-perimeter with*

$$F_\varepsilon \Delta E_\varepsilon \subset B(x,R). \tag{5.12}$$

Then, there holds

$$P_t(F_\varepsilon, B(x,R)) \leqslant \frac{1+c_0R^{t-s}}{1-c_0R^{t-s}}\, P_t(E_\varepsilon, B(x,R)), \tag{5.13}$$

for any $R < R_0 = R_0(N,\delta_0)$.

Proof. We observe that, by direct calculations, from (5.12), follows

$$\begin{aligned} P_t(F_\varepsilon) - P_t(E_\varepsilon) &= P_t(F_\varepsilon, B(x,R)) - P_t(E_\varepsilon, B(x,R)) \\ \text{and} \qquad P_t(F_\varepsilon \Delta E_\varepsilon) &\leqslant P_t(F_\varepsilon, B(x,R)) + P_t(E_\varepsilon, B(x,R)). \end{aligned} \tag{5.14}$$

Furthermore, thanks to Lemma 5.2 we know that F_ε is also a minimizer of (5.3), with $\Lambda = \Lambda_0$. From (5.10) and the fractional isoperimetric inequality (3.7), we then get

$$\begin{aligned} (1-t)P_t(F_\varepsilon, B(x,R)) \;&\leqslant\; (1-t)P_t(E_\varepsilon, B(x,R)) \\ &\quad +\varepsilon_0\, c_N \left(1-\frac{s}{t}\right)^{-1} [c_N\, t]^{1-\frac{s}{t}} |F_\varepsilon \Delta E_\varepsilon|^{\frac{t-s}{N}} (1-t) P_t(F_\varepsilon \Delta E_\varepsilon) \\ &\quad +\Lambda_0 \big||E_\varepsilon| - \omega_N\big|. \end{aligned} \tag{5.15}$$

Moreover, again from the fractional isoperimetric inequality and using (5.12),

$$\begin{aligned} \Lambda_0\big||E_\varepsilon| - \omega_N\big| \;&=\; \Lambda_0 \Big|\big||E_\varepsilon| - \omega_N\big| - \big||F_\varepsilon| - \omega_N\big|\Big| \\ &\leqslant\; \Lambda_0 |F_\varepsilon \Delta E_\varepsilon|^{\frac{N-t}{N}} |F_\varepsilon \Delta E_\varepsilon|^{\frac{t}{N}} \\ &\leqslant\; c_N\, \Lambda_0\, t\,(1-t) P_t(F_\varepsilon \Delta E_\varepsilon)\, |F_\varepsilon \Delta E_\varepsilon|^{\frac{t}{N}} \\ &\leqslant\; c_N\, \Lambda_0\, t\,(1-t) P_t(F_\varepsilon \Delta E_\varepsilon) R^t. \end{aligned}$$

From this, (5.14) and (5.15) we arrive at

$$\begin{aligned} (1-t)P_t(F_\varepsilon, B(x,R)) \;&\leqslant\; (1-t)P_t(E_\varepsilon, B(x,R)) \\ &\quad +\varepsilon_0\, c_N \left(1-\frac{s}{t}\right)^{-1} [c_N\, t]^{1-\frac{s}{t}} R^{t-s} (1-t) P_t(F_\varepsilon \Delta E_\varepsilon) \\ &\quad +\Lambda_0\, c_N\, t R^t\, (1-t) P_t(F_\varepsilon \Delta E_\varepsilon) \\ &\leqslant\; (1-t)P_t(E_\varepsilon, B(x,R)) \\ &\quad + c_0 R^{t-s} (1-t) \left[P_t(F_\varepsilon, B(x,R)) + P_t(E_\varepsilon, B(x,R))\right] \end{aligned}$$

which gives (5.13), if $R < \min\{1, 1/c_0^{\frac{1}{t-s}}\} =: R_0$, with

$$c_0 := \varepsilon_0\, c_N \left(1-\frac{s}{t}\right)^{-1} [c_N\, t]^{1-\frac{s}{t}} + \Lambda_0\, c_N\, t.$$

□

Lemma 5.5. *There exists* $\Theta = \Theta(N,\delta_0) > 0$ *and* $R_0 = R_0(N,\delta_0) > 0$ *such that, for any* $x \in \partial^m F_\varepsilon$ *and* $R < R_0$, *there holds*

$$(1-t)P_t(F_\varepsilon, B(x,R)) \leqslant \Theta\, R^{N-t}. \tag{5.16}$$

Proof. Let $E_\varepsilon = F_\varepsilon \setminus B(x,R)$, and observe that $P_t(E_\varepsilon, B(x,R)) \leqslant P_t(B(x,R))$. From (5.13), possibly reducing R_0, we then get

$$(1-t)P_t(F_\varepsilon, B(x,R)) \leqslant (1 + c_0\, R^{t-s})(1-t)P_t(B(x,R)) \leqslant \Theta\, R^{N-t}.$$

□

From Lemma 5.5 it follows that F_ε is also an additive ω-minimizer for the t-perimeter.

Corollary 5.6. *Let ε_0 be as in Lemma 5.2. Let F_ε be a minimizer of* (5.1) *with $\varepsilon < \varepsilon_0$, let $x \in \partial^m F_\varepsilon$, and let E_ε be a set of finite t-perimeter with*

$$F_\varepsilon \Delta E_\varepsilon \subset B(x,R). \tag{5.17}$$

There holds

$$(1-t)P_t(F_\varepsilon, B(x,R)) \leqslant (1-t)P_t(E_\varepsilon, B(x,R)) + c_0\, R^{N-s} \tag{5.18}$$

for any $R < R_0$, with R_0, c_0 depending only on N, δ_0.

Proof. By (5.13) and (5.16), possibly increasing the constant c_0 we have

$$(1-t)P_t(E_\varepsilon, B(x,R)) \geqslant \left(1 - c_0 R^{t-s}\right)(1-t)P_t(F_\varepsilon, B(x,R)) \geqslant (1-t)P_t(F_\varepsilon, B(x,R)) - c_0 \Theta\, R^{N-s}$$

for any $R < R_0$. □

From Corollary 5.6 we derive the $C^{1,\beta}$ regularity minimizer of (5.1) following standard arguments that can be found in [12, Theorem 1], [34] (see also [18, Corollary 3.5]).

Corollary 5.7. *There exists $\beta = \beta(N, \delta_0) < 1$ such that any minimizer F_ε of* (5.1)*, with $\varepsilon < \varepsilon_0$, as in Lemma 5.2, has boundary of class $C^{1,\beta}$ outside of a closed singular set of Hausdorff dimension at most $N-3$.*

Remark 5.8. As shown in [34], the singular set is finite if $N = 3$. Moreover if $t = 1$, by the general regularity theory for ω-minimizers of the classical perimeter developed in [4, 35] we have that F_ε has boundary of class $C^{1,\beta}$ outside of a closed singular set of Hausdorff dimension at most $N-8$. As a consequence of [11], the same holds even if t is sufficiently close to 1.

We are in the position of completing the proof of Theorem 1.2.

Proof of Theorem 1.2. The existence follows from Theorem 4.2. The regularity of ∂F follows from Corollary 5.4 and Corollary 5.7. □

6. RIGIDITY OF MINIMIZERS FOR SMALL VOLUMES AND PROOF OF THE MAIN THEOREM

We now develop the rigidity theory needed to prove Theorems 1.3 and 1.1.

Theorem 6.1. *For any $\eta > 0$ there exists $\bar\varepsilon = \bar\varepsilon(\eta, N, \delta_0) > 0$ such that any minimizer F_ε of* (5.1)*, with $\varepsilon < \bar\varepsilon$, can be written as*

$$\partial F_\varepsilon = \{(1 + u_\varepsilon(x))x : x \in \partial B\}, \tag{6.1}$$

where B is the ball of radius 1 having the same barycenter of F_ε, and $u_\varepsilon : \partial B \to \mathbb{R}$ satisfies

$$\|u_\varepsilon\|_{C^1(\partial B)} \leqslant \eta.$$

Proof. From Lemma 3.13, putting $m = \varepsilon^{\frac{N}{t-s}} \omega_N$ there, it follows that $|F_\varepsilon \Delta B| \to 0$ as $\varepsilon \to 0$. From the density lower bound proved in Proposition 3.11 it then follows that $\partial F_\varepsilon \to \partial B$ in the Hausdorff distance (i.e., in the Kuratowski sense).

The result now follows via a standard argument based on the ω-minimality of F_ε and on the regularity of the limit set B (see [18, Corollary 3.6] and, for $t = 1$, [35, Theorem 1] and [30, Theorem 26.6]). □

Theorem 6.2. *There exist $\tau_0, c_1, c_2 > 0$ depending only on N, with $c_1 < c_2$, with the following property. Suppose that E_τ is such that, for $\tau \in [0, \tau_0]$, ∂E_τ takes the form*

$$\partial E_\tau = \{(1 + \tau u(x))x : x \in \partial B\},$$

where $u : \partial B \to \mathbb{R}$ satisfies

$$\|u\|_{C^1(\partial B)} \leqslant 1/2.$$

Suppose moreover that the barycenter of E_τ is the same of that of B, say 0, and that $|E_\tau| = |B|$. Then, for all $\alpha \in (0, 1)$ it holds true that

$$c_1 \tau^2 \left([u]^2_{H^{\frac{1+\alpha}{2}}(\partial B)} + \alpha P_\alpha(B) \|u\|^2_{L^2(\partial B)} \right) \leqslant P_\alpha(E_\tau) - P_\alpha(B) \leqslant c_2 \tau^2 [u]^2_{H^{\frac{1+\alpha}{2}}(\partial B)}. \tag{6.2}$$

Proof. The first inequality in (6.2) has been proved in [18, Theorem 2.1]. It remains to prove the second inequality.

As in [18, Formula (2.20)], after some calculations we get that

$$P_\alpha(E_\tau) = \frac{\tau^2}{2} g(\tau) + \frac{P_\alpha(B)}{P(B)} h(\tau), \tag{6.3}$$

where we set

$$h(\tau) := \int_{\partial B} (1 + \tau u(x))^{N-\alpha} \, d\mathcal{H}^{N-1}(x),$$

$$\text{and} \qquad g(\tau) := \int_{\partial B} \int_{\partial B} \left(\int_{u(y)}^{u(x)} \int_{u(y)}^{u(x)} f_{|x-y|}(1 + \tau r, 1 + \tau \rho) \, dr \, d\rho \right) d\mathcal{H}^{N-1}(x) \, d\mathcal{H}^{N-1}(y),$$

being

$$f_\theta(a, b) := \frac{a^{N-1} b^{N-1}}{(|a - b|^2 + ab\theta^2)^{\frac{N+\alpha}{2}}}. \tag{6.4}$$

We observe that r and ρ in the definition of g range in $[-\|u\|_{L^\infty(\partial B)}, \|u\|_{L^\infty(\partial B)}] \subseteq [-1, 1]$, since $\|u\|_{L^\infty(\partial B)} \leqslant 1$. Hence, comparing with the definition of g, we notice that a and b in (6.4) range in $[1 - \tau, 1 + \tau]$, and therefore they are bounded and bounded away from zero. As a consequence, we get

$$f_\theta(a, b) \leqslant \frac{C_1}{(C_2 + C_3 \theta^2)^{\frac{N+\alpha}{2}}} \leqslant \frac{C_1}{(C_3 \theta^2)^{\frac{N+\alpha}{2}}} = \frac{C_4}{\theta^{N+\alpha}},$$

for suitable constants $C_1, \dots, C_4 > 0$. Therefore, up to renaming the constants, we have

$$g(\tau) \leqslant \int_{\partial B} \int_{\partial B} \left(\int_{u(y)}^{u(x)} \int_{u(y)}^{u(x)} \frac{c_N}{|x - y|^{N+\alpha}} \, dr \, d\rho \right) d\mathcal{H}^{N-1}(x) \, d\mathcal{H}^{N-1}(y) = c_N \, [u]^2_{H^{\frac{1+\alpha}{2}}(\partial B)}.$$

Thus, since $h(0) = P(B)$, by (6.3) we get

$$P_\alpha(E_\tau) - P_\alpha(B) \leqslant c_N \tau^2 [u]^2_{H^{\frac{1+\alpha}{2}}(\partial B)} + \frac{P_\alpha(B)}{P(B)} (h(\tau) - h(0)). \tag{6.5}$$

Now we want to estimate $h(\tau) - h(0)$. Since $|E_\tau| = |B|$, using polar coordinates, we get

$$\int_{\partial B} (1 + \tau u)^N \, d\mathcal{H}^{N-1} = N|E_\tau| = N|B| = P(B). \tag{6.6}$$

Thus

$$h(\tau) - h(0) = \int_{\partial B} (1 + \tau u)^{N-\alpha} \, d\mathcal{H}^{N-1} - P(B) = \int_{\partial B} (1 + \tau u)^N ((1 + \tau u)^{-\alpha} - 1) \, d\mathcal{H}^{N-1}. \tag{6.7}$$

By a Taylor expansion, we know that for any $x \geqslant 0$ small enough, it holds

$$\begin{aligned}&\big((1+x)^{-\alpha}-1)(1+x)^N\big)\\&\quad=\left(-\alpha x+\frac{\alpha(\alpha+1)}{2}x^2+\alpha\beta(x)\right)\left(1+Nx+\frac{N(N-1)}{2}x^2+\gamma(x)\right),\end{aligned}$$

with $|\beta(x)|+|\gamma(x)|\leqslant c_N x^3$, so that

$$\big((1+x)^{-\alpha}-1)(1+x)^N\big)\leqslant -\alpha x+\left(\frac{\alpha(\alpha+1)}{2}-N\alpha\right)x^2+\alpha\, c_N\, x^3.$$

By applying such an inequality to (6.7), and using the fact that $\|u\|_{L^\infty(\partial B)}<1$, we get

$$h(\tau)-h(0)\leqslant -\alpha\int_{\partial B}\left[\tau u+\left(N-\frac{\alpha+1}{2}\right)\tau^2u^2\right]\,d\mathcal{H}^{N-1}+\alpha\, c_N\,\tau^3\|u\|^2_{L^2(\partial B)}. \tag{6.8}$$

Also, from (6.6), we have

$$0=\int_{\partial B}((1+\tau u)^N-1)\,d\mathcal{H}^{N-1}\leqslant\int_{\partial B}(N\tau u+N(N-1)\tau^2u^2+c_N\,\tau^3u^3)\,d\mathcal{H}^{N-1}.$$

Hence, since $\|u\|_{L^\infty(\partial B)}<1$, we obtain

$$-\int_{\partial B}\tau u\,d\mathcal{H}^{N-1}\leqslant\frac{N-1}{2}\,\tau^2\|u\|^2_{L^2(\partial B)}+c_N\,\tau^3\|u\|^2_{L^2(\partial B)},$$

so that (6.8) gives

$$h(\tau)-h(0)\leqslant-\frac{\tau^2}{2}\alpha\,(N-\alpha)\|u\|^2_{L^2(\partial B)}+\alpha\,c_N\,\tau^3\|u\|^2_{L^2(\partial B)}\leqslant 0$$

for $\tau\leqslant\tau_0(N)$. By inserting this into (6.5) we obtain the second inequality in (6.2). □

We now complete the proof of Theorem 1.3.

Proof of Theorem 1.3. We have to show that there exists $\varepsilon_1=\varepsilon_1(N,\delta_0)\in(0,\varepsilon_0]$, ε_0 as in (5.4), and so $\bar m_1=\bar m_1(N,\delta_0)\in(0,\bar m_0]$, such that the ball B is the only minimizer of problem (5.1) for $\varepsilon<\varepsilon_1$. Let $\varepsilon<\varepsilon_1$ and let F_ε be a minimum of problem (5.1), which exists by Theorem 1.2. By the minimality of F_ε we have

$$(1-t)P_t(F_\varepsilon)-(1-t)P_t(B)\leqslant\varepsilon\,(sP_s(F_\varepsilon)-sP_s(B)) \tag{6.9}$$

where B has the same barycenter of F_ε. Possibly reducing ε we can assume that ∂F_ε can be written as in (6.1), with $\|u_\varepsilon\|_{C^1(\partial B)}\leqslant\tau_0/2$, where τ_0 is as in Theorem 6.2. Then, from (6.9) and (6.2) it follows

$$\begin{aligned}c_1(1-t)[u_\varepsilon]^2_{H^{\frac{1+t}{2}}(\partial B)}&\leqslant c_1(1-t)\left([u_\varepsilon]^2_{H^{\frac{1+t}{2}}(\partial B)}+tP_t(B)\|u_\varepsilon\|^2_{L^2(\partial B)}\right)\\&\leqslant\big((1-t)P_t(F_\varepsilon)-(1-t)P_t(B)\big)\\&\leqslant\varepsilon\,(sP_s(F_\varepsilon)-sP_s(B))\\&\leqslant\varepsilon\,s\,c_2\,[u_\varepsilon]^2_{H^{\frac{1+s}{2}}(\partial B)}.\end{aligned}\tag{6.10}$$

From (2.21) it then follows

$$c_1(1-t)[u_\varepsilon]^2_{H^{\frac{1+t}{2}}(\partial B)}\leqslant c_N\frac{\varepsilon s}{(1-s)}\,(1-t)[u_\varepsilon]^2_{H^{\frac{1+t}{2}}(\partial B)}$$

which implies $u_\varepsilon=0$, that is $F_\varepsilon=B$, whenever ε is sufficiently small. □

The next result is the counterpart to Theorem 1.3 for large volumes.

Theorem 6.3. *For all* $0 < s < t \leqslant 1$*, there exists a volume* $\bar{m}_2 = \bar{m}_2(N, s, t) \geqslant \bar{m}_1$ *such that, for* $m > m_2$*, the ball is not a local minimizer of problem* (1.7).

Proof. We have to show that there exists $\varepsilon_2 \geqslant \varepsilon_1$ such that the ball B is not a local minimizer of problem 5.1 for $\varepsilon > \varepsilon_2$. Let $F \neq B$ a nearly spherical set, that is a set which can be written as in (6.1), with $u \not\equiv 0$ and and $\|u\|_{C^1(\partial B)} \leqslant \tau_0/2$, where τ_0 is as in Theorem 6.2. We are going to show that there exists an ε depending on u (and thus on F), such that

$$(1-t)P_t(F) - \varepsilon s P_s(F) < (1-t)P_t(B) - \varepsilon s P_s(B).$$

As above, from (6.2) it follows

$$\begin{aligned} \big((1-t)P_t(F) - (1-t)P_t(B)\big) &\leqslant c_2(1-t)[u]^2_{H^{\frac{1+t}{2}}(\partial B)} \\ &< \varepsilon c_1 s [u]^2_{H^{\frac{1+s}{2}}(\partial B)} \\ &\leqslant \varepsilon\left(sP_s(F) - sP_s(B)\right), \end{aligned} \tag{6.11}$$

as soon as

$$\varepsilon > \varepsilon_2 := \frac{c_2(1-t)[u]^2_{H^{\frac{1+t}{2}}(\partial B)}}{c_1 s [u]^2_{H^{\frac{1+s}{2}}(\partial B)}}.$$

This shows that F has lower energy than B, so that the ball cannot be a local minimizer of problem (1.7). ☐

Notice that $\lim_{s\to 0} \bar{m}_2(N, s, t) = +\infty$ for all $t \in (0, 1]$, which is consistent with the fact that the ball is the unique minimizer of the t-perimeter, with volume constraint.

We conclude the section with the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $E \subset \mathbb{R}^N$. Since the deficit of a set is a 0 homogeneous quantity, we can suppose that the set E has measure ω_N. Because of this, inequality (1.6) is equivalent to prove that there exists $C(N, s, t) > 0$ such that the inequality

$$\widetilde{\delta}P_t(E) := P_t(E) - P_t(B) \geqslant C(N, s, t)\left(P_s(E) - P_s(B)\right) \tag{6.12}$$

holds true for any set E of measure ω_N. Indeed if this is true, then we get that for any $E \subset \mathbb{R}^N$ it holds

$$\begin{aligned} \delta P_t(E) = \delta P_t(E\omega_N/|E|) &= \frac{P_t(E\omega_N/|E|) - P_t(B)}{P_t(B)} \\ &\geqslant C(N, s, t)\frac{P_s(E\omega_N/|E|) - P_s(B)}{P_t(B)} = C(N, s, t)\frac{P_s(B)}{P_t(B)}\delta P_s(E) \end{aligned}$$

that is exactly 1.6 with $C = C(N, s, t)\frac{P_s(B)}{P_t(B)}$.

Let $E \subset \mathbb{R}^N$ be a set of measure ω_N. By Theorem 1.2 we know that there exists $\varepsilon_0 = \varepsilon_0(N, s, t)$ such that if $\varepsilon \leqslant \varepsilon_0$ then the only minimizer of the problem

$$\min\left\{(1-t)P_t(E) - \varepsilon s P_s(E) : |E| = \omega_N\right\}$$

is given by the unit ball B. This entails that

$$\widetilde{\delta}P_t(E) \geqslant \frac{\varepsilon_0 s}{1-t}\widetilde{\delta}P_s(E).$$

☐

7. A fractional isoperimetric problem

We recall from the Introduction the definition of the functional $\widetilde{\mathcal{F}_{s,t}}$ given by

$$\widetilde{\mathcal{F}_{s,t}}(E) = \begin{cases} \frac{((1-t)P_t(E))^{N-s}}{(sP_s(E))^{N-t}} & \text{if } 0 < s < t < 1 \\ \\ \frac{(N\omega_N P(E))^{N-s}}{(sP_s(E))^{N-1}} & \text{if } 0 < s < t = 1 \\ \\ \frac{(1-t)P_t(E)^N}{(N\omega_N|E|)^{N-t}} & \text{if } 0 = s < t < 1 \\ \\ N\omega_N \frac{P(E)^N}{|E|^{N-1}} & \text{if } s = 0 \text{ and } t = 1. \end{cases}$$

In this section we consider the generalized isoperimetric problem

$$\min_{E\subset\mathbb{R}^N} \widetilde{\mathcal{F}_{s,t}}(E), \qquad 0 \leqslant s < t \leqslant 1\,. \tag{7.1}$$

Remark 7.1. Notice that the quantity in (7.1) is scale invariant, hence without loss of generality we can look for minimizers E satisfying a volume constraint $|E| = \omega_N$.

The main aim of this section is the following existence theorem.

Theorem 7.2. *There exists a minimizer of problem* (7.1)*.*

Since its proof may result technical, for the reader's convenience we begin with a description of its strategy.
An usually successful argument to get existence for isoperimetric-type problems is the following: first apply the Direct Method in the Calculus of Variations on minimizing sequences which are equibounded, that is, whose elements are contained in a prescribed ball of fixed radius. Then try to show that starting from a given minimizing sequence E_n it is possible to construct another minimizing sequence F_n which is uniformly bounded and make use of the first step to conclude. Unfortunately it seems that the second step is not easily applicable to the functional $\widetilde{\mathcal{F}_{s,t}}$, for $s > 0$. An hint about what kind of difficulties may occur is the following: it is not even clear if a minimizer (if any) is connected or not. Indeed, up to the case $s = 0$, where sP_s reduces to the Lebesgue measure, which is translation invariant, the denominator of $\widetilde{\mathcal{F}_{s,t}}$ acts as a disaggregating term among different connected components. On the other hand it is not evident when the numerator (which tends to aggregate different connected components) can overcome such an effect. For this reason we adopt a different strategy from that described above, which can be divided as well into two steps.

Step 1 The first step is very easy and it is similar to that mentioned above. It reduces to show that for each given $R > 0$ there exists a minimizer E_R for $\widetilde{\mathcal{F}_{s,t}}$ among sets contained in a cube $Q_R = [-R, R]^N$ and that $P_t(E_R)$ is bounded from above independently from R. This is done in done in Lemmas 7.5 and 7.6.

Step 2 The second step, developed in the very proof of Theorem 7.2, is longer and needs a more careful analysis. Since, as mentioned above, we are not able to show that a minimizing sequence can be uniformly bounded, we adopt a different idea that may be seen as an adaptation of a concentration-compactness technique *à la Lions*, where the compactness is replaced by a sort of selection principle. More precisely we consider, for $n \in \mathbb{N}$, the minimizer $E_n \subseteq [-n,n]^N$ found in Step 1 and require them to have prescribed mass, say 1. Since we do not know, as n increases, the behavior of the E_n's (e.g. if they are connected, equibounded...) we select, for every n all the unitary cubes in $\mathbb{R}^N$ of the form $[-1,1]^N + z$, $z \in \mathbb{Z}^N$, which have a non-negligible intersection with E_n. So for each n we get a set of cubes $Q_{i,1} \dots, Q_{i,k_n}$ with non-empty intersection with E_n. Clearly for two fixed indexes $i \neq j$ we may have that $\operatorname{dist}(Q_{i,n}, Q_{j,n})$ diverges to $+\infty$. This means that two components of E_n will have infinity distance as $n \to \infty$ (this phenomenon may be seen as a *dichotomy phenomenon*). If this does not happens, then we say that $Q_{i,n}$ adn $Q_{j,n}$ have finite mutual distance at infinity and we (suitably) collect them together. Such a construction, thanks to the equiboundedness of $P_t(E_n)$, allows us to construct a sequence of limit points $\{G_i\}_{i\in\mathbb{N}}$ of the E_n's, where G_i is just one of the collections of cubes with finite mutual distance at infinity. Now the idea is simple: first we need to show that the amount of measure of all the G_i is the same as that of the E_n's (so, we want to eliminate the *vanishing phenomenon*). Then, we want to *select* a G_i such that $\widetilde{\mathcal{F}_{s,t}}(G_i)$ is the lowest possible. It is not difficult then to conclude that G_i is a minimizer for $\widetilde{\mathcal{F}_{s,t}}$.

A suitable version of the isoperimetric inequality (Lemma 7.5) and an existence result with uniform estimates for a constrained minimization problem (Lemma 7.6) are needed.

Remark 7.3. In what follows, with a slight abuse of notation, we extend the functionals $(1-t)P_t(\cdot)$ and $sP_s(\cdot)$ to $t=1$ and $s=0$ respectively, meaning that for $t=1$ it equals $N\omega_N P(\cdot)$, while for $s=0$ it equals $N\omega_N|\cdot|$.

We begin with a technical Lemma proved in [25, Lemma 4.2]. We report its proof for the reader convenience

Lemma 7.4. *Let $C>0$ and $\{x_i\}_{i\in\mathbb{N}}$ be a non-increasing sequence of positive numbers such that*

$$\sum_{i=1}^{\infty} x_i^{\frac{N-t}{N}} \leqslant C$$

and

$$\sum_{i=1}^{\infty} x_i = \frac{1}{2}.$$

Then there exists $k_0 \in \mathbb{N}$ such that if $k \geqslant k_0$ then

$$\sum_{i=k+1}^{\infty} x_i \leqslant \frac{1}{(2Ck)^{\frac{t}{N}}}.$$

Proof. Let k_0 such that

$$\frac{C}{(2k_0)^{\frac{t}{N}}} \leqslant \frac{1}{2},$$

$k \geqslant k_0$ and

$$\delta_k = \frac{C}{(2k)^{\frac{t}{N}}}.$$

Let moreover $M_k \in \mathbb{N}$ be such that

$$\sum_{i=M_k}^{\infty} x_i \geqslant \delta_k > \sum_{i=M_k+1}^{\infty} x_i.$$

Notice that such an M_k exists by our choice of k_0. We prove that

$$M_k \leqslant \frac{C^{\frac{N}{t}}}{2\delta_k^{\frac{N}{t}}} = k. \tag{7.2}$$

Then the previous inequality entails that

$$\frac{C}{(2k)^{\frac{t}{N}}} = \delta_k \geqslant \sum_{i=M_k+1}^{\infty} x_i \geqslant \sum_{i=k+1}^{\infty} x_i,$$

which is the statement of the Lemma. To prove (7.2) we notice that by the choice of M_k

$$\delta_k \leqslant \sum_{i=M_k}^{\infty} x_i = \sum_{i=M_k}^{\infty} x_i^{\frac{t}{N}} x_i^{\frac{N-t}{N}} \leqslant x_{M_k}^{\frac{t}{N}} \sum_{i=M_k}^{\infty} x_i^{\frac{N-t}{N}} \leqslant C x_{M_k}^{\frac{t}{N}}.$$

Moreover,

$$\frac{1}{2} \geqslant \sum_{i=1}^{M_k} x_i \geqslant M_k x_{M_k},$$

so that (7.2) follows. □

Lemma 7.5. *Let $s < t \in [0,1]$ satisfy (3.1). For any $E \subset \mathbb{R}^N$ there holds*

$$\frac{((1-t)P_t(E))^{\frac{N-s}{N-t}}}{(sP_s(E))} \geqslant c \tag{7.3}$$

for some $c = c(N, \delta_0) > 0$.

Proof. Let $s < t \in [0,1]$ and let $\delta_0 = t - s$. Notice that

$$\frac{t}{t-s} = 1 + \frac{s}{t-s} \leqslant 1 + \frac{1}{\delta_0}. \tag{7.4}$$

Then from (2.15), and since $\delta_0 < t$, it follows

$$|E|^{1-\frac{s}{t}} \leqslant C(N, \delta_0) \left((1-t)P_t(E)\right)^{\frac{N(t-s)}{(N-t)t}}.$$

Plugging this estimate into (2.3) (or (2.9) if $t = 1$) we get

$$sP_s(E) \ \leqslant \ C(N, \delta_0) \frac{t}{t-s} \left((1-t)P_t(E)\right)^{\frac{N-s}{N-t}},$$

which, together with (7.4) gives (7.3). □

We notice that, if $s = 0$, the claim is an immediate consequence of the the fractional isoperimetric inequality (2.15).

Lemma 7.6. *Let $s < t \in [0,1]$ satisfy (3.1). For $R > 1$ let $Q_R = [-R, R]^N$. Then, there exists a minimizer E_R of the problem*

$$\min_{E \subset Q_R \, |E| = \frac{1}{2}} \frac{((1-t)P_t(E))^{N-s}}{(sP_s(E))^{N-t}}. \tag{7.5}$$

Moreover

$$(1-t)P_t(E_R) \leqslant C \tag{7.6}$$

where C is independent of R.

Proof. We recall that, thanks to the notation introduced in Remark 7.3 we can deal at once with the cases $t < 1$ and $t = 1$. By Lemma 7.5 we know that

$$C(R) = \inf_{E \subset Q_R\, |E|=m} \frac{((1-t)P_t(E))^{\frac{N-s}{N-t}}}{(sP_s(E))}$$

is a strictly positive quantity. Clearly the map $R \mapsto C(R)$ is non-increasing. Let $C = C(1) + 1$ and let E_n be a minimizing sequence for (7.5), so that for n big enough it holds $(1-t)P_t(E_n) \leqslant C(sP_s(E_n))^{(N-t)/(N-s)}$. Possibly increasing the constant C, from (2.3) (or (2.9) if $t = 1$) it follows

$$(1-t)P_t(E_n) \leqslant C((1-t)P_t(E_n))^{\frac{s(N-t)}{t(N-s)}}$$

which gives

$$(1-t)P_t(E_n) \leqslant C \qquad \text{for all } n. \tag{7.7}$$

The existence of a minimizer now follows by the direct method the calculus of variations, since the compact embedding of $L^1(Q_R)$ and $H^s(Q_R)$ into $H^t(Q_R)$ and the estimate (7.6) directly follows from (7.7). □

We now prove Theorem 7.2.

Proof of Theorem 7.2. If $s = 0$ then the claim of the theorem is equivalent to that of the isoperimetric inequality (the fractional isoperimetric inequality if $t < 1$). Thus we consider just the case $s > 0$. Again, we shall always write $(t-1)P_t$ meaning that such a functional is equivalent to the classical perimeter if $t = 1$ (see Remark 7.3).
Let E_n be a minimizer of (7.5) with $R = n \in \mathbb{N}$ and $m = 1/2$. We divide Q_n into $(2n)^N$ unit cubes with vertices in $\mathbb{Z}^N$, and we let $\{Q_{i,n}\}_{i=1}^{I_n}$ be the unit cubes with non-negligible intersection with E_n, that is, $x_{i,n} = |E_n \cap Q_{i,n}| \in (0, 1/2]$ for all $i \in \{1, \dots, I_n\}$, for some $I_n \in \{1, \dots, (2n)^N\}$.

We remark that, from (2.10) (and omitting the integrands for simplicity), we have that

$$\sum_{i=1}^{\infty} P_t(E_n, Q_{i,n}) = \sum_{i=1}^{\infty} \int_{E_n \cap Q_{i,n}} \int_{\mathbb{R}^N \setminus E_n} + \int_{Q_{i,n} \setminus E_n} \int_{E_n \setminus Q_{i,n}} \leqslant \int_{E_n \cap Q_n} \int_{\mathbb{R}^N \setminus E_n} + \int_{Q_n \setminus E_n} \int_{E_n},$$

which implies that, for any $t < 1$,

$$\sum_{i=1}^{\infty} P_t(E_n, Q_{i,n}) \leqslant 2 \int_{E_n} \int_{\mathbb{R}^N \setminus E_n} = 2P_t(E_n). \tag{7.8}$$

The estimate in (7.8) also holds for $t = 1$ (as can be checked, for instance, by approximation).

Now, up to reordering the cubes $Q_{i,n}$ we can assume that the sequence $\{x_{i,n}\}_{i=1}^{I_n}$ is non-increasing in i, and we set $x_{i,n} := 0$ for $i > I_n$. We have that

$$\sum_{i=1}^{\infty} x_{i,n} = \frac{1}{2} \tag{7.9}$$

and, recalling (2.12), (7.6) and (7.8), and the fact that $x_{i,n} \leqslant |E_n| = 1/2 = |Q_{i,n}|/2$, we get

$$\sum_{i=1}^{\infty} x_{i,n}^{\frac{N-t}{N}} \leqslant C \sum_{i=1}^{\infty} (1-t)P_t(E_n, Q_{i,n}) \leqslant 2C\,(1-t)P_t(E_n) \leqslant C, \tag{7.10}$$

up to renaming C. By Lemma 7.4, from (7.9) and (7.10) it follows that

$$\sum_{i=k+1}^{\infty} x_{i,n} \;\leqslant\; C\, k^{-\frac{t}{N}} \tag{7.11}$$

for all $k \in \mathbb{N}$, where C depends only on (N, s, t).

Up to extracting a subsequence (using either a diagonal process or Tychonoff Theorem), we can suppose that $x_{i,n} \to \alpha_i \in [0, 1/2]$ as $n \to +\infty$ for every $i \in \mathbb{N}$, so that by (7.9) and (7.11) we have

$$\sum_i \alpha_i = \frac{1}{2}\,. \tag{7.12}$$

Fix now $z_{i,n} \in Q_{i,n}$. Up to extracting a further subsequence, we can suppose that $d(z_{i,n}, z_{j,n}) \to c_{ij} \in [0, +\infty]$, and (recalling (7.6)) that there exists $G_i \subseteq \mathbb{R}^N$ such that

$$(E_n - z_{i,n}) \to G_i \quad \text{in the } L^1_{\text{loc}}\text{-convergence} \tag{7.13}$$

for every $i \in \mathbb{N}$. We say that $i \sim j$ if $c_{ij} < +\infty$ and we denote by $[i]$ the equivalence class of i. Notice that G_i equals G_j up to a translation, if $i \sim j$. Let $\mathcal{A} := \{[i] : i \in \mathbb{N}\}$. We claim that

$$\sum_{[i]\in\mathcal{A}} P_t(G_i) \leqslant \liminf_{n\to+\infty} P_t(E_n) \qquad \text{and} \qquad \sum_{[i]\in\mathcal{A}} P_s(G_i) \leqslant \liminf_{n\to+\infty} P_s(E_n)\,. \tag{7.14}$$

To prove it, we first fix $M \in \mathbb{N}$ and $R > 0$. We take different equivalent classes $i_1, \ldots, i_M$ and we notice that if $i_k \neq i_j$ then the set $z_{i_k,n} + Q_R$ is drifting far apart from $z_{i_j,n} + Q_R$, and so

$$\lim_{n\to+\infty} \int_{z_{i_k,n}+Q_R} \int_{z_{i_j,n}+Q_R} \frac{dx\,dy}{|x-y|^{N+t}} = 0.$$

Accordingly, by (2.11), (7.13) and the lower semicontinuity of the perimeter,

$$\begin{aligned}
\sum_{i=1}^M P_t(G_i, Q_R) \;&\leqslant\; \liminf_{n\to+\infty} \sum_{k=1}^M P_t(E_n, (z_{i_k,n} + Q_R)) \\
&\leqslant\; \liminf_{n\to+\infty} P_t\left(E_n, \bigcup_{k=1}^N (z_{i_k,n} + Q_R)\right) + 2 \sum_{\substack{1\leqslant k,j\leqslant M\\ i_k\neq i_j}} \int_{z_{i_k,n}+Q_R} \int_{z_{i_j,n}+Q_R} \frac{dx\,dy}{|x-y|^{N+t}} \\
&\leqslant\; \liminf_{n\to+\infty} P_t(E_n).
\end{aligned}$$

By sending first $R \to +\infty$ and then $M \to +\infty$, this yields (7.14).

Now we claim that

$$\sum_{[i]\in\mathcal{A}} |G_i| = \frac{1}{2}. \tag{7.15}$$

Indeed, for every $i \in \mathbb{N}$ and $R > 0$ we have

$$|G_i| \geqslant |G_i \cap Q_R| = \lim_{n\to+\infty} |(E_n - z_{i,n}) \cap Q_R|.$$

If j is such that $j \sim i$ and $c_{ij} \leqslant \frac{R}{2}$, possibly enlarging R we have $Q_{j,n} - z_{i,n} \subset Q_R$ for all $n \in \mathbb{N}$, so that

$$\begin{aligned}
|(E_n - z_{i,n}) \cap Q_R| &= \sum_{j=1}^{I_n} |(E_n - z_{i,n}) \cap Q_R \cap (Q_{j,n} - z_{i,n})| \\
&\geqslant \sum_{j:\, c_{ij}\leqslant \frac{R}{2}} |(E_n - z_{i,n}) \cap Q_R \cap (Q_{j,n} - z_{i,n})| = \sum_{j:\, c_{ij}\leqslant \frac{R}{2}} |(E_n - z_{i,n}) \cap (Q_{j,n} - z_{i,n})| \\
&= \sum_{j:\, c_{ij}\leqslant \frac{R}{2}} |E_n \cap Q_{j,n}|,
\end{aligned}$$

and so

$$|G_i| \geqslant \lim_{n\to+\infty} |(E_n - z_{i,n}) \cap Q_R| \geqslant \lim_{n\to+\infty} \sum_{j:\, c_{ij} \leqslant \frac{R}{2}} |E_n \cap Q_{j,n}| = \sum_{j:\, c_{ij} \leqslant \frac{R}{2}} \alpha_j .$$

Letting $R \to +\infty$ we then have

$$|G_i| \geqslant \sum_{j:\, i\sim j} \alpha_j = \sum_{j \in [i]} \alpha_j \,,$$

hence, recalling (7.12),

$$\sum_{[i]\in\mathcal{A}} |G_i| \geqslant \frac{1}{2},$$

thus proving (7.15) (since the other inequality is trivial).

We now claim that

$$\sum_{[i]\in\mathcal{A}} P_s(G_i) \geqslant \limsup_{n\to+\infty} P_s(E_n). \tag{7.16}$$

Indeed, by (7.15) we have that for any $\varepsilon > 0$ there exist R, ℓ such that there exist ℓ distinct equivalence classes $[i_1], \dots, [i_\ell] \in \mathcal{A}$ such that

$$\frac{1}{2} - \varepsilon \leqslant \sum_{k=1}^{\ell} |G_{i_k} \cap B_R| = \lim_{n\to+\infty} \sum_{k=1}^{\ell} |(E_n - z_{i_k,n}) \cap B_R| \,. \tag{7.17}$$

For $\rho > 0$ we let

$$E^\rho_{n,1} = E_n \cap \bigcup_{k=1}^{\ell} (z_{i_k,n} + B_\rho) \qquad E^\rho_{n,2} = E_n \setminus E^\rho_{n,1} \,.$$

For n sufficiently large we have that the balls $z_{i_k,n} + B_R$ are disjoint (since the $z_{i_k,n}$ are drifting far away from each other, being each i_k in a different equivalence class). Therefore (7.17) gives that

$$|E^R_{n,1}| \geqslant \frac{1}{2} - 2\varepsilon \qquad \text{and} \qquad |E^R_{n,2}| \leqslant 2\varepsilon \tag{7.18}$$

if n is large enough. We claim that

$$\int_{E^{\bar\rho}_{n,1}} \int_{E^{\bar\rho}_{n,2}} \frac{dx\,dy}{|x-y|^{N+s}} \leqslant \frac{c_N}{s(1-s)} |E^{\bar\rho}_{n,2}|^{\frac{N-s}{N}} \qquad \text{for some } \bar\rho \in \Big[R, R + (2\delta)^{-\frac{1}{N}}\Big] , \tag{7.19}$$

where the constants c_N, δ depend only on N.

Indeed, if this is not the case, we would have that

$$\begin{aligned} &|E^\rho_{n,2}| > 0 \ \text{ and} \\ &\int_{E^\rho_{n,1}} \int_{E^\rho_{n,2}} \frac{dx\,dy}{|x-y|^{N+s}} > \frac{c_N}{s(1-s)} |E^\rho_{n,2}|^{\frac{N-s}{N}} \qquad \text{for every } \rho \in \Big[R, R + (2\delta)^{-\frac{1}{N}}\Big] . \end{aligned} \tag{7.20}$$

So we let

$$\mu(\rho) := |E^\rho_{n,1}| = |E_n| - |E^\rho_{n,2}| = \frac{1}{2} - |E^\rho_{n,2}|$$

and we obtain

$$\begin{aligned}\frac{c_N}{s(1-s)}\left(\frac{1}{2}-\mu(\rho)\right)^{\frac{N-s}{N}} &< \int_{E^{\rho}_{n,1}}\int_{E^{\rho}_{n,2}}\frac{dx\,dy}{|x-y|^{N+s}}\\ &\leqslant \int_{E^{\rho}_{n,1}}\int_{\mathbb{R}^N\setminus\bigcup_{k=1}^{\ell}(z_{i_k,n}+B_\rho)}\frac{dx\,dy}{|x-y|^{N+s}}\\ &\leqslant \frac{N\omega_N}{s}\int_{E^{\rho}_{n,1}}\frac{dx}{(\rho-|x-z_{i_{k(x)},n}|)^s}\\ &= \frac{N\omega_N}{s}\int_0^{\rho}\frac{\mu'(z)}{(\rho-z)^s}\,dz\end{aligned} \tag{7.21}$$

for all $\rho\in\left[R,R+(2\delta)^{-\frac{1}{N}}\right]$, where $k(x)\in\mathbb{N}$ is such that $x\in z_{i_{k(x)},n}+B_\rho$.

From (7.21) and Lemma 4.1 (used here with $m:=1/2$ and $\bar\rho:=R$), we obtain that $\mu(\rho)=1/2$ (and so $|E^{\rho}_{n,2}|=0$) for $\rho=R+(2\delta)^{-\frac{1}{N}}$, which leads to a contradiction with (7.20). We thus proved (7.19). Notice that inequality (7.19) holds also with t instead of s. So, by (7.19) and the fact that $|E^{\bar\rho}_{n,2}|\leqslant 2\varepsilon$ (recall (7.18)), we obtain that

$$\int_{E^{\bar\rho}_{n,1}}\int_{E^{\bar\rho}_{n,2}}\frac{dx\,dy}{|x-y|^{N+s}}\leqslant C\varepsilon^{\frac{N-s}{N}}\quad\text{and}\quad\int_{E^{\bar\rho}_{n,1}}\int_{E^{\bar\rho}_{n,2}}\frac{dx\,dy}{|x-y|^{N+t}}\leqslant C\varepsilon^{\frac{N-t}{N}}, \tag{7.22}$$

for some $C>0$, possibly depending on n, s and t.

From this, (2.1) and (7.6) we obtain

$$\begin{aligned}P_t(E^{\bar\rho}_{n,1})+P_t(E^{\bar\rho}_{n,2}) &= P_t(E_n)+2\int_{E^{\bar\rho}_{n,1}}\int_{E^{\bar\rho}_{n,2}}\frac{dx\,dy}{|x-y|^{N+t}}\\ &\leqslant P_t(E_n)+C\varepsilon^{\frac{N-t}{N}}\leqslant C\,.\end{aligned} \tag{7.23}$$

Now, by (2.3), we get

$$P_s(E^{\bar\rho}_{n,2})\leqslant C\,|E^{\bar\rho}_{n,2}|^{1-\frac{s}{t}}P_t(E^{\bar\rho}_{n,2})^{\frac{s}{t}}\,,$$

up to renaming C. Using this, (7.23) and then (7.18) once more, and possibly renaming C again, we conclude that

$$\begin{aligned}P_s(E^{\bar\rho}_{n,2}) &\leqslant C\,|E^{\bar\rho}_{n,2}|^{1-\frac{s}{t}}\Big(P_t(E^{\bar\rho}_{n,1})+P_t(E^{\bar\rho}_{n,2})\Big)^{\frac{s}{t}}\\ &\leqslant C\,|E^{\bar\rho}_{n,2}|^{1-\frac{s}{t}}\\ &\leqslant C\varepsilon^{1-\frac{s}{t}}\,.\end{aligned}$$

Consequently, using (2.1) and (7.22), we conclude that

$$\begin{aligned}P_s(E^{\bar\rho}_{n,1}) &= P_s(E_n)-P_s(E^{\bar\rho}_{n,2})+2\int_{E^{\bar\rho}_{n,1}}\int_{E^{\bar\rho}_{n,2}}\frac{dx\,dy}{|x-y|^{N+s}}\\ &\geqslant P_s(E_n)-C\varepsilon^{1-\frac{s}{t}}.\end{aligned} \tag{7.24}$$

Also, from (7.6), (7.13) and the compact embedding of $H^{\frac{t}{2}}$ into $H^{\frac{s}{2}}$ (see [16, Section 7]), we see that

$$\lim_{n\to+\infty}P_s\left((E_n-z_{i_k,n})\cap B_{\bar\rho}\right)=P_s(G_{i_k}\cap B_{\bar\rho}). \tag{7.25}$$

Now we recall that if K is a convex set, then $P_s(E \cap K) \leqslant P_s(E)$ (see for instance [18, Lemma B.1]). Together with (2.2) and (7.24), this implies

$$\begin{aligned}\sum_{[i]\in\mathcal{A}} P_s(G_i) &\geqslant \sum_{k=1}^{\ell} P_s(G_{i_k} \cap B_{\bar\rho}) \\ &= \lim_{n\to+\infty} \sum_{k=1}^{\ell} P_s\left((E_n - z_{i_k,n}) \cap B_{\bar\rho}\right) \\ &\geqslant \lim_{n\to+\infty} P_s(E_{n,1}^{\bar\rho}) \\ &\geqslant \limsup_{n\to+\infty} P_s(E_n) - C(N,s,t)\varepsilon^{\frac{t-s}{t}},\end{aligned}$$

which gives (7.16) by letting $\varepsilon \to 0^+$.

From (7.14) and (7.16) we obtain that

$$\frac{\sum_{[i]\in\mathcal{A}}(1-t)P_t(G_i)}{\left(s\sum_{[i]\in\mathcal{A}} P_s(G_i)\right)^{\frac{N-t}{N-s}}} \leqslant \liminf_{n\to+\infty} \frac{(1-t)P_t(E_n)}{(sP_s(E_n))^{\frac{N-t}{N-s}}}. \tag{7.26}$$

Let us now prove that the there exists j such that

$$\frac{(1-t)P_t(G_j)}{(sP_s(G_j))^{\frac{N-t}{N-s}}} \leqslant \frac{\sum_{[i]\in\mathcal{A}}(1-t)P_t(G_i)}{\left(s\sum_{[i]\in\mathcal{A}} P_s(G_i)\right)^{\frac{N-t}{N-s}}} =: S. \tag{7.27}$$

Indeed, if it is not the case, we get

$$\begin{aligned} S = \frac{\sum_{[i]\in\mathcal{A}}(1-t)P_t(G_i)}{\left(s\sum_{[i]\in\mathcal{A}} P_s(G_i)\right)^{\frac{N-t}{N-s}}} &= \frac{\sum_{[i]\in\mathcal{A}}\left(\frac{(1-t)P_t(G_i)}{(sP_s(G_i))^{\frac{N-t}{N-s}}}\right)(sP_s(G_i))^{\frac{N-t}{N-s}}}{\left(s\sum_{[i]\in\mathcal{A}} P_s(G_i)\right)^{\frac{N-t}{N-s}}} \\ > S\frac{\sum_{[i]\in\mathcal{A}}(sP_s(G_i))^{\frac{N-t}{N-s}}}{\left(s\sum_{[i]\in\mathcal{A}} P_s(G_i)\right)^{\frac{N-t}{N-s}}} \geqslant S, \end{aligned}$$

which is impossible. To get the last estimate we used the elementary inequality $\left(\sum_i c_i\right)^{\alpha} \leqslant \sum_i c_i^{\alpha}$ which holds true for $c_i \geqslant 0$ and $\alpha \in (0,1)$.

Now, let j be the index satisfying (7.27). Then, by (7.26) we get

$$\frac{(1-t)P_t(G_j)}{(sP_s(G_j))^{\frac{N-t}{N-s}}} \leqslant \liminf_{n\to\infty} \frac{(1-t)P_t(E_n)}{(sP_s(E_n))^{\frac{N-t}{N-s}}}. \tag{7.28}$$

Then, given any set E, fixed any $\epsilon > 0$, we intersecate E with a big ball B_{R_ϵ} in such a way that

$$\frac{(1-t)P_t(E\cap B_{R_\epsilon})}{(sP_s(E\cap B_{R_\epsilon}))^{\frac{N-t}{N-s}}} \leqslant \frac{(1-t)P_t(E)}{(sP_s(E))^{\frac{N-t}{N-s}}} + \epsilon.$$

Then, by the minimality of E_n,

$$\frac{(1-t)P_t(E\cap B_{R_\epsilon})}{(sP_s(E\cap B_{R_\epsilon}))^{\frac{N-t}{N-s}}} \geqslant \frac{(1-t)P_t(E_n)}{(sP_s(E_n))^{\frac{N-t}{N-s}}}$$

for any $n \geqslant n_\epsilon$. Thus, by (7.28) we obtain

$$\frac{(1-t)P_t(E)}{(sP_s(E))^{\frac{N-t}{N-s}}} + \epsilon \geqslant \liminf_{n\to\infty} \frac{(1-t)P_t(E_n)}{(sP_s(E_n))^{\frac{N-t}{N-s}}} \geqslant \frac{(1-t)P_t(G_j)}{(sP_s(G_j))^{\frac{N-t}{N-s}}}.$$

By sending $\epsilon \searrow 0$ we see that G_j is a desired minimizer. Eventually, we notice that by the homogeneity of the functional, a dilation $\widetilde{G}_j$ of G_j of measure $\frac{1}{2}$ has the same energy of G_j. Thus $\widetilde{G}_j$ is the desired minimizer. This conclude the proof.

□

Proposition 7.7. *Let F be a minimizer of (7.1). Then F is a multiplicative ω-minimizer of the t-perimeter, that is, for any set E such that $F\Delta E \subset B(x,R)$, there holds*

$$P_t(F, B(x,R)) \leqslant (1 + CR^{t-s})\, P_t(E, B(x,R)) \qquad \text{for any } R < R_0\,,$$

where R_0, C depend only on N, δ_0 and $|F|$.

Proof. First, if $\alpha \in (0,1)$, one sees that, for any $r \geqslant 0$,

$$1 - r^\alpha \leqslant |1-r|. \tag{7.29}$$

Also, from (5.11), we know that

$$P_s(F) - P_s(E) \leqslant P_s(F\Delta E),$$

for any sets E and F, and so, by possibly exchanging the roles of E and F we obtain

$$|P_s(E) - P_s(F)| \leqslant P_s(F\Delta E) \tag{7.30}$$

Now, letting E be such that $F\Delta E \subset B(x,R)$, using the minimality of F, (7.29) and (7.30) we see that

$$\begin{aligned}
P_t(E) &\geqslant P_s(E)^{\frac{N-t}{N-s}} \frac{P_t(F)}{P_s(F)^{\frac{N-t}{N-s}}} \\
&= P_t(F) + \left(\frac{P_s(E)^{\frac{N-t}{N-s}}}{P_s(F)^{\frac{N-t}{N-s}}} - 1 \right) P_t(F) \\
&\geqslant P_t(F) - \left| \frac{P_s(E)}{P_s(F)} - 1 \right| P_t(F) \\
&\geqslant P_t(F) - \frac{P_t(F)}{P_s(F)} |P_s(E) - P_s(F)| \\
&\geqslant P_t(F) - \frac{P_t(F)}{P_s(F)} P_s(F\Delta E).
\end{aligned}$$

Hence, by applying the fractional isoperimetric inequality (2.15) to $P_s(F)$, we obtain that

$$P_t(E) \geqslant P_t(F) - C(N,\delta_0)|F|^{-\frac{N-s}{N}} P_s(F\Delta E).$$

As in (5.15), by means of (2.3) and again the fractional isoperimetric inequality we then get

$$\begin{aligned}
P_t(E, B(x,R)) &\geqslant P_t(F, B(x,R)) - C(N,\delta_0)|F|^{-\frac{N-s}{N}} |F\Delta E|^{\frac{t-s}{N}} P_t(F\Delta E) \\
&= \left(1 - C(N,\delta_0)|F|^{-\frac{N-s}{N}} R^{t-s}\right) P_t(F, B(x,R))\,,
\end{aligned}$$

which gives

$$P_t(F, B(x,R)) \leqslant \frac{|F|^{-\frac{N-s}{N}}}{1 - C(N,\delta_0)R^{t-s}}\, P_t(E, B(x,R))\,.$$

□

Reasoning as in Section 5, from Proposition 7.7 we obtain the following regularity result.

Corollary 7.8. *There exists $\beta = \beta(N, \delta_0) < 1$ such that any minimizer F of (7.1) is bounded and has boundary of class $C^{1,\beta}$, outside of a closed singular set of Hausdorff dimension at most $N - 3$ (respectively $N - 8$ if $t = 1$).*

Proof of Theorem 1.4. The existence claim is a consequence of Theorem 7.2 and the regularity follows from Corollary 7.8. ☐

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