Convergence for long–times of a semidiscrete Perona-Malik equation in one dimension

G. Bellettini * , M. Novaga † , M. Paolini ‡

Abstract

We rigorously prove that the semidiscrete schemes of a Perona-Malik type equation converge, in a long time scale, to a suitable system of ordinary differential equations defined on piecewise constant functions. The proof is based on a formal asymptotic expansion argument, and on a careful construction of discrete comparison functions. Despite the equation has a region where it is backward parabolic, we prove a discrete comparison principle, which is the key tool for the convergence result.

Key words and phrases: nonconvex functionals, forward-backward parabolic equations, barriers, semidiscrete schemes.

1 Introduction

The one-dimensional Perona-Malik equation is the forward-backward parabolic equation

$$u_t = \left(\frac{u_x}{1+u_x^2}\right)_x, \qquad u(0) = u_0, \tag{1.1}$$

in $I \times (0,T)$, $I = (0,\ell) \subset \mathbb{R}$, and it is obtained as the gradient flow of the nonconvex functional

$$F(u) := \frac{1}{2}\int_I \log\left(1+u_x^2\right)\ dx. \tag{1.2}$$

The analog of equation (1.1) in two space dimensions was introduced in [17] in the context of image segmentation in computer vision: the convex-concave behaviour of the function $\log(1+p^2)$, depending on whether $|p|$ is smaller or larger than one, is, roughly speaking, the motivation for the following reasonable picture. Regions of the graph of the solution where the slope is less than one are expected to be further smoothened, in view of the heat-type character of the equation. On the other hand, in the remaining regions the solution should even increase its slope, thanks to the antiparabolic character of the equation. Contours of the image are therefore expected to be enhanced, at least for short times. However, even considering the simpler case of one space dimension, from a mathematical viewpoint we cannot still make rigorous the above picture. Indeed, the backward parabolic character of (1.1) in

*Dipartimento di Matematica, Università di Roma Tor Vergata, via della Ricerca Scientifica, 00133 Roma, Italy, and INFN Laboratori Nazionali di Frascati, via E. Fermi 40, 00044 Frascati, Roma, Italy. E-mail: Giovanni.Bellettini@lnf.infn.it

†Dipartimento di Matematica Pura e Applicata, Università di Padova, Via Trieste 63, 35121 Padova, Italy. E-mail: novaga@math.unipd.it

‡Dipartimento di Matematica e Fisica, Università Cattolica di Brescia, via Trieste 41, 25121 Brescia, Italy. E-mail: paolini@dmf.unicatt.it

regions where $|u_x| > 1$ makes the equation ill-posed. Despite this fact, in [11] it was recently proved that sufficiently smooth local solutions do exist for a dense class of initial data. These solutions seem not to necessarily reproduce the instabilities observed in numerical experiments: numerical solutions obtained with different regularization schemes exhibit, for short times, microstructures where $|u_x| > 1$, and nonsmooth solutions could also be expected. We refer to [6], [9], [14], [19], [15], [21], [13], [10], [22], [8] for a discussion on the subject.

Let us consider a subdivision of I in $N = \ell/h$ equal intervals with nodes $x_i^h = x_i := ih$, and let us fix Dirichlet boundary conditions $u(0) = 0$, $u(\ell) = \mathcal{M} \neq 0$. Let $\phi : \mathbb{R} \to [0, +\infty[$ be a smooth even function, increasing on $[0, +\infty[$, and satisfying the conditions

$$\phi(p) = p^2/2 \quad \textit{for } |p| \ll 1 \qquad \text{and} \qquad \phi(p) = \log|p| \quad \text{for } |p| \gg 1. \tag{1.3}$$

We are interested here in the long-time behavior of the semi-discrete solutions to the equation

$$\frac{\partial u}{\partial t} = (\phi'(u_x))_x, \tag{1.4}$$

obtained as the gradient flow of the functional (1.2), once one replaces $\frac{1}{2}\log(1+|p|^2)$ with the function $\phi(p)$. As in the case of (1.1), the main qualitative features of ϕ are the sublinearity at infinity, the strict convexity for $|p|$ small and the *strict concavity* for $|p|$ large. Our choice of assuming $\phi(p)$ to be exactly equal to $p^2/2$ near the origin and $\log|p|$ near infinity simplifies computations; we expect our results not to change qualitatively in the case of the Perona-Malik integrand in (1.2). We also note that values of the gradients of order $\mathcal{O}(1)$ are never achieved in our analysis. With a small abuse of notation, we shall still refer to (1.4) as to the Perona-Malik equation.

Numerical simulations show that, on a sufficiently long time scale, the numerical solution to (1.1) tends to be piecewise constant, and to move its plateaus in vertical direction. Let us consistently introduce the rescaled time variable $\tau := ht$; discretizing equation (1.1) in space in a standard way, we obtain

$$h\dot{u}^h - D^+\phi'(D^-u^h) = 0, \qquad u^h(0) = u_0, \tag{1.5}$$

with Dirichlet boundary conditions

$$u^h(0, \tau) = 0, \qquad u^h(\ell, \tau) = \mathcal{M}. \tag{1.6}$$

Here we identify a function with the vector of its nodal values, and the discrete derivative operators $D^{\pm}$ are defined in (2.4). Moreover, in view of the previous discussion, we assume the initial datum u_0 to be piecewise constant. Our main result is the following convergence theorem (see Theorem 5.4 for a precise statement):

Assume that u_0 is piecewise constant. Then the solutions u^h to (1.5) *converge locally uniformly in $\overline{I} \times [0, T_{\rm sing}[$ to a limit function $u_{\lim}$, which is piecewise constant in space, and is such that the height a_j of its $j^{\rm th}$-plateau with endpoints $y_{j-1} < y_j$ evolves with vertical velocity given by*

$$\dot{a}_j(\tau) = \frac{1}{y_j - y_{j-1}} \left(\frac{1}{a_{j+1}(\tau) - a_j(\tau)} - \frac{1}{a_j(\tau) - a_{j-1}(\tau)} \right). \tag{1.7}$$

The time $T_{\rm sing} > 0$ is the first time when (1.7) singularizes, namely when there is a collision of two adjacent plateaus. The proof of this result is based on an asymptotic expansion and on a comparison argument. More precisely, in Section 3 we perform a formal discrete asymptotic expansion of the solutions to (1.5) up to the order two. This requires a rather delicate computation, especially at the extremal nodes of the grid. Despite the fact that the reasoning is formal, it is a crucial tool, since the expansion suggests both the limit evolution law (1.7) and the form of suitably adapted comparison functions to make rigorous the convergence argument. Comparison functions $\Theta^{\pm h}$ are modeled on the notion of central solution (Definitions 4.1 and 4.4), with an addition of a vertical shift of order h^2

and a shape correction of order h^3. In Theorem 5.3 we prove a sort of comparison principle, and in Theorem 5.4 we prove the convergence result. The fact that the convergence is based on a comparison argument may be surprising, in view of the concavity of ϕ for large slopes. Technically, in the proof we consider separately the subintervals of I close to the jump points, where the slope of the solution falls in the concavity region of ϕ.

For clarity of exposition, before considering the general case, we prove the main results in the case of two jumps only (see Theorems 4.3, 5.1 and 5.2), when the comparison functions are actually strict sub/supersolutions of (1.5).

Eventually, we point out that the piecewise constant functions are the natural class of initial data for equation (1.5). Indeed, in [10, 4] it is proven that the solutions to the discretized version of (1.4) converge to a piecewise constant function, as $t \to +\infty$.

2 Notation and preliminaries

We consider a subdivision of $I :=]0, \ell[$ in $N = \ell/h$ equal intervals $K_i, i = 1, ..., N$ with nodes $x_i^h = x_i := ih$, so that $K_i^h = K_i = [x_{i-1}, x_i]$, where we set $x_0 := 0$ and $x_N := \ell$. The nodes $x_2, \ldots, x_{N-2}$ will be called inner nodes, and x_1, x_{N-1} extremal nodes. The intervals $K_2, \ldots, K_{N-1}$ will be called inner intervals, and K_1, K_N extremal intervals. For notational simplicity we omit the dependence on h of the nodes and of the intervals. Unless otherwise specified, we fix the Dirichlet boundary condition $u(0) = 0$, $u(\ell) = \mathcal{M}$. We denote by V_N^D the $(N-1)$-dimensional space of Lipschitz piecewise affine functions on the grid $\{x_1, \ldots, x_{N-1}\}$ assuming the Dirichlet boundary datum.

Throughout the paper we assume that $\phi : \mathbb{R} \to [0, +\infty)$ is an even function of class $\mathcal{C}^2$, and increasing on $[0, +\infty)$. Moreover, we shall assume that there exist $0 < p_0 < p_1$ such that

$$\begin{aligned} &\phi(p) = p^2/2 && \text{for } |p| < p_0, \\ &\phi(p) = \log|p| && \text{for } |p| > p_1. \end{aligned} \tag{2.1}$$

The Perona-Malik-type equation (1.4) is discretized in space using standard piecewise linear finite elements with mass lumping as done in [17], thus obtaining

$$\frac{\partial u^h}{\partial t} = D^+\phi'(D^-u^h) \tag{2.2}$$

with boundary conditions

$$u^h(0,t) = 0, \qquad u^h(\ell,t) = \mathcal{M} \tag{2.3}$$

(here we identify a function with the vector of its nodal values). The operators $D^\pm$ are defined as

$$\begin{aligned} h(D^-u)_i &= \quad u_i - u_{i-1}, && i = 1, ..., N, \\ h(D^+w)_i &= \quad w_{i+1} - w_i, && i = 1, ..., N-1, \end{aligned} \tag{2.4}$$

where the Dirichlet values (2.3) are taken into account.

Let us introduce the rescaled time variable $\tau := ht$. The derivative $\dot{u}^h$ of u^h with respect to τ solves the rescaled semidiscrete Perona-Malik equation

$$\mathcal{H}_{\rm PM}(u^h) := h\dot{u}^h - D^+\phi'(D^-u^h) = 0, \tag{2.5}$$

with Dirichlet boundary conditions

$$u^h(0,\tau) = 0, \qquad u^h(\ell,\tau) = \mathcal{M}. \tag{2.6}$$

Definition 2.1. *Let $\mathcal{T} \subseteq [0, +\infty[$ be an interval. We say that a smooth function $\tau \in \mathcal{T} \to v(\tau) \in V_N^D$ (or, with a slight abuse of notation, the vector $\{v_i(\tau)\}_{i=1}^{N-1}$ of its nodal values) is a strict supersolution (resp. strict subsolution) of the semidiscrete Perona-Malik equation (2.5) in $I \times \mathcal{T}$ if*

$$\mathcal{H}_{\rm PM}(v) > 0 \qquad (\text{resp. } \mathcal{H}_{\rm PM}(v) < 0), \tag{2.7}$$

where the inequalities are required to hold in the whole of $I \times \mathcal{T}$, or equivalently at all inner and extremal nodes for $\tau \in \mathcal{T}$.

3 Formal asymptotic analysis

In this section we perform a formal asymptotic expansion of the solution u^h of (2.5), (2.6), under the assumption that the solution $u_{\lim}$ of the limit problem as $h \to 0^+$ has only two jumps, located exactly at the two boundary points $\{x_0, x_N\}$ of I. To begin the analysis, we fix an initial height

$$a_0 \notin \{0, \mathcal{M}\}, \tag{3.1}$$

which is the initial value of $u_{\lim}$ in $]0, \ell[$. Since $u_{\lim}$ must assume the Dirichlet boundary datum, it has initially only two jumps at x_0 and x_N.
We expect an evolution with a plateau evolving vertically in time: $u_{\lim}(x, \tau) = a(\tau)$, with $a(0) = a_0$. We are interested in determining the vertical speed $\dot{a}(\tau)$. Our aim is to determine the first terms of an asymptotic expansion of the discrete solution u^h in terms of powers of h, valid for small values of $h > 0$. A key point is that we cannot simply expand the nodal values u_i^h, since the number of nodes also depends on h, and x_i assumes different positions as $h \to 0^+$. Therefore we will suppose that there are functions of x of which we use only the nodal values. We stress that the asymptotic expansion is formal; nevertheless, it is useful to prove the *rigorous* convergence result of Section 5.
We shall assume that the discrete solution $u^h(\tau)$, which for simplicity we occasionally denote also by $u(\tau) = (u_i(\tau))_i$, can be expanded in terms of h with functions U_j which are independent of h, and so that $U_j(\cdot, \tau)$ are smooth in $\overline{I}$. Precisely, we assume

$$u^h(x_i, \tau) = a(\tau) + hU_1(x_i, \tau) + h^2 U_2(x_i, \tau) + ..., \qquad i = 1, \dots, N-1. \tag{3.2}$$

We also assume that during the evolution the space derivative of u^h is always $<< 1$ in the inner intervals K_i, $i = 2, ..., N-1$, whereas it is $>> 1$ in the two extremal intervals K_1 and K_N. In this way equation (2.5) will coincide with the space-discretized heat equation at all inner nodes x_i, $i = 2, ..., N-2$ (recall (2.1)).
Using (3.2) the derivative of u^h with respect to τ can be expressed as

$$\dot{u}^h(x_i, \tau) = \dot{a}(\tau) + h\dot{U}_1(x_i, \tau) + h^2 \dot{U}_2(x_i, \tau) + \mathcal{O}(h^3). \tag{3.3}$$

3.1 Analysis at the inner nodes in case of two jumps

We need to express the discrete space derivatives in terms of the continuous ones, using Taylor expansion. If $v = v(x)$ is smooth in $\overline{I}$ we have $v(x_{i\pm1}) = v(x_i) \pm hv'(x_i) + \frac{h^2}{2} v''(x_i) \pm \frac{h^3}{6} v'''(x_i) + \mathcal{O}(h^4)$, and

$$(D^+D^-v)_i = \frac{1}{h^2}(v_{i+1} - 2v_i + v_{i-1}) = v''(x_i) + \mathcal{O}(h^2). \tag{3.4}$$

Substituting in (2.5), using (3.3) and (3.2) we get

$$h\dot{a}(\tau) + h^2 \dot{U}_1(x_i, \tau) = hU_1''(x_i, \tau) + h^2 U_2''(x_i, \tau) + \mathcal{O}(h^3), \qquad i = 2, \dots, N-2, \tag{3.5}$$

where again we denote by $'$ the derivative with respect to x.

Note that if we want to extend (3.5) to the order 3 we need to make use of the fourth derivative of v in (3.4).
We now equate the coefficients of the same powers of h.
Order 1. From (3.5) it follows that U_1 has constant second space derivative at the inner nodes. It is then natural to impose for U_1 the requirement

$$U_1''(x,\tau) = \dot{a}(\tau) \qquad \forall x \in \overline{I}. \tag{3.6}$$

Integrating twice we can write, for any $x \in \overline{I}$,

$$U_1'(x,\tau) = \dot{a}(\tau)\left(x - \frac{\ell}{2}\right) + \beta(\tau), \tag{3.7}$$

$$U_1(x,\tau) = \frac{1}{2}\dot{a}(\tau)\left(x - \frac{\ell}{2}\right)^2 + \beta(\tau)\left(x - \frac{\ell}{2}\right) + \gamma(\tau) =: \widetilde{U}_1(x,\tau) + \gamma(\tau), \tag{3.8}$$

where the time depending functions β and γ are still to be determined.
Order 2. We get

$$U_2''(x,\tau) = \dot{U}_1(x,\tau) = \frac{1}{2}\ddot{a}(\tau)\left(x - \frac{\ell}{2}\right)^2 + \dot{\beta}(\tau)\left(x - \frac{\ell}{2}\right) + \dot{\gamma}(\tau) \qquad \forall x \in I, \tag{3.9}$$

so that U_2 is a polynomial of degree 4 having fourth space derivative given by $\ddot{a}(\tau)$.
The functions a and β will be determined at the end of the discussion of the 0-order terms in Section 3.2, see formulae (3.21), (3.23). The function γ will be determined in (3.29).

3.2 Analysis at the extremal nodes in case of two jumps

It is essential to analyze what happens at the two estremal nodes x_1 and x_{N-1}. Let us concentrate on the node x_1; we need to accurately compute the two incremental quotients $(D^-u)_i$, $i = 1, 2$. It is here convenient to Taylor-expand about the point $x = 0$, whose position does not depend on h. If v is smooth in $\overline{I}$, by expanding the values $v(x_2) = v(2h)$ and $v(x_1) = v(h)$ about 0, and subtracting we end up with

$$(D^-v)_2 = v'(0) + \frac{3}{2}hv''(0) + \mathcal{O}(h^2). \tag{3.10}$$

This formula can be used to compute, using (3.2),

$$(D^-u)_2 = hU_1'(0,\tau) + h^2\left(U_2'(0,\tau) + \frac{3}{2}U_1''(0,\tau)\right) + \mathcal{O}(h^3). \tag{3.11}$$

Similarly, expanding about ℓ,

$$(D^-u)_{N-1} = hU_1'(\ell,\tau) + h^2\left(U_2'(\ell,\tau) - \frac{3}{2}U_1''(\ell,\tau)\right) + \mathcal{O}(h^3). \tag{3.12}$$

In the first extremal interval K_1 the situation is different, since we have to take into account the Dirichlet value $u(0) = 0$. Moreover, since we end up with a very large $\mathcal{O}(1/h)$ space derivative, we need to use the definition $\phi'(s) = 1/s$ (see (2.1)). Using Taylor expansion about 0 for the value of u at x_1, from (3.7) we have

$$\begin{aligned}\phi'\left((D^-u)_1\right) &= \left[\frac{a(\tau)}{h} + U_1(0,\tau) + h\left(U_2(0,\tau) + U_1'(0,\tau)\right) + \mathcal{O}(h^2)\right]^{-1} \\ &= \left[\frac{a(\tau)}{h} + U_1(0,\tau) + \mathcal{O}(h)\right]^{-1} = \frac{h}{a(\tau)} - h^2\frac{U_1(0,\tau)}{a^2(\tau)} + \mathcal{O}(h^3).\end{aligned} \tag{3.13}$$

Similarly, we find

$$\begin{aligned}\phi'\left((D^-u)_N\right) &= \left[\frac{\mathcal{M}-a(\tau)}{h} - U_1(\ell,\tau) + \mathcal{O}(h)\right]^{-1} \\ &= \frac{h}{\mathcal{M}-a(\tau)} + h^2\frac{U_1(\ell,\tau)}{(\mathcal{M}-a(\tau))^2} + \mathcal{O}(h^3).\end{aligned} \tag{3.14}$$

Finally, using (3.11), (3.13) and (2.1), we have

$$\begin{aligned}(D^+\phi'(D^-u))_1 =& \frac{1}{h}\left(\phi'((D^-u)_2) - \phi'(D^-u)_1\right) = \frac{1}{h}\left((D^-u)_2 - \phi'(D^-u)_1\right) \\ =& -\frac{1}{a(\tau)} + U_1'(0,\tau) + h\left(U_2'(0,\tau) + \frac{3}{2}U_1''(0,\tau) + \frac{U_1(0,\tau)}{a^2(\tau)}\right) + \mathcal{O}(h^2),\end{aligned} \tag{3.15}$$

and similarly

$$\begin{aligned}(D^+\phi'(D^-u))_{N-1} =& \frac{1}{\mathcal{M}-a(\tau)} - U_1'(\ell,\tau) \\ &+ h\left(-U_2'(\ell,\tau) + \frac{3}{2}U_1''(\ell,\tau) + \frac{U_1(\ell,\tau)}{(\mathcal{M}-a(\tau))^2}\right) + \mathcal{O}(h^2).\end{aligned} \tag{3.16}$$

Regarding the time derivative in (2.5), we express (3.3) at the node x_1 in terms of $x = 0$ by Taylor expansion, e.g.

$$\dot U_1(x_1,\tau) = \dot U_1(0,\tau) + h\dot U_1'(0,\tau) + \mathcal{O}(h^2).$$

Therefore, substituting in (3.3)

$$h\dot u_1 = h\dot a(\tau) + h^2\dot U_1(0,\tau) + h^3(\dot U_2(0,\tau) + \dot U_1'(0,\tau)) + \mathcal{O}(h^4) = h\dot a(\tau) + h^2\dot U_1(0,\tau) + \mathcal{O}(h^3). \tag{3.17}$$

Similarly

$$h\dot u_{N-1} = h\dot a(\tau) + h^2\dot U_1(\ell,\tau) + h^3(\dot U_2(\ell,\tau) - \dot U_1'(\ell,\tau)) + \mathcal{O}(h^4) = h\dot a(\tau) + h^2\dot U_1(\ell,\tau) + \mathcal{O}(h^3). \tag{3.18}$$

Equating (3.17) to (3.15) and (3.18) to (3.16), we get equations for various orders in powers of h that we now specify.

Order 0. Terms of order 0 are only present on the right-hand-side; they give

$$U_1'(0,\tau) = \frac{1}{a(\tau)}, \tag{3.19}$$

$$U_1'(\ell,\tau) = \frac{1}{\mathcal{M}-a(\tau)}. \tag{3.20}$$

Integrating (3.6) on the whole of I and equating the result with (3.20) minus (3.19) we get the most important result:

$$\dot a(\tau) = \frac{1}{\ell}\left(\frac{1}{\mathcal{M}-a(\tau)} - \frac{1}{a(\tau)}\right), \qquad \tau \in [0, T_{\text{sing}}[, \tag{3.21}$$

which is sufficient to recover $a(\tau)$ in terms of the initial value

$$a(0) = a_0. \tag{3.22}$$

Here $T_{\text{sing}} = T_{\text{sing}}(\mathcal{M}, a_0) \le +\infty$ is the first time when

$$\text{either} \quad \lim_{\tau\to T_{\text{sing}}^-} a(\tau) = 0 \quad \text{or} \quad \lim_{\tau\to T_{\text{sing}}^-} (\mathcal{M} - a(\tau)) = 0.$$

Remark 3.1. Let $\mathcal{M} > 0$. If $a_0 = \mathcal{M}/2$ then $a(\tau) \equiv \mathcal{M}/2$ is an unstable equilibrium ($T_{\rm sing} = +\infty$). In the other cases, we have $T_{\rm sing} < +\infty$. Moreover if $a_0 \in]0, \mathcal{M}/2[$ (resp. $a_0 \in]\mathcal{M}/2, \mathcal{M}[$) then $\dot a < 0$ (resp. $\dot a > 0$) in $[0, T_{\rm sing}[$, and $\lim_{\tau \to T_{\rm sing}^-} \dot a(\tau) = -\infty$ (resp. $\lim_{\tau \to T_{\rm sing}^-} \dot a(\tau) = +\infty$).

We can also substitute (3.19) (evaluated at $x = \ell$) in (3.7) to uniquely recover, using (3.21), the function β:

$$\beta(\tau) = \frac{1}{2}\left(\frac{1}{a(\tau)} + \frac{1}{\mathcal{M} - a(\tau)}\right), \qquad \tau \in [0, T_{\rm sing}[. \tag{3.23}$$

Order 1. Collecting all terms of order h in (3.15) and (3.17) we get, for $\tau \in [0, T_{\rm sing}[$,

$$U_2'(0,\tau) + \frac{U_1(0,\tau)}{a^2(\tau)} = \dot a(\tau) - \frac{3}{2} U_1''(0,\tau) = -\frac{1}{2}\dot a(\tau), \tag{3.24}$$

where we use (3.6) evaluated at $x = 0$. In (3.24) we have two unknowns, since the value of $U_1(0,\tau)$ depends on $\gamma(\tau)$ which we do not know yet. However, we still have to enforce a similar equation on the rightmost node x_{N-1}, which gives another independent relation for $\gamma(\tau)$ and $U_2'(\ell,\tau)$, the latter being related to $U_2'(0,\tau)$ via (3.9). From (3.16), (3.18) and (3.6) we get

$$-U_2'(\ell,\tau) + \frac{U_1(\ell,\tau)}{(\mathcal{M} - a(\tau))^2} = \dot a(\tau) - \frac{3}{2} U_1''(\ell,\tau) = -\frac{1}{2}\dot a(\tau). \tag{3.25}$$

Adding (3.24) and (3.25) we get

$$U_2'(0,\tau) - U_2'(\ell,\tau) + \frac{U_1(0,\tau)}{a^2(\tau)} + \frac{U_1(\ell,\tau)}{(\mathcal{M} - a(\tau))^2} = -\dot a(\tau). \tag{3.26}$$

From (3.8) we have

$$\ell\dot\gamma = \int_I \dot U_1 \, dx - \int_I \dot{\widetilde U}_1 \, dx. \tag{3.27}$$

Furthermore, from (3.9), (3.26) and (3.8) we obtain

$$\begin{aligned} \int_I \dot U_1 \, dx = {}& U_2'(\ell,\tau) - U_2'(0,\tau) = \dot a(\tau) + \frac{U_1(0,\tau)}{a^2(\tau)} + \frac{U_1(\ell,\tau)}{(\mathcal{M} - a(\tau))^2} \\ = {}& \dot a(\tau) + \frac{\widetilde U_1(0,\tau)}{a^2(\tau)} + \frac{\widetilde U_1(\ell,\tau)}{(\mathcal{M} - a(\tau))^2} + \left(\frac{1}{a^2(\tau)} + \frac{1}{(\mathcal{M} - a(\tau))^2}\right)\gamma(\tau). \end{aligned} \tag{3.28}$$

From (3.27) and (3.28) we deduce

$$\ell\,\dot\gamma(\tau) = \left(\frac{1}{a^2(\tau)} + \frac{1}{(\mathcal{M} - a(\tau))^2}\right)\gamma(\tau) - \int_I \dot{\widetilde U}_1(x,\tau)\, dx + \dot a(\tau) + \frac{\widetilde U_1(0,\tau)}{a^2(\tau)} + \frac{\widetilde U_1(\ell,\tau)}{(\mathcal{M} - a(\tau))^2}. \tag{3.29}$$

Since a and β have already been determined, also the function $\widetilde U_1$ defined in (3.8) is determined. Then (3.29) is a linear ordinary differential equation in γ, with smooth coefficients in $[0, T_{\rm sing}[$. Therefore, γ cannot singularize before $T_{\rm sing}$, and is uniquely defined in $[0, T_{\rm sing}[$, once $\gamma(0)$ has been assigned. Consequently, the expressions of U_1 and U_2 in (3.2) in $[0, T_{\rm sing}[$ at the inner and extremal nodes are given by (3.8) and (3.9).

Figure 1: The points in (3.30) and (3.31) and the function φ_j and ψ_j defined in (3.37), (3.40) respectively.

3.3 Asymptotic analysis in case of many jumps

In order to prove the convergence result in Section 5.2 in case of many jumps, we need an asymptotic expansion localized on a generic subinterval of I having bundary points which are not grid points. Let $k \geq 1$ be an integer independent of h, and $y_0, y_1, \ldots, y_k$ be points of $\overline{I}$ with

$$y_0 := 0 < y_1 < \ldots < y_k := \ell. \tag{3.30}$$

Define $I_j :=]y_{j-1}, y_j[$, and assume that y_{j-1} and y_j do not belong to the grid. We denote by $i_j \in \mathbb{N}$ the index satisfying

$$y_j \in]x_{i_j}, x_{i_j} + h[, \tag{3.31}$$

which is unique for $h > 0$ sufficiently small.
We now repeat the formal analysis of Section 3 when I is replaced by I_j, assuming the validity of the expansion

$$u^h(x_i, \tau) = a_j(\tau) + hU_{1,j}(x_i, \tau) + h^2 U_{2,j}(x_i, \tau) + ..., \qquad i = i_{j-1} + 1, \ldots, i_j, \tag{3.32}$$

with the conventions

$$a_0(\tau) \equiv 0 \qquad \text{and} \qquad a_{k+1}(\tau) \equiv \mathcal{M}.$$

Arguing similarly to Section 3.1 we obtain

$$U_{1,j}(x, \tau) = \frac{1}{2}\dot{a}_j(\tau)\left(x - \frac{y_j + y_{j-1}}{2}\right)^2 + \beta_j(\tau)\left(x - \frac{y_j + y_{j-1}}{2}\right) + \gamma_j(\tau) =: \widetilde{U}_{1,j}(x, \tau) + \gamma_j(\tau) \tag{3.33}$$

and

$$U''_{2,j}(x, \tau) = \dot{U}_{1,j}(x, \tau) = \frac{1}{2}\ddot{a}_j(\tau)\left(x - \frac{y_j + y_{j-1}}{2}\right)^2 + \dot{\beta}_j(\tau)\left(x - \frac{y_j + y_{j-1}}{2}\right) + \dot{\gamma}_j(\tau), \tag{3.34}$$

for any $x \in \overline{I_j}$. In particular

$$U''_{1,j} = \dot{a}_j \qquad \text{in } \overline{I_j}. \tag{3.35}$$

Some changes are required in the arguments of Section 3.2, since now the boundary points of I_j are not points of the grid, and the Dirichlet condition is not satisfied anymore. Taking this remark into account, and observing that the boundary points y_{j-1}, y_j of the interval do not change with h, expanding about y_{j-1} yields that the analog of (3.10) reads as

$$(D^- v)_{i_{j-1}+2} = v'(y_{j-1}) + \left(\frac{1}{2} + \varphi_{j-1}(h)\right) hv''(y_{j-1}) + \mathcal{O}(h^2), \tag{3.36}$$

where

$$\varphi_{j-1}(h) := \frac{x_{i_{j-1}} + h - y_{j-1}}{h}. \tag{3.37}$$

Similarly the analog of (3.11), using (3.32), reads as

$$(D^-u)_{i_{j-1}+2} = hU'_{1,j}(y_{j-1},\tau) + h^2\left[U'_{2,j}(y_{j-1},\tau) + \left(\frac{1}{2} + \varphi_{j-1}(h)\right)U''_{1,j}(y_{j-1},\tau)\right] + \mathcal{O}(h^3), \quad (3.38)$$

Similarly, formula (3.12) is replaced by

$$(D^-u)_{i_j} = hU'_{1,j}(y_j,\tau) + h^2\left[U'_{2,j}(y_j,\tau) - \left(\frac{1}{2} + \psi_j(h)\right)U''_{1,j}(y_j,\tau)\right] + \mathcal{O}(h^3), \quad (3.39)$$

where

$$\psi_j(h) := \frac{y_j - x_{i_j}}{h} = 1 - \varphi_j(h). \quad (3.40)$$

Let us now compute $(D^-u)_{i_{j-1}+1}$ and $(D^-u)_{i_j+1}$. We have

$$\begin{aligned}(D^-u)_{i_{j-1}+1} =& \frac{a_j(\tau) - a_{j-1}(\tau)}{h} + U_{1,j}(y_{j-1},\tau) - U_{1,j-1}(y_{j-1},\tau) + \mathcal{O}(h)\\ (D^-u)_{i_j+1} =& \frac{a_{j+1}(\tau) - a_j(\tau)}{h} - U_{1,j+1}(y_j,\tau) - U_{1,j}(y_j,\tau) + \mathcal{O}(h).\end{aligned} \quad (3.41)$$

Hence, recalling (2.1),

$$\phi'((D^-u)_{i_{j-1}+1}) = \frac{h}{a_j(\tau) - a_{j-1}(\tau)} - h^2\frac{U_{1,j}(y_{j-1},\tau) - U_{1,j-1}(y_{j-1},\tau)}{(a_j(\tau) - a_{j-1}(\tau))^2} + \mathcal{O}(h^3), \quad (3.42)$$

and

$$\phi'((D^-u)_{i_{j-1}}) = \frac{h}{a_{j+1}(\tau) - a_j(\tau)} + h^2\frac{U_{1,j+1}(y_j,\tau) - U_{1,j}(y_j,\tau)}{(a_{j+1}(\tau) - a_j(\tau))^2} + \mathcal{O}(h^3), \quad (3.43)$$

Therefore, from (3.38) and (3.42) we get

$$\begin{aligned}(D^+\phi'(D^-u))_{i_{j-1}+1} =& -\frac{1}{a_j(\tau) - a_{j-1}(\tau)} + U'_{1,j}(y_{j-1},\tau)\\ &+ h\Big[U'_{2,j}(y_{j-1},\tau) + \left(\frac{1}{2} + \varphi_{j-1}(h)\right)U''_{1,j}(y_{j-1},\tau)\\ &\quad + \frac{U_{1,j}(y_{j-1},\tau) - U_{1,j-1}(y_{j-1},\tau)}{(a_j(\tau) - a_{j-1}(\tau))^2}\Big] + \mathcal{O}(h^2),\end{aligned} \quad (3.44)$$

and from (3.39) and (3.43)

$$\begin{aligned}(D^+\phi'((D^-u))_{i_j} =& \frac{1}{a_{j+1}(\tau) - a_j(\tau)} - U'_{1,j}(y_j,\tau)\\ &+ h\Big[-U'_{2,j}(y_j,\tau) + \left(\frac{1}{2} + \psi_j(h)\right)U''_{1,j}(y_j,\tau)\\ &\quad + \frac{U_{1,j+1}(y_j,\tau) - U_{1,j}(y_j,\tau)}{(a_{j+1}(\tau) - a_j(\tau))^2}\Big] + \mathcal{O}(h^2).\end{aligned} \quad (3.45)$$

Recalling (2.5), using the expansion

$$\dot U_{1,j}(x_{i_{j-1}} + h,\tau) = \dot U_{1,j}(y_{j-1},\tau) + h\varphi_j(h)\dot U'_{1,j}(y_{j-1},\tau) + \mathcal{O}(h^2),$$

and

$$h\dot u_h = h\dot a_j + h^2\dot U_{1,j} + \mathcal{O}(h^3), \quad (3.46)$$

we get, equating the terms of order 0 in h,

$$U'_{1,j}(y_{j-1},\tau)=\frac{1}{a_j(\tau)-a_{j-1}(\tau)},\qquad U'_{1,j}(y_j,\tau)=\frac{1}{a_{j+1}(\tau)-a_j(\tau)}. \tag{3.47}$$

Taking the difference of the two equations in (3.47), and integrating in I_j equation (3.35) it follows

$$\dot{a}_j(\tau)=\frac{1}{y_j-y_{j-1}}\left(\frac{1}{a_{j+1}(\tau)-a_j(\tau)}-\frac{1}{a_j(\tau)-a_{j-1}(\tau)}\right),\qquad j=1,\dots,k. \tag{3.48}$$

Using (3.33) and (3.48) we have

$$\begin{aligned}\frac{1}{a_{j+1}(\tau)-a_j(\tau)}=U'_1(y_j,\tau)=&\frac{1}{2}\dot{a}_j(\tau)\,(y_j-y_{j-1})+\beta_j(\tau)\\ =&\frac{1}{2}\left(\frac{1}{a_{j+1}(\tau)-a_j(\tau)}-\frac{1}{a_j(\tau)-a_{j-1}(\tau)}\right)+\beta_j(\tau),\end{aligned}$$

which implies

$$\beta_j(\tau)=\frac{1}{2}\left(\frac{1}{a_{j+1}(\tau)-a_j(\tau)}+\frac{1}{a_j(\tau)-a_{j-1}(\tau)}\right),\quad j=1,\dots,k. \tag{3.49}$$

Moreover, using (3.46), equating the terms of order 1 in h, and adding the two formulae in (3.44), we get

$$\begin{aligned}&U'_{2,j}(y_j,\tau)-U'_{2,j}(y_{j-1},\tau)=(\varphi_{j-1}(h)+\psi_j(h)-1)\dot{a}_j\\ &+\frac{U_{1,j}(y_{j-1},\tau)-U_{1,j-1}(y_{j-1},\tau)}{(a_j(\tau)-a_{j-1}(\tau))^2}+\frac{U_{1,j+1}(y_j,\tau)-U_{1,j}(y_j,\tau)}{(a_{j+1}(\tau)-a_j(\tau))^2}.\end{aligned} \tag{3.50}$$

Reasoning as in Section 3.2 and using (3.35) we obtain

$$\begin{aligned}(1-\varphi_{j-1}(h)-\psi_j(h))\dot{a}_j=&U'_{2,j}(y_j,\tau)-U'_{2,j}(y_{j-1},\tau)\\ &+\frac{U_{1,j}(y_{j-1},\tau)-U_{1,j-1}(y_{j-1},\tau)}{(a_j(\tau)-a_{j-1}(\tau))^2}+\frac{U_{1,j+1}(y_j,\tau)-U_{1,j}(y_j,\tau)}{(a_{j+1}(\tau)-a_j(\tau))^2}.\end{aligned} \tag{3.51}$$

Using also (3.34) we deduce

$$\begin{aligned}\int_{I_j}U''_{2,j}(x,\tau)\;dx=&\int_{I_j}\dot{U}_{1,j}(x,\tau)\;dx=U'_{2,j}(y_{j+1},\tau)-U'_{2,j}(y_j,\tau)\\ =&(1-\varphi_{j-1}(h)-\psi_j(h))\dot{a}_j-\frac{U_{1,j}(y_{j-1},\tau)-U_{1,j-1}(y_{j-1},\tau)}{(a_j(\tau)-a_{j-1}(\tau))^2}\\ &-\frac{U_{1,j+1}(y_j,\tau)-U_{1,j}(y_j,\tau)}{(a_{j+1}(\tau)-a_j(\tau))^2}\\ =&\int_{I_j}\dot{\widetilde{U}}_{1,j}(x,\tau)\;dx+\dot{\gamma}_j(\tau)(y_{j+1}-y_j).\end{aligned} \tag{3.52}$$

Since a_j and β_j have been already computed, (3.52) is a system of equations which determines the unknown γ_j. Once also γ_j has been found, we have the complete expression of $U_{1,j}$ and hence of $U_{2,j}$.

4 Construction of barriers and comparison functions

Based on the asymptotic analysis developped in Section 3 we construct a strict supersolution and a strict subsolution of (2.5), (2.6) (that we call barriers) in case of two jumps. In case of many jumps, we will construct suitable comparison functions, which will be strict super/subsolutions only in certain subintervals of I. Although potential ϕ is not convex, we shall nevertheless be able to prove a version of the maximum principle suitable for our situation. We will denote by C a positive constant the value of which may vary from line to line.

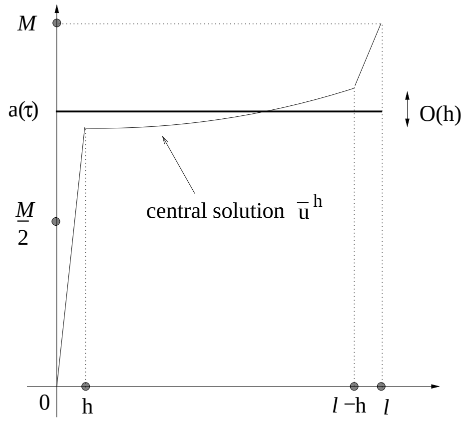

Figure 2: Graph of the central solution $\overline{u}^h$ in (4.1) when $a_0 \in\,]\mathcal{M}/2, \mathcal{M}[$, for $\mathcal{M} > 0$. In view of (3.21) and (3.23), if $a(\tau) \in\,]\mathcal{M}/2, \mathcal{M}[$, then the abscissa of the vertex of the parabola defining $U_1(\cdot, \tau)$ in (3.8) belongs to $]-\infty, 0[$, so that $U_1(\cdot, \tau)$ is increasing in $]h, \ell - h[$.

4.1 Central solution in case of two jumps

The construction of barriers is based on a *central* solution $\overline{u}^h$ defined using the asymptotic analysis of Section 3, to which we shall add/subtract appropriately selected terms.
More precisely, recalling that $a(\tau)$ is the solution of (3.21) and (3.22), we give the following

Definition 4.1. *We define the central solution $\overline{u}^h$ as*

$$\overline{u}^h(x_i, \tau) := a(\tau) + hU_1(x_i, \tau) + h^2\widetilde{U}_2(x_i, \tau), \qquad i = 1, \ldots, N-1,\ \tau \in [0, T_{\text{sing}}[, \tag{4.1}$$

where $\widetilde{U}_2$ is obtained by integrating twice (3.9), choosing the first integrating constant in such a way that (3.24) and (3.25) are satisfied, and the second integrating constant in any way, e.g. by imposing that $\widetilde{U}_2$ vanishes at $x = \ell/2$.

Note that $\overline{u}^h$ satisfies the Dirichlet boundary conditions, namely

$$\overline{u}^h(0, \tau) = 0, \qquad \overline{u}^h(\ell, \tau) = \mathcal{M}.$$

Upon substituting $\overline{u}^h$ in (2.5) we get a (small) residual $\mathcal{H}_{\text{PM}}(\overline{u}^h)$ that we must compute separately at the inner nodes $x_2, ..., x_{N-2}$ and at the extremal nodes x_1 and x_{N-1}.
Let now $T \in\,]0, T_{\text{sing}}[$. Recalling the analysis up to the order two in Section 3.1 we readily get

$$|\mathcal{H}_{\text{PM}}(\overline{u}^h)_i| \leq C\, h^3, \qquad i = 2, ..., N-2,\ \tau \in [0, T], \tag{4.2}$$

with a constant C depending only on T and ℓ. It is thus possible to control this residual using an appropriate time-dependent, constant in space, vertical shift of order $\mathcal{O}(h^2)$.
Furthermore, recalling the analysis up to the order one in Section 3.2 we also get

$$|\mathcal{H}_{\text{PM}}(\overline{u}^h)_i| \leq C\, h^2, \qquad i = 1 \text{ and } i = N-1,\ \tau \in [0, T], \tag{4.3}$$

with a constant C depending only on T and ℓ.

Remark 4.2. Due to the Dirichlet boundary condition, the $\mathcal{O}(h^2)$ vertical shift (see the term $C_1(\tau)h^2$ in (4.4) below) that we will need to control the residual at the inner nodes in (4.2) produces a change in the derivative at the extremal intervals which in turn produces an $\mathcal{O}(h^2)$ modification of the computed residual that needs to be compensated if it has the wrong sign. This is the reason of the presence of an additional shape correction of order $\mathcal{O}(h^3)$ in the form of our super/subsolutions (see the term $C_1(\tau)C_2h^3(x_1-\ell/2)^2$ in (4.4)) that we define in the next section.

4.2 Discrete sub and supersolution in case of two jumps

The supersolution u^{+h} and the subsolution u^{-h} of (2.5), (2.6) are defined using the first two terms of the formal asymptotic expansion with the addition of a vertical shift $C_1(\tau)h^2$ and a shape correction $C_1(\tau)C_2h^3(x-\ell/2)^2$. More precisely, recalling the expression of $\overline{u}^h$ in (4.1), and given $T \in]0, T_{\rm sing}[$, for any $\tau \in [0,T]$ we define $u^{\pm h}(\tau) \in V_N^D$ as

$$u^{\pm h}(x_i,\tau) \quad := \quad \overline{u}^h(x_i,\tau) \pm C_1(\tau)h^2 \pm C_1(\tau)C_2h^3\left(x_i-\frac{\ell}{2}\right)^2, \quad i=1,\dots,N-1, \tag{4.4}$$

and

$$u^{\pm h}(0,\tau) := 0, \qquad u^{\pm h}(\ell,\tau) := \mathcal{M}. \tag{4.5}$$

The positive function $C_1(\tau)$ and the positive constant C_2 will be chosen later on independently of h (but possibly depending on T). Having shift terms of order at least $\mathcal{O}(h^2)$ is important to ensure that all functions between the two barriers, which are at mutual distance $\mathcal{O}(h^2)$, have spatial derivatives that differ of $\mathcal{O}(h)$ from the derivative of the central solution $\overline{u}^h$, which in turn ensures that the regions of the domain of ϕ where $\phi'' < 0$ are not actually used.
The following result confirms the usefulness of the formal asymptotic expansion made in Section 3. We point out that we use the symbol $\mathcal{O}(h^\sigma)$, $\sigma \geq 0$, to denote an infinitesimal of order h^σ, which is independent of $\tau \in [0,T]$.

Theorem 4.3. *Fix $T \in]0, T_{\rm sing}[$. Then there exist a smooth positive function $C_1 : [0,T] \to]0,+\infty[$, a constant $C_2 > 0$, and $h_0 > 0$ such that the function u^{+h} (resp. u^{-h}) defined in* (4.4) *is a strict supersolution (resp. strict subsolution) of the semidiscrete Perona-Malik equation* (2.5) *in $I \times [0,T]$, for all $h \in]0, h_0[$.*

Proof. We only consider the function u^{+h}, since a similar argument works for u^{-h}. We set

$$\mathcal{S}_1 := C_1(\tau)h^2, \qquad \mathcal{S}_2 := C_1(\tau)C_2h^3(x-\ell/2)^2, \qquad (x,\tau) \in I \times [0,T].$$

We call $\mathcal{S}_1$ the vertical shift and $\mathcal{S}_2$ the shape correction to the central solution $\overline{u}^h$.
Inner nodes. Let us first analyze the situation at the nodes $x_2, ..., x_{N-2}$. The vertical shift $\mathcal{S}_1$ contributes only to the time derivative $\dot{u}^{+h}$, with contribution $C_1'(\tau)h^3$. The shape correction $\mathcal{S}_2$ has a contribution to $\dot{u}^{+h}$, bounded by $C_1'(\tau)C_2\ell^2h^4$, which can be neglected if h is small enough, and a contribution (with negative sign) to the discrete space derivative which is bounded by $C_1(\tau)C_2\ell h^3$. Therefore, in order to have the inequality

$$\mathcal{H}_{\rm PM}(u^{+h}) > 0 \qquad \text{at the inner nodes,} \tag{4.6}$$

we need

$$C_1'(\tau) \geq C\Big(1 + C_1(\tau)C_2\Big), \tag{4.7}$$

for some $C > 0$ big enough, depending only on T and ℓ, where the extra term 1 in (4.7) is needed in order to control the residual in (4.2).
Extremal nodes. Let us consider the extremal node x_1. Recall that, thanks to the Dirichlet boundary datum (4.5), the value of u^{+h} at $x = 0$ is not modified with the vertical shift. Hence, in the interval

K_1, the space derivative of u^{+h} differs from the space derivative of $\overline{u}^h$ essentially of an amount $C_1(\tau)h$, which is due to the term $\mathcal{S}_1$. More precisely, we have

$$D^- u^{+h}(x_1,\tau) = D^-\overline{u}^h(x_1,\tau) + C_1(\tau)h + C_1(\tau)C_2\mathcal{O}(h^2),$$

with $|\mathcal{O}(h^2)| \leq \ell\, h^2$. Being

$$\left| D^-\overline{u}^h(x_1,\tau) - \frac{a(\tau)}{h} \right| \leq \|U_1(x_1,\cdot) + h\,\widetilde{U}_2(x_1,\cdot)\|_{L^\infty([0,T])},$$

we get, using the second assumption in (2.1),

$$|\phi'\left(D^- u^{+h}\right)(x_1,\tau) - \phi'\left(D^-\overline{u}^h\right)(x_1,\tau)| \leq CC_1(\tau)\left(h^3 + C_2 h^4\right), \tag{4.8}$$

where the constant C depends only on T and ℓ. This term needs a compensation, which is given by $\mathcal{S}_2$ that in turn influences the discrete space derivative of u^{+h} in K_2. Recalling that $x_1 = h$ and $x_2 = 2h$, we have, using $(2h - \frac{\ell}{2})^2 - (h - \frac{\ell}{2})^2 = -\ell h + 3h^2$,

$$D^- u^{+h}(x_2,\tau) = D^-\overline{u}^h(x_2,\tau) - \ell C_1(\tau)C_2 h^3 + 3C_1(\tau)C_2 h^4,$$

which gives, using the first assumption in (2.1),

$$\phi'\left(D^- u^{+h}\right)(x_2,\tau) = \phi'\left(D^-\overline{u}^h\right)(x_2,\tau) - \ell C_1(\tau)C_2 h^3 + 3C_1(\tau)C_2 h^4. \tag{4.9}$$

From (4.8), (4.9) and the definition of $\mathcal{H}_{\rm PM}$ we then obtain

$$\mathcal{H}_{\rm PM}(u^{+h}(x_1,\tau)) \geq \mathcal{H}_{\rm PM}\left(\overline{u}^h(x_1,\tau)\right) + C_1(\tau)\Big(C_2\ell - C - CC_2 h\Big)h^2,$$

where the term $CC_1(\tau)C_2h^3$ absorbs the last addendum on the right-hand side of (4.8) and (4.9). Recalling from (4.3) that $\mathcal{H}_{\rm PM}(\overline{u}^h(x_1,\tau))$ is of order h^2, we obtain

$$\mathcal{H}_{\rm PM}(u^{+h}(x_1,\tau)) > 0, \tag{4.10}$$

provided we choose $h_0 > 0$ small enough and $C_2 > 0$ and $C_1(0) > 0$ large enough.
Once we have chosen C_2 and $C_1(0)$ we can define the increasing function $C_1(\tau)$ such that also (4.7) holds, so that the thesis follows from (4.6) and (4.10).
In the case of the extremal node x_{N-1} one can reason in a similar way, observing that (4.9) is replaced by

$$\phi'\left(D^- u^{+h}\right)(x_{N-1},\tau) = \phi'\left(D^-\overline{u}^h\right)(x_{N-1},\tau) + \ell C_1(\tau)C_2h^3 + 2C_1(\tau)C_2h^3 - 3C_1(\tau)C_2h^4.$$

□

4.3 Discrete comparison functions in case of many jumps

The super and subsolutions defined in (4.4) are perturbations of the central solution $\overline{u}^h$, which at time 0 has only two jumps at the boundary of I. In order to deal with more general initial data, we shall patch together the super and subsolutions previously defined, thus obtaining what we will call the comparison functions.
Let $k \geq 1$ be an integer independent of h, and $y_0, y_1, \ldots, y_k$ be points of I with $y_0 := 0 < y_1 < \ldots < y_k := \ell$. Let $u_0 : I \to \mathbb{R}$ be a function which is constantly equal to some constant a_j^0 in the interval $]y_{j-1}, y_j[$, for $1 \leq j \leq k$. Let also $a_0^0 := 0$, $a_{k+1}^0 := \mathcal{M}$. We assume that $a_j^0 \neq a_{j+1}^0$ for $j = 1, \ldots, k-1$. Set

$$u_0(x) := \sum_{j=0}^{k-1} a_{j+1}^0 1_{[y_j, y_{j+1}[}(x), \qquad x \in I, \tag{4.11}$$

where, given a subinterval A of I, we indicate by 1_A its characteristic function, namely $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$ if $x \in I \setminus A$. We think of u_0 as assuming the Dirichlet boundary datum (2.6), therefore if $a_1^0 \neq 0$ (resp. $a_k^0 \neq \mathcal{M}$) the point y_0 (resp. y_k) is considered a jump point of u_0. For simplicity, in the sequel we shall assume that y_j is not a grid point, so that in particular $u_0(x_{i_j})$ is well defined (see Figure 3). The general case can be treated in a similar way, with only minor modifications.
As in Section 3.3, for all $1 \leq j \leq k$ we define

$$I_j :=]y_{j-1}, y_j[. \tag{4.12}$$

and we denote by i_j the index for which $y_j \in]x_{i_j}, x_{i_j} + h[$, see Figure 1. Let $a_0(\tau) \equiv 0$, $a_{k+1}(\tau) \equiv \mathcal{M}$, and $(a_1(\tau), \dots, a_k(\tau))$, $\tau \in [0, T_{\text{sing}}[$, be the solution of the ODE's system

$$\dot{a_j}(\tau) = \frac{1}{y_j - y_{j-1}} \left(\frac{1}{a_{j+1}(\tau) - a_j(\tau)} - \frac{1}{a_j(\tau) - a_{j-1}(\tau)} \right), \quad j = 1, \dots, k, \tag{4.13}$$

with

$$a_j(0) = a_j^0, \qquad j = 0, \dots, k+1. \tag{4.14}$$

The time $T_{\text{sing}} \in]0, +\infty]$ is the first time for which at least one index $j \in \{0, \dots, k\}$ is such that $\liminf_{\tau \to T_{\text{sing}}^-} (a_{j+1}(\tau) - a_j(\tau)) = 0$.
System (4.13) is a generalization of equation (3.21), and reduces to it when $k = 1$, and has been formally derived in Section 3.3.
Define the piecewise constant function $u_{\text{lim}} : I \times [0, T_{\text{sing}}[\to \mathbb{R}$ as

$$u_{\text{lim}}(x, \tau) := \sum_{j=0}^{k-1} a_{j+1}(\tau) 1_{[y_j, y_{j+1}[}(x), \qquad (x, \tau) \in I \times [0, T_{\text{sing}}[. \tag{4.15}$$

Again, we assume that u_{lim} satisfies (2.6), therefore if $a_1(\tau) \neq 0$ (resp. $a_k(\tau) \neq \mathcal{M}$) the point y_0 (resp. y_k) is considered a jump point of $u_{\text{lim}}(\cdot, \tau)$ (see Figure 3).
Observe that the vertical velocity $\dot{a}_j$ of each horizontal plateau of the graph of $u_{\text{lim}}(\cdot, \tau)$ depends on the behaviour of its contiguous plateaus. Set $J_{u_{\text{lim}}} := \emptyset$ if $k = 1$ and $J_{u_{\text{lim}}} := \{y_1, \dots, y_{k-1}\}$ if $k \geq 2$. The set $J_{u_{\text{lim}}}$ is therefore the set of jump points of $u_{\text{lim}}(\cdot, \tau)$ which are inside I. The case $k = 1$ is when the jumps of $u_{\text{lim}}(\cdot, \tau)$ are both located at the boundary of I, and has been considered in Theorem 4.3.

Definition 4.4. *Let $T \in]0, T_{\text{sing}}[$. For $\tau \in [0, T_{\text{sing}}[$ and $j = 1, \dots, k$, we define the central solution $\overline{u}^h$ as*

$$\overline{u}^h(x_i, \tau) := a_j(\tau) + hU_{1,j}(x_i, \tau) + h^2 \widetilde{U}_{2,j}(x_i, \tau), \qquad i = i_{j-1} + 1, \dots, i_j, \tag{4.16}$$

where $\widetilde{U}_{2,j}$ is obtained by integrating twice (3.34), choosing the integrating constants as in Definition 4.1.

In analogy with (4.4), we also give the following

Definition 4.5. *Let $T \in]0, T_{\text{sing}}[$. For $\tau \in [0, T_{\text{sing}}[$ and $j = 1, \dots, k$, we define the comparison functions $\Theta^{\pm h}(\tau) \in V_N^D$ as*

$$\begin{aligned} \Theta^{\pm h}(x_i, \tau) &:= \overline{u}^h(x_i, \tau) \pm (-1)^{j+1} C_1(\tau) h^2 \pm (-1)^{j+1} C_1(\tau) C_2 h^3 \left(x_i - \frac{y_{j-1} + y_j}{2} \right)^2, \\ \Theta^{\pm h}(0, \tau) &:= 0, \qquad \Theta^{\pm h}(\ell, \tau) := \mathcal{M}, \end{aligned} \tag{4.17}$$

for all $i = i_{j-1} + 1, \dots, i_j$.

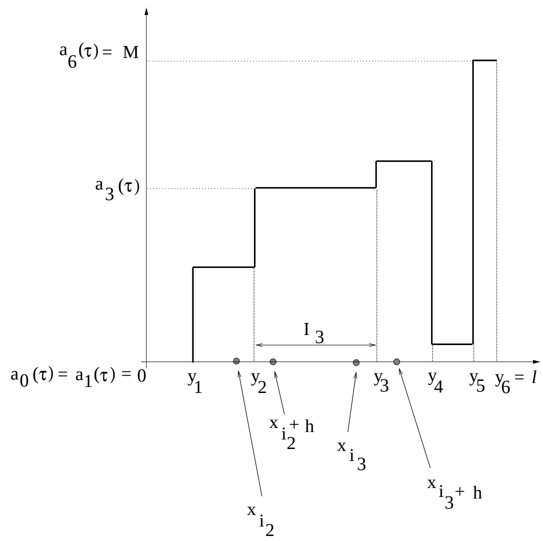

Figure 3: Jump points ($k = 6$) and the values of the function $u_{\rm lim}(\cdot, \tau)$.

The positive function $C_1(\tau)$ and the positive constant C_2 will be chosen later on independently of h (but possibly depending on T).
The functions $\Theta^{\pm h}$ are no longer super/subsolutions in the whole of $I \times [0, T]$, as shown in the following theorem.

Theorem 4.6. *Fix $T \in]0, T_{\rm sing}[$. There exist a smooth increasing function $C_1 : [0, T] \;\to]0, +\infty[$, a constant $C_2 > 0$, and $h_0 > 0$ such that, for all $h \in]0, h_0[$, the function Θ^{+h} (resp. Θ^{-h}) defined in* (4.17) *is a strict supersolution (resp. subsolution) of* (2.5) *in $I_j \times [0, T]$ for $j \in \{1, \ldots, k\}$ odd (resp. even), while it is a strict subsolution (resp. supersolution) in $I_j \times [0, T]$ for $j \in \{1, \ldots, k\}$ even (resp. odd).*

Proof. It is enough to consider the function Θ^{+h}. We also fix $j \in \{1, \ldots, k\}$ odd, the argument being similar when j is odd.

Inner nodes. The analysis at the inner nodes is exactly as in the proof of Theorem 4.3, and we omit the details.

Extremal nodes. It is enough to consider the extremal node $x_{i_{j-1}+1}$, since the arguments for x_{i_j} are the same. In the interval $K_{i_{j-1}+1}$, the space derivative of Θ^{+h} differs from the space derivative of $\overline{u}^h$ of an amount $2hC_1(\tau)$. More precisely, we have

$$D^-\Theta^{+h}(x_{i_{j-1}+1}, \tau) = D^-\overline{u}^h(x_{i_{j-1}+1}, \tau) + 2C_1(\tau)h + 2C_1(\tau)C_2\mathcal{O}(h^2),$$

with $|\mathcal{O}(h^2)| \leq \ell\, h^2$. Being

$$\begin{aligned} \left| D^-\overline{u}^h(x_{i_{j-1}+1}, \tau) - \frac{a_j(\tau) - a_{j-1}(\tau)}{h} \right| \;&\leq\; \|U_{1,j-1}(x_{i_{j-1}+1}, \cdot) + h\,\widetilde{U}_{2,j-1}(x_{i_{j-1}+1}, \cdot)\|_{L^\infty([0,T])} \\ &+\; \|U_{1,j}(x_{i_{j-1}+1}, \cdot) + h\,\widetilde{U}_{2,j}(x_{i_{j-1}+1}, \cdot)\|_{L^\infty([0,T])}, \end{aligned} \tag{4.18}$$

we get

$$|\phi'\left(D^-\Theta^{+h}\right)(x_{i_{j-1}+1}, \tau) - \phi'\left(D^-\overline{u}^h\right)(x_{i_{j-1}+1}, \tau)| \leq 2C_1(\tau)C\,(h^3 + C_2h^4), \tag{4.19}$$

where the constant C depends only on T and ℓ. Moreover, we have

$$D^-\Theta^{+h}(x_{i_{j-1}+2},\tau) = D^-\overline{u}^h(x_{i_{j-1}+2},\tau) - (y_j + y_{j-1})C_1(\tau)C_2h^3 + 3C_1(\tau)C_2h^4,$$

which gives

$$\phi'\left(D^-\Theta^{+h}\right)(x_{i_{j-1}+2},\tau) = \phi'\left(D^-\overline{u}^h\right)(x_{i_{j-1}+2},\tau) - (y_j + y_{j-1})C_1(\tau)C_2h^3 + 3C_1(\tau)C_2h^4. \quad (4.20)$$

From (4.19), (4.20) and the definition of $\mathcal{H}_{\rm PM}$ we then obtain

$$\mathcal{H}_{\rm PM}(\Theta^{+h}(x_{i_{j-1}+1},\tau)) \geq \mathcal{H}_{\rm PM}\left(\overline{u}^h(x_{i_{j-1}+1},\tau)\right) + C_1(\tau)\Big(C_2(y_j + y_{j-1}) - C - CC_2h\Big)h^2,$$

where the term $CC_1(\tau)C_2h^3$ absorbs the last terms on the right-hand side of (4.19) and (4.20). Recalling that $\mathcal{H}_{\rm PM}(\overline{u}^h(x_{i_{j-1}+1},\tau))$ is of order h^2, we obtain

$$\mathcal{H}_{\rm PM}(\Theta^{+h}(x_{i_{j-1}+1},\tau)) > 0, \quad (4.21)$$

provided we choose h_0 small enough, C_2 big enough and $C_1(0) > 0$. □

5 The convergence result

5.1 Comparison and convergence in case of two jumps

We want to prove a comparison principle between the super and subsolutions of (2.5), (2.6) constructed in Section 4.2.

Theorem 5.1. *Let $u_0 \equiv a_0 \notin \{0, \mathcal{M}\}$ in I. Assume that u_0 satisfies the Dirichlet boundary condition $u_0(0) = 0$ and $u_0(\ell) = \mathcal{M} \neq 0$, so that 0 and ℓ are the two jump points of u_0. Let $u^h : I \times [0, +\infty[\to \mathbb{R}$ be the solution of (2.5), (2.6), with*

$$u^h(x_i, 0) = a_0, \qquad i = 1, \ldots, N-1.$$

Let $u^{\pm h} : I \times [0,T] \to \mathbb{R}$ be the strict super and subsolution of (2.5) constructed in Section 4.2. Then, there exists $h_0 > 0$ such that

$$u^{-h}(x,\tau) \leq u^h(x,\tau) \leq u^{+h}(x,\tau), \quad (5.1)$$

for all $(x,\tau) \in \overline{I} \times [0,T]$, and $h \in]0, h_0[$.

Proof. We argue by contradiction. Let $\tau_0 \in [0,T]$ be the infimum of the times when one of the inequalities in (5.1) is not satisfied. Notice that $\tau_0 > 0$, since $|u^h(x,0) - u^{\pm h}(x,0)| \geq ch^2$ for some positive constant $c > 0$, if h is small enough. Hence at time τ_0 the graph of u^h touches the graph of one of the barriers $u^{\pm h}$. Suppose for instance that at $\tau = \tau_0$ the graph of u^h touches the graph of u^{+h} at some (possibly nonunique) node x_{i_0}, so that $\dot{u}^h(x_{i_0},\tau_0) \geq \dot{u}^{+h}(x_{i_0},\tau_0)$. Since $\dot{u}^{+h}$ is a strict supersolution of (2.5) we then have

$$h\dot{u}^h(x_{i_0},\tau_0) = D^+\phi'(D^-u^h)(x_{i_0},\tau_0) \geq h\dot{u}^{+h}(x_{i_0},\tau_0) > D^+\phi'(D^-u^{+h})(x_{i_0},\tau_0). \quad (5.2)$$

We now divide the proof into two cases.

Case 1. Assume that $i_0 \in \{2, \ldots, N-2\}$, namely x_{i_0} is an inner node of I. Note that, by construction, there exists a constant $C > 0$ (depending on T) such that

$$|u^{+h}(x,\tau) - u^{-h}(x,\tau)| \leq Ch^2, \qquad (x,\tau) \in [h, \ell - h] \times [0,T].$$

Hence, since the grid size is h and $u^{-h} \leq u^h \leq u^{+h}$ in $[h, \ell-h] \times [0,\tau_0]$, it follows that

$$|D^-u^h(x,\tau)| \leq Ch, \qquad (x,\tau) \in [2h, \ell - h] \times [0,\tau_0]. \quad (5.3)$$

Recalling that $\phi'(p) = p$ in a neighbourhood of 0, (5.2) implies

$$D^+D^-u^h(x_{i_0}, \tau_0) > D^+D^-u^{+h}(x_{i_0}, \tau_0), \tag{5.4}$$

which gives a contradiction, since $u^{+h}(\cdot, \tau_0) \geq u^h(\cdot, \tau_0)$ on $[0, \ell]$ and $u^{+h}(x_{i_0}, \tau_0) = u^h(x_{i_0}, \tau_0)$.
Case 2. Assume now that x_{i_0} is an extremal node. Without loss of generality, we suppose $i_0 = 1$, since a similar reasoning holds if $i_0 = N-1$. Then, as u^{+h} and u^h satisfy the same Dirichlet boundary condition, we have that $u^{+h}(\cdot, \tau_0)$ and $u^h(\cdot, \tau_0)$ coincide in $[0, x_1]$, and therefore

$$D^-u^h(x_1, \tau_0) = D^-u^{+h}(x_1, \tau_0). \tag{5.5}$$

On the other hand, we have $u^h(x_2, \tau_0) \leq u^{+h}(x_2, \tau_0)$, so that

$$D^-u^h(x_2, \tau_0) \leq D^-u^{+h}(x_2, \tau_0). \tag{5.6}$$

Inequalities (5.6) and (5.3) imply

$$\phi'(D^-u^h)(x_2, \tau_0) = D^-u^h(x_2, \tau_0) \leq D^-u^{+h}(x_2, \tau_0) = \phi'(D^-u^{+h})(x_2, \tau_0). \tag{5.7}$$

From (5.5) and (5.7) we deduce

$$D^+\phi'(D^-u^h)(x_1, \tau_0) \leq D^+\phi'(D^-u^{+h})(x_1, \tau_0), \tag{5.8}$$

which contradicts (5.2). □

As a consequence of Theorem 5.1 and the explicit form of the barriers, we have the following convergence result of the solutions of (2.5) to the solution of (3.21).

Theorem 5.2. *Let $u_0 \equiv a_0 \notin \{0, \mathcal{M}\}$ in I. Assume that u_0 satisfies the Dirichlet boundary conditions $u_0(0) = 0$ and $u_0(\ell) = \mathcal{M} \neq 0$, so that 0 and ℓ are the two jump points of u_0. Let $u^h : I \times [0, +\infty[\to \mathbb{R}$ be the solution of* (2.5), (2.6), *with*

$$u^h(x_i, 0) = a_0, \qquad i = 1, \dots, N-1.$$

Let $u_{\lim}(x, \tau) := a(\tau)$ for all $x \in I$, where $a(\tau)$ solves (3.21), (3.22), *and $u_{\lim}$ satisfies the Dirichlet boundary conditions. Then*

$$\lim_{h \to 0^+} u^h = u_{\lim} \qquad \textit{uniformly in the compact subsets of } \overline{I} \times [0, T_{\text{sing}}[. \tag{5.9}$$

5.2 Comparison and convergence in case of many jumps

We now extend the comparison result of Theorem 5.1 to the more general comparison functions constructed in Section 4.3.

Theorem 5.3. *Let u_0 be as in* (4.11) *and let $u^h : I \times [0, +\infty[\to \mathbb{R}$ be the solution of* (2.5), (2.6), *with initial datum*

$$u^h(x_i, 0) = a_j^0, \qquad j = 1, \dots, k, \ i = i_{j-1} + 1, \dots, i_j.$$

Let $T \in]0, T_{\text{sing}}[$, and let $u^{\pm h} : I \times [0, T] \to \mathbb{R}$ be the comparison functions defined in (4.17). *Then there exists $h_0 > 0$ such that*

$$\begin{aligned} &\Theta^{-h}(x, \tau) \leq u^h(x, \tau) \leq \Theta^{+h}(x, \tau), \qquad (x, \tau) \in \overline{I_j} \times [0, T], \ j \text{ odd}, \\ &\Theta^{+h}(x, \tau) \leq u^h(x, \tau) \leq \Theta^{-h}(x, \tau), \qquad (x, \tau) \in \overline{I_j} \times [0, T], \ j \text{ even} \end{aligned} \tag{5.10}$$

for all $h \in]0, h_0[$.

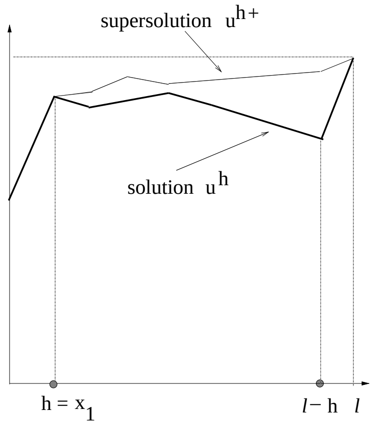

Figure 4: The comparison principle in the proof of case 2 in Theorem 5.1. The supersolution touches the solution (the graph in bold) in correspondence of the first extremal point x_1 of the grid.

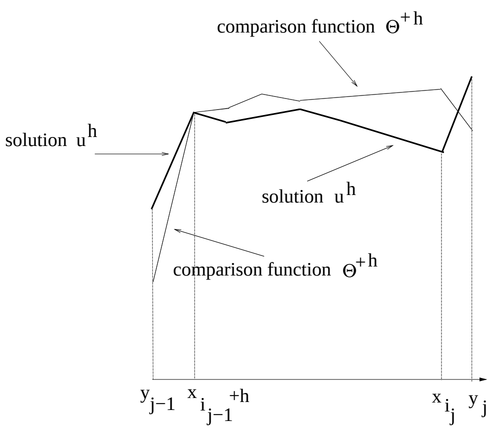

Figure 5: The comparison principle in case 2 of Theorem 5.3. When j is odd, the comparison function touches the solution (the graph in bold) in correspondence of the extremal point $x_{i_{j-1}} + h$, and it is partly below and partly above the solution.

Proof. The proof is similar to the proof of Theorem 5.1. Arguing by contradiction, we let $\tau_0 \in]0, T]$ be the infimum of the times when one of the inequalities in (5.10) is not satisfied. We fix for simplicity j odd, being the argument analogous in the case of j even, by exchanging the role of Θ^{+h} and Θ^{-h}. Hence at time τ_0 the graph of u^h touches the graph of one of the comparison functions, say Θ^{+h}, at some (possibly nonunique) point $x_{i_0} \in I_j$, so that

$$h\dot{u}^h(x_{i_0}, \tau_0) = D^+\phi'(D^-u^h)(x_{i_0}, \tau_0) \geq h\dot{\Theta}^{+h}(x_{i_0}, \tau_0) > D^+\phi'(D^-\Theta^{+h})(x_{i_0}, \tau_0). \tag{5.11}$$

We now divide the proof into two cases.

Case 1. Assume that $x_{i_0} \notin \{x_{i_{j-1}} + h, x_{i_j}\}$, namely x_{i_0} is an inner node of I_j. Note that, by construction, there exists a constant $C > 0$ (depending on T) such that

$$|\Theta^{+h}(x, \tau) - \Theta^{-h}(x, \tau)| \leq Ch^2, \qquad (x, \tau) \in [x_{i_{j-1}} + h, x_{i_j}] \times [0, T].$$

Hence, since the grid size is h and $\Theta^{-h} \leq u^h \leq \Theta^{+h}$ in $[x_{i_{j-1}} + h, x_{i_j}] \times [0, \tau_0]$, it follows that

$$|D^-u^h(x, \tau)| \leq Ch, \qquad (x, \tau) \in [x_{i_{j-1}} + 2h, x_{i_j}] \times [0, T]. \tag{5.12}$$

Then, being $\phi'(p) = p$ in a neighbourhood of 0, and being Θ^{+h} a strict supersolution of (2.5) in $I_j \times [0, \tau_0]$ by Theorem 4.6, we reach a contradiction as in *Case 1* of the proof of Theorem 5.1.

Case 2. Assume now that x_{i_0} is an extremal node, for instance $x_{i_0} = x_{i_{j-1}} + h$. Then, as

$$\Theta^{+h} \leq u^h \qquad \text{in } [x_{i_{j-2}} + h, x_{i_{j-1}}] \times [0, \tau_0],$$

we get

$$D^-u^h(x_{i_0}, \tau_0) \leq D^-\Theta^{+h}(x_{i_0}, \tau_0). \tag{5.13}$$

Since (4.18) implies that $|D^-\Theta^{+h}(x_{i_0}, \tau_0)| \geq c/h$ for some $c = c(T) > 0$, and $\phi'(p) = 1/p$ for $|p| > p_1$, from (5.13) we get

$$\phi'(D^-u^h(x_{i_0}, \tau_0)) \geq \phi'(D^-\Theta^{+h}(x_{i_0}, \tau_0)). \tag{5.14}$$

On the other hand, we have $u^h(x_{i_0} + h, \tau_0) \leq \Theta^{+h}(x_{i_0} + h, \tau_0)$, so that

$$D^-u^h(x_{i_0} + h, \tau_0) \leq D^-\Theta^{+h}(x_{i_0} + h, \tau_0). \tag{5.15}$$

Inequalities (5.15), (5.12) and $\phi'(p) = p$ for $|p| < p_0$, imply

$$\phi'(D^-u^h)(x_{i_0} + h, \tau_0) = D^-u^h(x_{i_0} + h, \tau_0) \leq D^-\Theta^{+h}(x_{i_0} + h, \tau_0) = \phi'(D^-\Theta^{+h})(x_{i_0} + h, \tau_0). \tag{5.16}$$

Subtracting (5.14) from (5.16) we then deduce

$$D^+\phi'(D^-u^h)(x_{i_0}, \tau_0) \leq D^+\phi'(D^-\Theta^{+h})(x_{i_0}, \tau_0),$$

which contradicts (5.11). $\square$

As before, from Theorem 5.3 and the explicit form of the comparison functions, we get the following convergence result.

Theorem 5.4. *Let u_0 be as in (4.11), $u_{\lim}$ as in (4.15), and let $u^h : I \times [0, +\infty[\to \mathbb{R}$ be the solution of (2.5), (2.6), with initial datum*

$$u^h(x_i, 0) = a_j^0, \qquad j = 1, \ldots, k, \ i = i_{j-1} + 1, \ldots, i_j.$$

Then

$$\lim_{h \to 0^+} u^h = u_{\lim} \qquad \text{uniformly on the compact subsets of } \overline{I} \times [0, T_{\mathrm{sing}}[. \tag{5.17}$$

We conclude the paper with two observations. Firstly, we note that the evolution law (4.13) coincides with the gradient flow of the Γ-limit of the functionals $\int_I \phi(u_x)\, dx$ restricted to piecewise linear functions and suitably rescaled (see [5]). Secondly, we observe that, if the Dirichlet boundary conditions (1.6) are replaced by homogeneous Neumann (or periodic) boundary conditions, then the analysis is simplified, and we expect the convergence of the solutions to a piecewise constant limit function satisfying (4.13), with homogeneous Neumann (or periodic) boundary conditions in place of (4.14).

References

[1] G. Bellettini, G. Fusco: *T*he Γ-limit and the related gradient flow for singular perturbation functionals of Perona-Malik type, Trans. Amer. Math. Soc. **360** (2008), 4929–4987.

[2] G. Bellettini, G. Fusco, N. Guglielmi, *A concept of solution for forward-backward equations of the form $u_t = \frac{1}{2}(\phi'(u_x))_x$ and numerical experiments for the singular perturbation $u_t = -\epsilon^2 u_{xxxx} + \frac{1}{2}(\phi'(u_x))_x$*, Discrete Contin. Dyn. Syst., **16** (2006), 783-842.

[3] G. Bellettini, M. Novaga, E. Paolini, *Global solutions to the gradient flow equation of a nonconvex functional*, SIAM J. Math. Anal., **37** (2006), 1657–1687.

[4] G. Bellettini, M. Novaga, M. Paolini, C. Tornese, *Convergence of discrete schemes for the Perona-Malik equation*, J. Differential Equations **245** (2008), 892–924.

[5] G. Bellettini, M. Novaga, M. Paolini, C. Tornese, *Classification of the equilibria for the semi-discrete Perona-Malik equation*, Calcolo, **46**(4) (2009), 221–243.

[6] F. Catté, P.-L. Lions, J.-M. Morel, T. Coll, *Image selective smoothing and edge detection by nonlinear diffusion*, SIAM J. Numer. Anal. **29** (1992), 182–193.

[7] S. Esedoglu, *An analysis of the Perona-Malik scheme*, Comm. Pure Apll. Math. **LIV** (2001), 1442–1487.

[8] S. Esedoglu, J. Greer, *Upper bounds on the coarsening rate of discrete, ill-posed, nonlinear diffusion equations*, Comm. Pure Appl. Math. **62** (2009), 57-81.

[9] F. Fierro, R. Goglione, M. Paolini, *Numerical simulations of mean curvature flow in presence of a nonconvex anisotropy*, Math. Models Methods Appl. Sci. **8** (1998), 573–601.

[10] M. Ghisi, M. Gobbino, *Gradient estimates for the Perona-Malik equation*, Math. Ann. **3** (2007), 557–590.

[11] M. Ghisi, M. Gobbino, *A class of local classical solutions for the one-dimensional Perona-Malik equation*, Trans. Amer. Math. Soc. **361** (2009), 6429–6446.

[12] M. Gobbino, *Gradient flow for the one-dimensional Mumford-Shah functional*, Ann. Sc. Norm. Sup. Pisa Cl. Sci. (4), **27** (1998), 145–193.

[13] M. Gobbino, *Entire solutions to the one-dimensional Perona-Malik equation*, Comm. Partial Differential Equations **32** (2007), 719–743.

[14] B. Kawhol, N. Kutev, *Maximum and comparison principle for the one-dimensional anisotropic diffusion* , Math. Ann., **311** (1998), 107-123.

[15] S. Kichenassamy, *The Perona-Malik paradox*, SIAM J. Appl. Math. **57** (2001), 1328–1342.

[16] M. Nitzberg, D. Mumford and T. Shiota, Filtering, Segmentation and Depth, *Lecture Notes in Computer Science* 662, Springer-Verlag, Berlin (1993).

[17] P. Perona, M. Malik, *Scale space and edge detection using anisotropic diffusion*, IEEE Trans. Pattern Anal. Mach. Intell. **12** (1990), 629–639.

[18] D.G. Schaeffer, M. Shearer, T.P Witelski, *A discrete model for an ill-posed non linear parabolic PDE*, Physica D **160** (2001), 189–221.

[19] O. Scherzer, J. Weickert, *Relations between regularization and diffusion filtering* , J. Math. Imaging Vision **12** (2000), 43–63.

[20] S. Slemrod, *Dynamics of Measure Valued Solutions to a Backward-Forward Heat Equation*, J. Dynam. Differential Equations **3** (2001), 1–28.

[21] S. Taheri, Q. Tang, K. Zhang, *Young measure solutions and instability of the one-dimensional Perona-Malik equation*, J. Math. Anal. Appl. **308** (2005), 467–490.

[22] K. Zhang, *Existence of infinitely many solutions for the one-dimensional Perona-Malik problem*, Calc. Var. Partial Differential Equations **326** (2006), 171–199.