Brunn-Minkowski inequality for the 1-Riesz capacity and level set convexity for the 1/2-Laplacian

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Abstract

We prove that that the 1-Riesz capacity satisfies a Brunn-Minkowski inequality, and that the capacitary function of the 1/2-Laplacian is level set convex.

Keywords: fractional Laplacian; Brunn-Minkowski inequality; level set convexity; Riesz capacity.

1 Introduction

In this paper we consider the following problem

$$\begin{cases} (-\Delta)^s u = 0 & \text{on } \mathbb{R}^N \setminus K \\ u = 1 & \text{on } K \\ \lim_{|x|\to+\infty} u(x) = 0 \end{cases} \tag{1}$$

where $N \geq 2$, $s \in (0, N/2)$, and $(-\Delta)^s$ stands for the s-fractional Laplacian, defined as the unique pseudo-differential operator $(-\Delta)^s : \mathcal{S} \mapsto L^2(\mathbb{R}^N)$, being $\mathcal{S}$ the Schwartz space of functions with fast decay to 0 at infinity, such that

$$\mathcal{F}(-\Delta)^s f = |\xi|^{2s}\mathcal{F}(f)(\xi),$$

where $\mathcal{F}$ denotes the Fourier transform. We refer to the guide [12, Section 3] for more details on the subject. A quantity strictly related to Problem (1) is the so-called *Riesz potential energy* of a set E, defined as

$$I_\alpha(E) = \inf_{\mu(E)=1} \int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{d\mu(x)\, d\mu(y)}{|x-y|^{N-\alpha}} \qquad \alpha \in (0, N). \tag{2}$$

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It is possible to prove (see [18]) that if E is a compact set, then the infimum in the definition of $\mathcal{I}_\alpha(E)$ is achieved by a Radon measure μ supported on the boundary of E if $\alpha \geq 2$, and with support equal to E if $\alpha \in (0,2)$. If μ is the optimal measure for the set E, we define the *Riesz potential* of E as

$$v(x) = \int_{\mathbb{R}^N} \frac{d\mu(y)}{|x-y|^{N-\alpha}}, \tag{3}$$

so that

$$I_\alpha(E) = \int_{\mathbb{R}^N} v(x) d\mu(x).$$

It is not difficult to check (see [18, 15]) that the potential v satisfies

$$(-\Delta)^{\frac{\alpha}{2}} v = c(\alpha, N)\, \mu,$$

where $c(\alpha, N)$ is a positive constant, and that $v = I_\alpha(E)$ on E. In particular, if $s = \alpha/2$, then $v_K = v/I_{2s}(K)$ is the unique solution of Problem (1).

Following [18], we define the *α-Riesz capacity* of a set E as

$$\mathrm{Cap}_\alpha(E) := \frac{1}{I_\alpha(E)}. \tag{4}$$

We point out that this is not the only concept of capacity present in literature. Indeed, another one is given by the 2-capacity of a set E, defined by

$$\mathcal{C}_2(E) = \min \left\{ \int_{\mathbb{R}^N} |\nabla \varphi|^2 : \varphi \in C^1(\mathbb{R}^N, [0,1]),\ \varphi \geq \chi_E \right\} \tag{5}$$

where χ_A is the characteristic function of the set A. It is possible to prove that, if E is a compact set, then the minimum in (5) is achieved by a function u satisfying

$$\begin{cases} \Delta u = 0 & \text{on } \mathbb{R}^N \setminus E \\ u = 1 & \text{on } E \\ \lim_{|x| \to +\infty} u(x) = 0. \end{cases} \tag{6}$$

It is worth stressing that the 2-capacity and the α-Riesz capacity share several properties, and coincide if $\alpha = 2$. We refer the reader to [19, Chapter 8] for a discussion of this topic.

In a series of works (see for instance [5, 10, 17] and the monography [16]) it has been proved that the solutions of (6) are level set convex provided E is a convex body, that is, a compact convex set with non-empty interior. Moreover, in [1] (and later in [9] in a more general setting and in [8] for the logarithmic capacity in 2 dimensions) it has been proved that the 2-capacity satisfies a suitable version of the Brunn-Minkowski inequality: given two convex bodies K_0 and K_1 in $\mathbb{R}^N$, for any $\lambda \in [0,1]$ it holds

$$\mathcal{C}_2(\lambda K_1 + (1-\lambda)K_0)^{\frac{1}{N-2}} \geq \lambda\, \mathcal{C}_2(K_1)^{\frac{1}{N-2}} + (1-\lambda)\, \mathcal{C}_2(K_0)^{\frac{1}{N-2}}.$$

We refer to [20, 14] for a comprehensive survey on the Brunn-Minkowski inequality.

The main purpose of this paper is to show the analogous of these results in the fractional setting $\alpha = 1$, that is, $s = 1/2$ in Problem (1). More precisely, we shall prove the following result.

Theorem 1.1. *Let $K \subset \mathbb{R}^N$ be a convex body and let u be the solution of Problem* (1) *with $s = 1/2$. Then*

(i) u is level set convex, that is, for every $c \in \mathbb{R}$ the set $\{u > c\}$ is convex;

(ii) the 1*-Riesz capacity* $\mathrm{Cap}_1(K)$ *satisfies the following Brunn-Minkowski inequality: for any couple of convex bodies K_0 and K_1 and for any $\lambda \in [0,1]$ we have*

$$\mathrm{Cap}_1(\lambda K_1 + (1-\lambda)K_0)^{\frac{1}{N-1}} \geq \lambda \mathrm{Cap}_1(K_1)^{\frac{1}{N-1}} + (1-\lambda)\mathrm{Cap}_1(K_0)^{\frac{1}{N-1}}. \tag{7}$$

The proof of the Theorem 1.1 will be given in Section 2, and relies on the results in [11, 9], and on the following theorem due to L. Caffarelli and L. Silvestre.

Theorem 1.2 ([7])**.** *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a measurable function and let $U : \mathbb{R}^N \times [0,+\infty)$ be the solution of*

$$\Delta_{(x,t)} U(x,t) = 0, \quad \text{on } \mathbb{R}^N \times (0,+\infty) \qquad U(x,0) = f(x).$$

Then, for any $x \in \mathbb{R}^N$ there holds

$$\lim_{t \to 0^+} \partial_t U(x,t) = (-\Delta)^s f(x).$$

Eventually, in Section 3 we provide an application of Theorem 1.1 and we state some open problems.

2 Proof of the main result

This section is devoted to the proof of Theorem 1.1.

Lemma 2.1. *Let K be a compact convex set with positive* 2*-capacity and let $(K_\varepsilon)_{\varepsilon>0}$ be a family of compact convex sets with positive* 2*-capacity such that $K_\varepsilon \to K$ in the Hausdorff distance, as $\varepsilon \to 0$. Letting u_ε and u be the capacitary functions of K_ε and K respectively, we have that u_ε converges uniformly on $\mathbb{R}^N$ to u as $\varepsilon \to 0$. As a consequence, we have that the sequence $\mathcal{C}_2(K_\varepsilon)$ converges to $\mathcal{C}_2(K)$, and that the sets $\{u_\varepsilon > s\}$ converge to $\{u > s\}$ for any $s > 0$, with respect to the Hausdorff distance.*

Proof. We only prove that $u_\varepsilon \to u$ uniformly as $\varepsilon \to 0$ since this immediately implies the other claims. Let $\Omega_\varepsilon = K \cup K_\varepsilon$. Since $u_\varepsilon - u$ is a harmonic function on $\mathbb{R}^N \setminus \Omega_\varepsilon$, we have that

$$\sup_{\mathbb{R}^N \setminus \Omega_\varepsilon} |u_\varepsilon - u| \le \sup_{\partial\Omega_\varepsilon} |u_\varepsilon - u| \le \max\left\{1 - \min_{\partial\Omega_\varepsilon} u, 1 - \min_{\partial\Omega_\varepsilon} u_\varepsilon\right\}. \tag{8}$$

Moreover, by Hausdorff convergence, we know that there exists a sequence $(r_\varepsilon)_\varepsilon$ infinitesimal as $\varepsilon \to 0$ such that $K_\varepsilon \subset K + B_{r_\varepsilon}$, where $B(r)$ indicates the ball of radius r centred at the origin. Thus

$$\min\left\{\min_{\partial\Omega_\varepsilon} u, \min_{\partial\Omega_\varepsilon} u_\varepsilon\right\} \ge \min\left\{\min_{K+B(2r_\varepsilon)} u, \min_{K_\varepsilon+B(2r_\varepsilon)} u_\varepsilon\right\}. \tag{9}$$

Since the right-hand side of (9) converges to 1 as $\varepsilon \to 0$, from (8) we obtain

$$\lim_{\varepsilon\to 0} \sup_{\mathbb{R}^N \setminus \Omega_\varepsilon} |u_\varepsilon - u| = 0,$$

which gives the thesis. □

Remark 2.2. Notice that a compact convex set has positive 2-capacity if and only if its $\mathcal{H}^{N-1}$-measure is non-zero (see [13]).

Proof of Theorem 1.1. We start by proving claim (i). Let us consider the problem

$$\begin{cases} -\Delta_{(x,t)} U(x,t) = 0 & \text{in } \mathbb{R}^N \times (0,\infty) \\ U(x,0) = 1 & x \in K \\ U_t(x,0) = 0 & \text{in } x \in \mathbb{R}^N \setminus K \\ \lim_{|(x,t)|\to\infty} U(x,t) = 0. \end{cases} \tag{10}$$

By Theorem 1.2 we have that $U(x,0) = u(x)$ for every $x \in \mathbb{R}^N$. Notice also that, for any $c \in \mathbb{R}$, we have

$$\{u \ge c\} = \{(x,t) : U(x,t) \ge c\} \cap \{t = 0\}$$

which entails that is u is level set convex, provided that U is level set convex. In order to prove this we introduce the problem

$$\begin{cases} \Delta_{(x,t)} V(x,t) = 0 & \text{in } \mathbb{R}^{N+1} \setminus K \\ V = 1 & x \in K \\ \lim_{|(x,t)|\to\infty} V(x,t) = 0 \end{cases} \tag{11}$$

whose solution is given by the capacitary function of the set K in $\mathbb{R}^{N+1}$, that is, the function which achieves the minimum in Problem (5).

Since K is symmetric with respect to the hyperplane $\{t = 0\}$ (where it is contained), it follows, for instance by applying a suitable version of the Pólya-Szegö inequality for the Steiner symmetrization (see for instance [2, 4]), that V is symmetric as well with respect to the same hyperplane. In particular we have that $\partial_t V(x, 0) = 0$ for all $x \in \mathbb{R}^N \setminus K$. This implies that $V(x,t) = U(x,t)$ for every $t \geq 0$. To conclude the proof, we are left to check that V is level set convex. To prove this we recall that the capacitary function of a convex body is level set convex, as proved in [9]. Moreover, by Lemma 2.1 applied to the sequence of convex bodies $K_\varepsilon = K + B(\varepsilon)$ we get that V is level set convex as well. This concludes the proof of (i).

To prove (ii) we start by noticing that the 1-Riesz capacity is a $(N-1)$-homogeneous functional, hence inequality (7) can be equivalently stated (see for instance [1]) by requiring that, for any couple of convex sets K_0 and K_1 and for any $\lambda \in [0,1]$, the inequality

$$\mathrm{Cap}_1(\lambda K_1 + (1-\lambda)K_0) \geq \min\{\mathrm{Cap}_1(K_0), \mathrm{Cap}_1(K_1)\} \tag{12}$$

holds true.

We divide the proof of (12) into two steps.

Step 1.
We characterize the 1-Riesz capacity of a convex set K as the behaviour at infinity of the solution of the following PDE

$$\begin{cases} (-\Delta)^{1/2} v_K = 0 & \text{in } \mathbb{R}^N \setminus K \\ v_K = 1 & \text{in } K \\ \lim_{|x|\to\infty} |x|^{N-1} v_K(x) = \mathrm{Cap}_1(K) \end{cases}$$

We recall that, if μ_K is the optimal measure for the minimum problem in (2), then the function

$$v(x) = \int_{\mathbb{R}^N} \frac{d\mu_K(y)}{|x-y|^{N-1}}$$

is harmonic on $\mathbb{R}^N \setminus K$ and is constantly equal to $I_1(K)$ on K (see for instance [15]). Moreover the optimal measure μ_K is supported on K, so that $|x|^{N-1} v(x) \to \mu_K(K) = 1$ as $|x| \to \infty$. The claim follows by letting $v_K = v / I_1(K)$.

Step 2.
Let $K_\lambda = \lambda K_1 + (1-\lambda)K_0$ and $v_\lambda = v_{K_\lambda}$. We want to prove that

$$v_\lambda(x) \geq \min\{v_0(x), v_1(x)\}$$

for any $x \in \mathbb{R}^N$. To this aim we introduce the auxiliary function

$$\widetilde{v}_\lambda(x) = \sup\big\{ \min\{v_0(x_0), v_1(x_1)\} : x = \lambda x_1 + (1-\lambda)x_0 \big\},$$

and we notice that Step 2 follows if we show that $v_\lambda \geq \widetilde{v}_\lambda$. An equivalent formulation of this statement is to require that for any $s > 0$ we have

$$\{\widetilde{v}_\lambda > s\} \subseteq \{v_\lambda > s\}. \tag{13}$$

A direct consequence of the definition of $\widetilde{v}_\lambda$ is that

$$\{\widetilde{v}_\lambda > s\} = \lambda\{v_1 > s\} + (1-\lambda)\{v_0 > s\}.$$

For all $\lambda \in [0,1]$, we let V_λ be the harmonic extension of v_λ on $\mathbb{R}^N \times [0,\infty)$, which solves

$$\begin{cases} -\Delta_{(x,t)} V_\lambda(x,t) = 0 & \text{in } \mathbb{R}^N \times (0,\infty) \\ V_\lambda(x,0) = v_\lambda(x) & \text{in } \mathbb{R}^N \times \{0\} \\ \lim_{|(x,t)|\to\infty} V_\lambda(x,t) = 0. & \end{cases} \tag{14}$$

Notice that V_λ is the capacitary function of K_λ in $\mathbb{R}^{N+1}$, restricted to $\mathbb{R}^N \times [0,+\infty)$. Letting $H = \{(x,t) \in \mathbb{R}^N \times \mathbb{R} : t = 0\}$, for any $\lambda \in [0,1]$ and $s \in \mathbb{R}$ we have

$$\{V_\lambda > s\} \cap H = \{v_\lambda > s\}.$$

Letting also

$$\widetilde{V}_\lambda(x,t) = \sup\{\min\{V_0(x_0,t_0), V_1(x_1,t_1)\} : (x,t) = \lambda(x_1,t_1) + (1-\lambda)(x_0,t_0)\}, \tag{15}$$

as above we have that

$$\{\widetilde{V}_\lambda > s\} = \lambda\{V_1 > s\} + (1-\lambda)\{V_0 > s\}.$$

By applying again Lemma 2.1 to the sequences $K_0^\varepsilon = K_0 + B(\varepsilon)$ and $K_1^\varepsilon = K_1 + B(\varepsilon)$, we get that the corresponding capacitary functions, denoted respectively as V_0^ε and V_1^ε, converge uniformly to V_0 and V_1 in $\mathbb{R}^N$, and that $\widetilde{V}_\lambda^\varepsilon$, defined as in (15), converges uniformly to $\widetilde{V}_\lambda$ on $\mathbb{R}^N \times [0,+\infty)$.

Since $\widetilde{V}_\lambda^\varepsilon(x,t) \leq V_\lambda^\varepsilon(x,t)$ for any $(x,t) \in \mathbb{R}^N \times [0,+\infty)$, as shown in [9, pages $474-476$], we have that $\widetilde{V}_\lambda(x,t) \leq V_\lambda(x,t)$. As a consequence, we get

$$\begin{aligned} \{v_\lambda > s\} = \{V_\lambda > s\} \cap H \supseteq \{\widetilde{V}_\lambda > s\} \cap H &= \Big[\lambda\{V_1 > s\} + (1-\lambda)\{V_0 > s\}\Big] \cap H \\ \supseteq \lambda\{V_1 > s\} \cap H + (1-\lambda)\{V_0 > s\} \cap H &= \lambda\{v_1 > s\} + (1-\lambda)\{v_0 > s\} \end{aligned}$$

for any $s > 0$, which is the claim of *Step 2*.

We conclude by observing that inequality (12) follows immediately, by putting together *Step* 1 and *Step* 2. This concludes the proof of (ii), and of the theorem. □

Remark 2.3. The equality case in the Brunn-Minkowski inequality (7) is not easy to address by means of our techniques. The problem is not immediate even in the case of the 2-capacity, for which it has been studied in [6, 9].

3 Applications and open problems

In this section we state a corollary of Theorem 1.1. To do this we introduce some tools which arise in the study of convex bodies. The *support function* of a convex body $K \subset \mathbb{R}^N$ is defined on the unit sphere centred at the origin $\partial B(1)$ as

$$h_K(\nu) = \sup_{x \in \partial K} \langle x, \nu \rangle.$$

The *mean width* of a convex body K is

$$M(K) = \frac{2}{\mathcal{H}^{N-1}(\partial B(1))} \int_{\partial B(1)} h_K(\nu)\, d\mathcal{H}^{N-1}(\nu).$$

We refer to [20] for a complete reference on the subject. We observe that, if $N = 2$, then $M(K)$ coincides up to a constant with the perimeter $P(K)$ of K (see [3]).

We denote by $\mathcal{K}_N$ the set of convex bodies of $\mathbb{R}^N$ and we set

$$\mathcal{K}_{N,c} = \{K \in \mathcal{K}_N,\, M(K) = c\}.$$

The following result has been proved in [3].

Theorem 3.1. *Let $F : \mathcal{K}_N \to [0, \infty)$ be a q-homogeneous functional which satisfies the Brunn-Minkowski inequality, that is, such that $F(K + L)^{1/q} \geq F(K)^{1/q} + F(L)^{1/q}$ for any $K, L \in \mathcal{K}_N$. Then the ball is the unique solution of the problem*

$$\min_{K \in \mathcal{K}_N} \frac{M(K)}{F^{1/q}(K)}\,. \tag{16}$$

An immediate consequence of Theorem 3.1, Theorem 1.1 and Definition 4 is the following result.

Corollary 3.2. *The minimum of I_1 on the set $\mathcal{K}_{N,c}$ is achieved by the ball of mean width c. In particular, if $N = 2$, the ball of radius r solves the isoperimetric type problem*

$$\min_{K \in \mathcal{K}_2, P(K) = 2\pi r} I_1(K). \tag{17}$$

Motivated by Theorem 1.1 and Corollary 3.2 we conclude the paper with the following conjecture:

Conjecture 3.3. *For any $N \geq 2$ and $\alpha \in (0, N)$, the α-Riesz capacity $\mathrm{Cap}_\alpha(K)$ satisfies the following Brunn-Minkowski inequality:*
for any couple of convex bodies K_0 and K_1 and for any $\lambda \in [0, 1]$ we have

$$\mathrm{Cap}_\alpha(\lambda K_1 + (1 - \lambda)K_0)^{\frac{1}{N-\alpha}} \geq \lambda \mathrm{Cap}_\alpha(K_1)^{\frac{1}{N-\alpha}} + (1 - \lambda)\mathrm{Cap}_\alpha(K_0)^{\frac{1}{N-\alpha}}. \tag{18}$$

In particular, for any $\alpha \in (0, 2)$ the ball of radius r is the unique solution of the isoperimetric type problem

$$\min_{K \in \mathcal{K}_2, P(K) = 2\pi r} I_\alpha(K). \tag{19}$$

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