

Curve shortening flow in heterogeneous media

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Abstract

In recent years, there has been a growing interest in geometric evolutions in heterogeneous media. Here we consider curvature driven flows of planar curves with an additional space-dependent forcing term, and we look for estimates which depend only on the L^∞ -norm of the forcing term. Our motivation comes from a homogenization problem, which we can rigorously solve in the special case when the initial curve is a graph and the forcing term does not depend on the vertical direction. In such case, we are also able to define a solution of the evolution even if the forcing term is just a bounded function, not necessarily continuous.

1 Introduction

In this paper we consider the curvature shortening flow of planar curves in a heterogeneous medium, which is modeled by a spatially-dependent additive forcing term. The evolution law reads:

$$v = (\kappa + g)\nu, \tag{1}$$

where ν is inward normal vector to the curve, κ is the curvature of the curve, v is the normal velocity vector, and $g \in L^\infty(\mathbb{R}^2)$ represents the forcing term.

The original motivation for our analysis comes from a homogenization problem related to the averaged behaviour of an interface moving by curvature plus a rapidly oscillating forcing term. More precisely, the evolution law is given by

$$v = \left(\kappa + g\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \right) \nu, \tag{2}$$

where g is a 1-periodic Lipschitz continuous function.

When the forcing term is periodic, Equation (1) was recently considered in [6], where the authors prove existence and uniqueness of planar pulsating waves in every direction of propagation. This result leads to the homogenization of (2) for plane-like initial data (see Section 3). Related results on the homogenization of interfaces moving with normal velocity given by

$$v = \left(\varepsilon\kappa + g\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \right) \nu,$$

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have been obtained in [5] and [12], under suitable assumptions on the forcing term including the fact that it does not change sign, and in [4] under more general assumptions. In particular, the authors show that the homogenized evolution law, when it exists, is a first order anisotropic geometric law of the form $v = \bar{c}(\nu)\nu$.

Coming back to our problem, as a first step we look for geometric estimates for solutions to (1), which depend only on the L^∞ -norm of g . In particular, reasoning as in the case of the unperturbed curvature flow [9, 2, 11], in Section 2 we classify all possible singularities which can arise during the evolution. As a consequence, in Section 2.5 we can show that, when g is smooth and the initial curve is embedded, the existence time of a regular solution to (1) is bounded below by a quantity depending only on $\|g\|_\infty$ and on the initial curve. Unfortunately, since we have no estimates on the curvature in terms of $\|g\|_\infty$, we are not able to obtain a general existence result for (1) in the nonsmooth case, i.e. when $g \in L^\infty$.

However, in Section 2.6 we overcome this difficulty by assuming that the initial curve is the graph of a function u , for instance in the vertical direction. In this case Equation (1) becomes

$$u_t = \frac{u_{xx}}{1 + u_x^2} + g(x, u(x))\sqrt{1 + u_x^2}. \quad (3)$$

In Lemma 2.18 we establish an L^p -estimate on u_x which depends only on $\|g\|_\infty$. In Proposition 2.21 we consider a sequence of smooth forcings g_n weakly converging to $g \in L^\infty$. Using the estimate on u_x and the results of the Section 2.5, and letting u_n be the solution corresponding to g_n , we can pass to the limit as $n \rightarrow \infty$ and obtain that $u_n \rightarrow u \in H^1([0, T], L^2([0, 1])) \cap L^\infty([0, T], H^1([0, 1]))$, for some time $T > 0$ depending only on $\|g\|_\infty$ and on the initial datum. When g does not depend on u , we obtain a stronger estimate on $\|u_t\|_\infty$, which allows us to show that $u \in W^{1,\infty}([0, T], L^\infty([0, 1])) \cap L^\infty([0, T], W^{2,\infty}([0, 1]))$.

As a first application, this leads to an existence and uniqueness result for solutions to (3), when g is a L^∞ -function which is independent of u (see Theorems 2.23 and 2.25).

The second application of our result is to the homogenization problem (2). In section 3, under the assumptions of Theorem 2.23, that is, when the curve is a graph and g is independent of the vertical direction, we can pass to the limit in (2) as $\varepsilon \rightarrow 0$, and show that the limit curve moves according to the evolution law

$$v = \left(\kappa + \int_{[0,1]^2} g(x, y) dx dy \right) \nu.$$

2 Local existence of solutions

In this section we are concerned with the local existence for (1), under the assumption that the forcing term g is smooth and bounded, i.e. $g \in C^\infty(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. If we parametrize counterclockwise the evolving curve with a function $\gamma = (\gamma^1, \gamma^2) : [0, 1] \times [0, T] \rightarrow \mathbb{R}^2$, with $\gamma(0, \cdot) = \gamma(1, \cdot)$, problem (1) becomes

$$\gamma_t = (\kappa + g)\nu = \frac{\gamma_{xx}^\perp}{|\gamma_x|^2} + g(\gamma) \frac{(-\gamma_x^2, \gamma_x^1)}{|\gamma_x|}, \quad (4)$$

where ξ^\perp denotes the component of the vector ξ orthogonal to γ_x . As usual we let τ, ν, κ be respectively the unit tangent vector, the unit normal vector and the curvature of the

evolving curve. Denoting by s the arclength parameter of the curve, so that $\partial_s = \partial_x/|\gamma_x|$, by the classical Frenet–Serret formulas we have

$$\gamma_s = \tau, \quad \gamma_{ss} = \tau_s = \kappa \nu, \quad \nu_s = -\kappa \tau. \quad (5)$$

We recall the following local in time existence result for (4), proved in [7, Thm 4.1].

Theorem 2.1. *Let $\gamma_0 : [0, 1] \rightarrow \mathbb{R}^2$ be a smooth map such that $\gamma_0(0) = \gamma_0(1)$ and $|\gamma_0'(x)| > 0$ for all $x \in [0, 1]$. Then there exist $T > 0$ and a smooth solution to (4), defined on $[0, 1] \times [0, T]$, such that $\gamma(x, 0) = \gamma_0(x)$ for all $x \in [0, 1]$.*

2.1 Estimates on the curvature and its derivatives

Lemma 2.2. *The following commutation rule holds:*

$$\partial_t \partial_s = \partial_s \partial_t + \kappa(\kappa + g) \partial_s. \quad (6)$$

Moreover,

$$\tau_t = (\kappa + g)_s \nu, \quad (7)$$

$$\nu_t = -(\kappa + g)_s \tau \quad (8)$$

$$\kappa_t = (\kappa + g)_{ss} + \kappa^2(\kappa + g). \quad (9)$$

Proof. By definition of arclength, we have

$$\partial_s = \frac{\partial_x}{|\gamma_x|}.$$

Therefore, from (4) and (5),

$$\begin{aligned} \partial_t \partial_s - \partial_s \partial_t &= -|\gamma_x|^{-3} \gamma_x \cdot \gamma_{xt} \partial_x \\ &= -|\gamma_x|^{-2} \tau \cdot \gamma_{xt} \partial_x \\ &= -|\gamma_x| \tau \cdot ((\kappa + g)_s \nu)_s \partial_x \\ &= -|\gamma_x| \tau \cdot (\kappa + g) \nu_s \partial_x \\ &= \kappa(\kappa + g) \partial_s, \end{aligned}$$

that is (6). Now, applying (6) to (4) and (5), we obtain

$$\begin{aligned} \tau_t &= (\gamma_s)_t = (\gamma_t)_s + \kappa(\kappa + g) \gamma_s \\ &= (\kappa + g)_s \nu + (\kappa + g) \nu_s + \kappa(\kappa + g) \tau = (\kappa + g)_s \nu, \end{aligned}$$

which is (7).

Also, since $|\nu| = 1$,

$$0 = \frac{(\nu \cdot \nu)_t}{2} = \nu \cdot \nu_t$$

and so, from (7),

$$\begin{aligned} \nu_t &= (\nu_t \cdot \nu) \nu + (\nu_t \cdot \tau) \tau = (\nu_t \cdot \tau) \tau \\ &= ((\nu \cdot \tau)_t - \nu \cdot \tau_t) \tau = -(\nu \cdot \tau_t) \tau = -(\kappa + g)_s \tau, \end{aligned}$$

that is (8), and

$$\begin{aligned}\kappa_t &= (\kappa\nu)_t \cdot \nu = (\tau_s)_t \cdot \nu = (\tau_t)_s \cdot \nu + \kappa(\kappa + g)\tau_s \cdot \nu \\ &= ((\kappa + g)_s\nu)_s \cdot \nu + \kappa^2(\kappa + g) = (\kappa + g)_{ss} + \kappa^2(\kappa + g),\end{aligned}$$

that is (9). \square

Let us compute the evolution for the spatial derivatives of the curvature. We denote by $p_{j,k}(\partial_s^\ell \kappa, \partial_s^m g)$ a generic polynomial depending on the derivatives up to order j of κ and the derivatives up to order k of g .

Lemma 2.3. *For all $j \in \mathbb{N}$, $j \geq 1$, it holds*

$$\partial_t \partial_s^j \kappa = (\partial_s^j \kappa)_{ss} + ((j+3)\kappa^2 + (j+2)\kappa g) \partial_s^j \kappa + p_{j-1, j+2}(\partial_s^\ell \kappa, \partial_s^m g). \quad (10)$$

Proof. The proof is by induction on j . When $j = 1$ from (6) and (9) we easily get

$$\partial_t \kappa_s = (\kappa_s)_{ss} + (4\kappa^2 + 3\kappa g)\kappa_s + (\kappa^2 g_s + g_{sss}).$$

Assume now (10) for some $j \in \mathbb{N}$. Using (6), we compute recursively

$$\begin{aligned}\partial_t \partial_s^{j+1} \kappa &= \partial_s \partial_t \partial_s^j \kappa + \kappa(\kappa + g) \partial_s^{j+1} \kappa \\ &= (\partial_s^{j+1} \kappa)_{ss} + ((j+3)\kappa^2 + (j+2)\kappa g + \kappa(\kappa + g)) \partial_s^{j+1} \kappa + p_{j, j+3}(\partial_s^\ell \kappa, \partial_s^m g),\end{aligned}$$

which gives (10) for all j . \square

We recall that in [7] similar equations as (10) for the evolution of the second fundamental form and its derivatives are obtained for forced mean curvature flow of hypersurfaces in any dimension.

We now compute the evolution equation of $w := \log |\gamma_x|$.

Lemma 2.4. *There holds*

$$w_t = -\kappa(\kappa + g). \quad (11)$$

Proof. A direct computation using (4) gives

$$w_t = \frac{\gamma_x \cdot \gamma_{xt}}{|\gamma_x|^2} = \tau \cdot (\partial_s \gamma_t) = -\kappa(\kappa + g). \quad \square$$

Lemma 2.5. *Assume that (4) admits a smooth solution on $[0, \bar{t}]$, with $\bar{t} > 0$. Then*

$$\max_{[0,1] \times [0, \bar{t}]} (\partial_s^j \kappa)^2 \leq C_j$$

for all $j \in \mathbb{N}$, where the constants C_j depend only on the initial curve, on \bar{t} , on $\max_{[0,1] \times [0, \bar{t}]} \kappa^2$ and on $\|g\|_{C^{j+2}}$.

Proof. Following [8], we let

$$K_j(x, t) := (\partial_s^j \kappa(x, t))^2 \quad M_j(t) := \max_{x \in [0,1]} K_j(x, t).$$

For all \bar{x} such that $K_j(\bar{x}, t) = M_j(t)$ we have

$$\begin{aligned}\partial_s K_j &= 0 \\ \partial_{ss} K_j &= 2\left((\partial_s^j \kappa)_s^2 + \partial_s^j \kappa (\partial_s^j \kappa)_{ss}\right) \leq 0\end{aligned}$$

and so

$$\partial_s^j \kappa(\bar{x}, t) (\partial_s^j \kappa)_{ss}(\bar{x}, t) \leq 0. \quad (12)$$

Recalling (10), for a.e. $t \in [0, \bar{t}]$ we have

$$\begin{aligned}\dot{M}_j(t) &= \max_{\bar{x}: K_j(\bar{x}, t) = M_j(t)} \partial_t K_j \\ &= \max_{\bar{x}: K_j(\bar{x}, t) = M_j(t)} 2\partial_s^j \kappa (\partial_s^j \kappa)_t \\ &= \max_{\bar{x}: K_j(\bar{x}, t) = M_j(t)} 2\partial_s^j \kappa \left((\partial_s^j \kappa)_{ss} + A_j \partial_s^j \kappa + B_j \right)\end{aligned}$$

where the constants A_j, B_j depend on M_ℓ and $\|g\|_{C^k}$, with $\ell < j$ and $k \leq j + 2$. Hence, using (12) we get

$$\dot{M}_j \leq 2A_j M_j + 2B_j.$$

By Gronwall's Lemma it then follows that the quantities M_j are uniformly bounded on $[0, \bar{t}]$. \square

Since the existence result in Theorem 2.1 is first established in the usual Hölder parabolic spaces $C^{k+\alpha, 2(k+\alpha)}([0, 1] \times [0, T])$ (see [14]), if we still denote by T the maximal existence time of the evolution, we have that, if $T < +\infty$, either $|\gamma_x|^{-1}$ or $|\partial_s^j \kappa|$ blow up as $t \rightarrow T$, for some $j \in \mathbb{N}$.

Proposition 2.6. *Let T be the maximal existence time of the evolution (4), and assume $T < +\infty$. Then*

$$\lim_{t \rightarrow T} \|\kappa^2\|_{L^\infty} = +\infty. \quad (13)$$

Proof. Assume by contradiction that κ^2 is uniformly bounded for all $t \in [0, T)$ and $x \in [0, 1]$. Equation (11) implies that $|\gamma_x|$ and $1/|\gamma_x|$ are also uniformly bounded on $[0, 1] \times [0, T)$. But Lemma 2.5 implies that also the quantities $(\partial_j \kappa(x, t))^2$ are uniformly bounded on $[0, 1] \times [0, T)$ for all $j \in \mathbb{N}$, thus reaching a contradiction. We then proved

$$\limsup_{t \rightarrow T} \|\kappa^2\|_{L^\infty} = +\infty.$$

Notice that the limsup is indeed a full limit due to (9). \square

The following Lemma provides a lower bound to (13).

Lemma 2.7. *Let T as above and assume $T < +\infty$. The following curvature lower bound holds:*

$$\liminf_{t \rightarrow T} \sqrt{T-t} \|\kappa\|_{L^\infty} \geq \frac{1}{\sqrt{2}}. \quad (14)$$

Proof. Notice that (9) can be written as

$$(\kappa + g)_t = (\kappa + g)_{ss} + (\kappa + g)\kappa^2 + (\kappa + g)\nabla g \cdot \nu. \quad (15)$$

Let $w := (\kappa + g)^2$. Observe that, for $\varepsilon > 0$, $\kappa^2 \leq (1 + \varepsilon)w + (1 + \frac{1}{\varepsilon})g^2$. So, from (15) it follows

$$\begin{aligned} w_t &= w_{ss} - 2(\kappa + g)_s^2 + 2w\kappa^2 + 2w\nabla g \cdot \nu \\ &\leq w_{ss} + 2w \left((1 + \varepsilon)w + \left(1 + \frac{1}{\varepsilon}\right)g^2 \right) + 2\|\nabla g\|_{L^\infty} w \\ &\leq w_{ss} + 2(1 + \varepsilon)w^2 + 2 \left(2 + \frac{1}{\varepsilon}\right) \|g\|_{W^{1,\infty}} w. \end{aligned}$$

Letting $M := \max_{x \in [0,1]} (\kappa + g)^2$, from (16) we get

$$\frac{d}{dt}(M + C) = \dot{M} \leq 2(1 + \varepsilon)M^2 + 2 \left(2 + \frac{1}{\varepsilon}\right) \|g\|_{W^{1,\infty}} M \leq 2(1 + \varepsilon)(M + C)^2, \quad (16)$$

where $C = [(1 + 1/(2\varepsilon))/(1 + \varepsilon)] \|g\|_{W^{1,\infty}}^2$, so that

$$-\frac{d}{dt} \frac{1}{M + C} \leq 2(1 + \varepsilon). \quad (17)$$

Integrating on $[t, s] \subset [0, T)$ we thus obtain

$$\frac{1}{M(t) + C} - \frac{1}{M(s) + C} \leq 2(1 + \varepsilon)(s - t).$$

Letting now $s \rightarrow T$ and recalling that $M(s) \rightarrow +\infty$ by Proposition 2.6, we get

$$\frac{1}{M(t) + C} \leq 2(1 + \varepsilon)(T - t),$$

that is

$$M(t) \geq \frac{1}{2(1 + \varepsilon)(T - t)} - C,$$

which gives the thesis. \square

From (17) and Proposition 2.6 we obtain the following estimate on the maximal existence time of the evolution.

Proposition 2.8. *Let T be the maximal existence time of (4), then*

$$T \geq c(\|\kappa_0\|_{L^\infty}, \|g\|_{W^{1,\infty}}).$$

Remark 2.9. Note that, differently from the case of curve shortening flow, in our case, due to the presence of the forcing term g , self-intersections may arise. Nevertheless if the initial curve is embedded then, thanks to Proposition 2.8, it remains embedded in a time interval $[0, T']$, with $T' > 0$ depending only on the initial datum and on $\|g\|_{W^{1,\infty}}$.

We think it is an interesting problem to determine whether or not the constant c in Proposition 2.8 depends only on the initial set and on the L^∞ -norm of g (see for instance Section 2.5 below for a special case).

2.2 Huisken's monotonicity formula

In the following we derive a monotonicity formula for curvature flow with a forcing term, and apply it to the analysis of singularities.

By a standard computation, using the fact that γ solves (4), we get the following formula.

Lemma 2.10. *Let $\tau > 0$ and let $f : \mathbb{R}^2 \times [0, \tau) \rightarrow \mathbb{R}$ be a smooth function. Then*

$$\frac{d}{dt} \int_{\gamma} f(\gamma(x(s), t), t) ds = \int_{\gamma} [f_t - \kappa(\kappa + g)f + (\kappa + g)\nabla f \cdot \nu] ds. \quad (18)$$

We denote by $L_t(\gamma)$ the length of the curve $\gamma([0, 1], t)$, that is

$$L_t(\gamma) := \int_0^1 |\gamma_x| dx = \int_{\gamma} ds.$$

When no confusion can arise, we write $L(\gamma)$ instead of $L_t(\gamma)$.

Corollary 2.11. *Let $\gamma : [0, 1] \times [0, T] \rightarrow \mathbb{R}^2$ be a solution to (4). Then*

$$L_t(\gamma) \leq L_0(\gamma) e^{\frac{\|g\|_{\infty}^2}{2} t} \quad \forall t \in [0, T]. \quad (19)$$

Proof. Taking $f \equiv 1$ in (18) we have

$$\frac{\partial}{\partial t} L_t(\gamma) = - \int_{\gamma} \kappa(\kappa + g) ds \leq \int_{\gamma} -\frac{\kappa^2}{2} + \frac{g^2}{2} ds, \quad (20)$$

which gives (19) by Gronwall's Lemma. \square

We now apply Lemma 2.10 with $f(p, t) = \frac{e^{-|p-p_0|^2/4(T-t)}}{\sqrt{4\pi(T-t)}}$, $p, p_0 \in \mathbb{R}^2$, and we get

$$\begin{aligned} & \frac{d}{dt} \int_{\gamma} \frac{e^{-|\gamma(x(s), t) - p_0|^2/4(T-t)}}{\sqrt{4\pi(T-t)}} ds \\ &= - \int_{\gamma} \frac{e^{-|\gamma - p_0|^2/4(T-t)}}{\sqrt{4\pi(T-t)}} \left[\frac{|\gamma - p_0|^2}{4(T-t)^2} - \frac{1}{2(T-t)} + \kappa(\kappa + g) + (\kappa + g) \frac{(\gamma - p_0) \cdot \nu}{2(T-t)} \right] ds \\ &= - \int_{\gamma} \frac{e^{-|\gamma - p_0|^2/4(T-t)}}{\sqrt{4\pi(T-t)}} \left[\frac{\gamma - p_0}{2(T-t)} + \left(\kappa + \frac{g}{2} \right) \nu \right]^2 ds + \frac{1}{4} \int_{\gamma} \frac{e^{-|\gamma - p_0|^2/4(T-t)}}{\sqrt{4\pi(T-t)}} g^2 ds \\ & \quad + \int_{\gamma} \frac{e^{-|\gamma - p_0|^2/4(T-t)}}{\sqrt{4\pi(T-t)}} \left[\frac{1}{2(T-t)} + \kappa \frac{(\gamma - p_0) \cdot \nu}{2(T-t)} \right] ds. \end{aligned} \quad (21)$$

Following [9, Theorem 3.1], the last term can be actually written as

$$\begin{aligned} & \int_{\gamma} \frac{e^{-|\gamma - p_0|^2/4(T-t)}}{\sqrt{4\pi(T-t)}} \left[\frac{1}{2(T-t)} + \kappa \frac{(\gamma - p_0) \cdot \nu}{2(T-t)} \right] ds \\ &= \int_{\gamma} \frac{e^{-|\gamma - p_0|^2/4(T-t)}}{\sqrt{4\pi(T-t)}} \left| \frac{(\gamma - p_0) \cdot \nu}{2(T-t)} \right|^2 ds. \end{aligned}$$

Substituting this in (21) we obtain an analog of Huisken's monotonicity formula ([9])

$$\begin{aligned} & \frac{d}{dt} \int_{\gamma} \frac{e^{-|\gamma(x(s),t)-p_0|^2/4(T-t)}}{\sqrt{4\pi(T-t)}} ds \\ &= \int_{\gamma} \frac{e^{-|\gamma(x(s),t)-p_0|^2/4(T-t)}}{\sqrt{4\pi(T-t)}} \left(- \left[\kappa + \frac{(\gamma(x(s),t)-p_0) \cdot \nu}{2(T-t)} + \frac{g}{2} \right]^2 + \frac{1}{4}g^2 \right) ds. \end{aligned} \quad (22)$$

In the next paragraph we will apply this formula in the analysis of type I singularities.

2.3 Type I singularities

We assume that at time T the flow is developing a singularity of type I, i.e. there exists a constant $C_0 > 1$ such that

$$\max_{x \in [0,1]} |\kappa(x,t)| \leq \frac{C_0}{\sqrt{2(T-t)}}. \quad (23)$$

Observe that for every x and $0 \leq t \leq r < T$

$$|\gamma(x,r) - \gamma(x,t)| \leq C_0 \sqrt{2(T-t)} - C_0 \sqrt{2(T-r)} + \|g\|_{\infty}(r-t). \quad (24)$$

This implies that the functions $\gamma(\cdot, t)$ converge uniformly to a function γ_T as $t \rightarrow T$. Now we fix $x \in [0, 1]$ such that $\gamma(x, t) \rightarrow \gamma_T(x) =: \hat{p}$ and $\kappa(x, t)$ becomes unbounded as $t \rightarrow T$. We rescale the curve around the point \hat{p} as follows:

$$\tilde{\gamma}(x, z) := \frac{\gamma(x, t(z)) - \hat{p}}{\sqrt{2(T-t(z))}} \quad z(t) := -\log \sqrt{T-t}.$$

From (24), we deduce

$$|\tilde{\gamma}(x, z)| = \left| \frac{\gamma(x, t(z)) - \hat{p}}{\sqrt{2(T-t(z))}} \right| \leq C_0 + \frac{1}{\sqrt{2}} \|g\|_{\infty} e^{-z} \leq C_0 + \frac{1}{\sqrt{2}} \|g\|_{\infty},$$

so in particular $\tilde{\gamma}(x, z)$ remains bounded as $z \rightarrow +\infty$. The evolution law satisfied by the rescaled curve $\tilde{\gamma}$ is

$$\tilde{\gamma}_z = (\tilde{\kappa} + \sqrt{2}e^{-z}g)\tilde{\nu} + \tilde{\gamma} \quad z \in [-\log \sqrt{T}, +\infty). \quad (25)$$

We also have the rescaled version of the monotonicity formula (22): letting $y = \frac{x-\hat{p}}{\sqrt{2(T-t)}}$, we compute

$$\begin{aligned} & \frac{d}{dz} \int_{\tilde{\gamma}} e^{-|\tilde{\gamma}(x(\tilde{s}),z)|^2/2} d\tilde{s} = 2(T-t) \frac{d}{dt} \int_{\gamma} \frac{e^{-|\gamma(x(s),t)-\hat{p}|^2/4(T-t)}}{\sqrt{2(T-t)}} ds \\ &= \int_{\tilde{\gamma}} e^{-|\tilde{\gamma}(x(\tilde{s}),z)|^2/2} \left(- \left[\tilde{\kappa} + \langle \tilde{\gamma}(x(\tilde{s}),z) | \tilde{\nu} \rangle + e^{-z} \frac{g}{\sqrt{2}} \right]^2 + \frac{e^{-2z}}{2} g^2 \right) d\tilde{s}. \end{aligned} \quad (26)$$

Letting $F(z) := \int_{\tilde{\gamma}} e^{-|\tilde{\gamma}(x(\tilde{s}), z)|^2/2} d\tilde{s}$, Equation (26) gives

$$\frac{d}{dz} F(z) \leq \frac{\|g\|_\infty^2}{2} e^{-2z} F(z) \leq \frac{\|g\|_\infty^2}{2} e^{-2z}. \quad (27)$$

Integrating (27) we obtain

$$F(z) \leq e^{\|g\|_\infty^2 T/4} F(-\log \sqrt{T}) \quad \forall z \geq -\log \sqrt{T}.$$

In particular, we deduce that for every $R > 0$ there exists a uniform bound on $\mathcal{H}^1(\tilde{\gamma}([0, 1], z) \cap B(0, R))$, where \mathcal{H}^1 denotes the 1 dimensional Hausdorff measure. Indeed

$$\begin{aligned} \mathcal{H}^1(\tilde{\gamma}([0, 1], z) \cap B(0, R)) &= \int_{\tilde{\gamma}} \chi_{B(0, R)} d\tilde{s} \leq \int_{\tilde{\gamma}} \chi_{B(0, R)} e^{\frac{R^2 - |\tilde{\gamma}(x, z)|^2}{2}} d\tilde{s} \\ &\leq e^{\frac{|R|^2}{2}} F(z) \leq K, \end{aligned} \quad (28)$$

for some positive constant K .

Proposition 2.12. *Under Assumption (23), for each sequence $z_j \rightarrow +\infty$ there exists a subsequence z_{j_k} such that the curve $\tilde{\gamma}(\cdot, z_{j_k})$, rescaled around \hat{p} , locally smoothly converges to some smooth, nonflat limit curve $\tilde{\gamma}_\infty$, such that*

$$\tilde{\kappa} + \langle \tilde{\gamma}_\infty | \tilde{\nu} \rangle = 0. \quad (29)$$

Proof. The proof follows the same argument as in [9, Proposition 3.4]. Indeed, the limit curve is smooth thanks to (28), Proposition 2.6 and the fact that the rescaled curve $\tilde{\gamma}$ has uniformly bounded curvature. Moreover, it is nonflat by (14). Finally, the limit curve satisfies (29) thanks to (26) and (27). \square

Remark 2.13. Proposition 2.12 implies that the type I singularities of (1) are modeled by homothetic solutions of the flow, as for the spatially homogeneous case [9]. We recall that, among such solutions, the circle is the only embedded one [1], hence, under Assumption (23), T is actually the extinction time for the evolution. From this we can conclude that

$$T \geq c(\|\kappa_0\|_{L^\infty}, \|g\|_{L^\infty}).$$

2.4 Type II singularities

We consider now the case that at time T the flow is developing a singularity of type II, i.e.

$$\limsup_{t \rightarrow T} \max_{x \in [0, 1]} |\kappa(x, t)| \sqrt{T - t} = +\infty. \quad (30)$$

Proposition 2.14. *Under condition (30), there exists a sequence of points and times (x_n, t_n) on which the curvature blows up such that the rescaled curve along this sequence converges in \mathcal{C}^∞ to a planar, convex limiting solution, which moves by translation.*

Proof. By means of (9) and (19), an easy calculation implies that

$$\frac{d}{dt} \int_{\gamma} |\kappa| ds = -2 \sum_{x: \kappa(x,t)=0} |\kappa_s| + \int_{\gamma} \frac{\kappa}{|\kappa|} g_{ss} ds \leq \|\nabla^2 g\|_{\infty} L_0(\gamma) e^{\frac{\|g\|_{\infty}^2 t}{2}} + \|\nabla g\|_{\infty} \int_{\gamma} |\kappa| ds. \quad (31)$$

From (31), using Gronwall lemma, we obtain that $t \rightarrow \int_{\gamma} |\kappa| ds$ is uniformly bounded in $[0, T]$ and admits a bounded limit as $t \rightarrow T^-$.

Following [2] we choose a sequence (x_n, t_n) such that

- $t_n \in [0, T - \frac{1}{n})$ and $t_n < t_{n+1}$;
- $k_n = |\kappa(x_n, t_n)| \rightarrow +\infty$ and

$$k_n \sqrt{T - \frac{1}{n} - t_n} = \max_{t \in [0, T-1/n]} \left(\|\kappa\|_{\infty} \sqrt{T - \frac{1}{n} - t} \right) \rightarrow +\infty \quad \text{as } n \rightarrow +\infty. \quad (32)$$

We define the new parameter u as follows $u = k_n^2(t - t_n)$, $u \in [-k_n^2 t_n, k_n^2(T - t_n)]$ and the rescaled curve along the sequence (x_n, t_n) as $\gamma_n(x, u) = k_n(\gamma(x, t(u)) - \gamma(x_n, t_n))$, for $x \in [0, 1]$. Observe that $\gamma_n(x_n, 0) = (0, 0)$ and $\kappa_{\gamma_n}(x_n, 0) = 1$; moreover

$$v_{\gamma_n} = \frac{d}{du} \gamma_n = \frac{(\kappa + g(\gamma))\nu}{k_n} = (\kappa_{\gamma_n} + g_n)\nu_{\gamma_n} \quad (33)$$

where $g_n(y) = g(y/k_n + \gamma(x_n, t_n))/k_n$.

In the sequel, we shall write for simplicity κ_n instead of κ_{γ_n} . Note that for every $\varepsilon, \omega > 0$, there exists \bar{n} such that $\kappa_n^2 \leq 1 + \varepsilon$ for $u \in [-k_n^2 t_n, \omega]$. Indeed, using (32), we get

$$\kappa_n^2(x, u) = \frac{\kappa^2(x, t(u))}{k_n^2} \leq \frac{T - 1/n - t_n}{T - 1/n - t(u)} = \frac{T - 1/n - t_n}{T - 1/n - t_n - u/k_n^2}.$$

This implies that, on every bounded interval of time, the curvatures of the rescaled curves are uniformly bounded. Moreover, from this, we deduce uniform bounds also on the derivatives of the curvature, using Lemma 2.5 and recalling that γ_n satisfies (33) and the fact that $\|\nabla^j g_n\|_{\infty} = \|\nabla^j g\|_{\infty}/k_n^{j+1} \rightarrow 0$. By the same argument of [2, Theorem 7.3], this implies that there exists a subsequence along which the rescaled curves converge smoothly to a smooth, non trivial limit γ_{∞} defined in $(-\infty, +\infty)$. Moreover γ_{∞} evolves by mean curvature flow, $L_t(\gamma_{\infty}) = +\infty$ and $\|\kappa_{\infty}\|_{\infty} = 1 = |\kappa_{\infty}(0, 0)|$.

We prove now that γ_{∞} is convex.

Recall that from (31) we get that $t \rightarrow \int_{\gamma} |\kappa|$ is uniformly bounded and admits a limit as $t \rightarrow T^-$. The same also holds for $t \rightarrow \int_{\gamma_n} |\kappa_n|$. Moreover, from (31) we also obtain

$$\frac{d}{du} \int_{\gamma_n} |\kappa_n| ds = -2 \sum_{\kappa_n(x,u)=0} |(\kappa_n)_s| + \int_{\gamma_n} \frac{\kappa_n}{|\kappa_n|} (g_n)_{ss} ds.$$

So

$$\begin{aligned} -2 \int_{-M}^M \sum_{\kappa_n=0} |(\kappa_n)_s| du &= \int_{\gamma_n} (|\kappa_n(x, M)| - |\kappa_n(x, -M)|) ds - \int_{-M}^M \int_{\gamma_n} \frac{\kappa_n}{|\kappa_n|} (g_n)_{ss} ds du \\ &= \int_{\gamma} \left(\left| \kappa \left(x, t_n + \frac{M}{k_n^2} \right) \right| - \left| \kappa \left(x, t_n - \frac{M}{k_n^2} \right) \right| \right) ds - \int_{-M}^M \int_{\gamma_n} \frac{\kappa_n}{|\kappa_n|} (g_n)_{ss} ds du. \end{aligned}$$

Letting $n \rightarrow +\infty$ along the subsequence on which $\gamma_n \rightarrow \gamma_\infty$, we get

$$\int_\gamma \left| \kappa \left(x, t_n + \frac{M}{k_n^2} \right) \right| ds - \int_\gamma \left| \kappa \left(x, t_n - \frac{M}{k_n^2} \right) \right| ds \rightarrow 0.$$

We argue as in (31), using the definition of g_n and the fact that, by (19), $L_t(\gamma_n) \leq k_n K$ for some constant K just depending on $\|g\|_\infty$ and T ,

$$\left| \int_{-M}^M \int_{\gamma_n} \frac{\kappa_n}{|\kappa_n|} (g_n)_{ss} ds du \right| \leq \int_{-M}^M \frac{\|\nabla^2 g\|_\infty}{k_n^3} L_u(\gamma_n) + \frac{\|\nabla g\|_\infty}{k_n^2} \left(\int_{\gamma_n} |\kappa_n| ds \right) du \leq \frac{C}{k_n^2} \rightarrow 0$$

as $n \rightarrow +\infty$. In particular, this gives

$$-2 \int_{-M}^M \sum_{x, \kappa_n(x, u)=0} |(\kappa_n)_s| du \rightarrow 0 \quad \text{as } n \rightarrow +\infty,$$

and we can conclude as in [2, Theorem 7.7] that γ_∞ is a convex eternal solution to the curvature flow, that is, γ_∞ is the so-called *Grim Reaper*. \square

2.5 The embedded case

In this section we strengthen Proposition 2.8 in the case of embedded planar curves.

Following [10] we define

$$\eta(t) := \inf_{x < y} \frac{|\gamma(x, t) - \gamma(y, t)|}{L_{x, y}(t)}, \quad (34)$$

where

$$L_{x, y}(t) := \min \left(\int_x^y |\gamma_x(\sigma, t)| d\sigma, \int_0^x |\gamma_x(\sigma, t)| d\sigma + \int_y^1 |\gamma_x(\sigma, t)| d\sigma \right).$$

Notice that the infimum in (34) is in fact a minimum, moreover η is a continuous function in $[0, T)$, where T is the first singularity time. Since the initial curve is embedded, we also have $\eta(0) > 0$. Let now

$$\mathcal{E}(t) := \left\{ (x, y) : x < y \text{ and } \eta(t) = \frac{|\gamma(x, t) - \gamma(y, t)|}{L_{x, y}(t)} \right\}.$$

In the following we assume for simplicity that $L_{x, y}(t) = \int_x^y |\gamma_x(\sigma, t)| d\sigma$, the other case being analogous. Notice that, if $\eta(t) < \sqrt{2}/2$, we have the estimate

$$\int_x^y |\kappa| |\gamma_x(\sigma, t)| d\sigma \geq \max_{z, w \in [x, y]} |\tau(z, t) - \tau(w, t)| > \frac{\pi}{2} \quad (35)$$

for all $(x, y) \in \mathcal{E}(t)$. Indeed, if $|\tau(z, t) - \tau(w, t)| \leq \frac{\pi}{2}$ for all $z, w \in [x, y]$, then $\gamma([x, y])$ is the graph of a 1-Lipschitz function, which in turn implies $\eta(t) \geq \sqrt{2}/2$. Letting $c := \pi/2$, from

(35) we get

$$\begin{aligned} \int_x^y \kappa(\kappa + g)|\gamma_x(\sigma, t)| d\sigma &\geq \frac{1}{L_{x,y}} \left(\int_x^y |\kappa|\gamma_x(\sigma, t)| d\sigma \right)^2 - \|g\|_\infty \int_x^y |\kappa|\gamma_x(\sigma, t)| d\sigma \\ &\geq c \left(\frac{c}{L_{x,y}} - \|g\|_\infty \right) \geq 0 \end{aligned} \quad (36)$$

whenever $L_{x,y} \leq c/\|g\|_\infty$. Moreover, reasoning as in [10], from the minimality condition it follows that

$$(\kappa(x)\nu(x) - \kappa(y)\nu(y)) \cdot (x - y) \geq 0, \quad (37)$$

for all $(x, y) \in \mathcal{E}(t)$. When $\eta(t) < \sqrt{2}/2$, using (36), (37) and the so called Hamilton's trick (see [15]) we compute

$$\begin{aligned} \dot{\eta}(t) &= \min_{(x,y) \in \mathcal{E}(t)} \frac{1}{L_{x,y}} \left[[(\kappa(x) + g(x))\nu(x) - (\kappa(y) + g(y))\nu(y)] \cdot \frac{(x-y)}{|x-y|} + \eta \int_x^y \kappa(\kappa + g)|\gamma_x| d\sigma \right] \\ &\geq \min_{(x,y) \in \mathcal{E}(t)} \left[-\frac{2\|g\|_\infty}{L_{x,y}} + \frac{c\eta}{L_{x,y}} \left(\frac{c}{L_{x,y}} - \|g\|_\infty \right) \right] \\ &\geq \min_{(x,y) \in \mathcal{E}(t)} \left[-\left(2 + \frac{\pi\sqrt{2}}{4} \right) \frac{\|g\|_\infty}{L_{x,y}} + \frac{\pi^2}{4L_{x,y}^2} \eta \right]. \end{aligned} \quad (38)$$

Theorem 2.15. *Let γ_0 be an embedding and let T be the maximal existence time of (4). Then*

$$T \geq c(\gamma_0, \|g\|_\infty). \quad (39)$$

Proof. Remark 2.13 assures that the statement is true if the evolution develops a type I singularity at $t = T$. Now, we can assume that the evolution develops a type II singularity at $t = T$. In particular it follows that $\eta(T) = 0$. Let $\tau := \sup\{t \in [0, T] : \eta(s) > 0 \text{ on } [0, t]\}$. Notice that $\tau > 0$ due to the fact that γ_0 is an embedding.

The thesis will follow if we show that τ is bounded below by a constant depending only on γ_0 and $\|g\|_\infty$. Since $\eta(\tau) = 0$, we can find $0 \leq t_1 < t_2 \leq \tau$ such that

$$\eta(t_1) = \bar{\eta} := \min \left(\eta(0), \frac{\sqrt{2}}{2} \right) \quad \eta(t_2) = \frac{\bar{\eta}}{2} \quad \eta(t) \in \left(\frac{\bar{\eta}}{2}, \bar{\eta} \right) \text{ for all } t \in (t_1, t_2).$$

In particular, letting $a := (2 + \pi\sqrt{2}/4)\|g\|_\infty$ and $b := \pi^2/4$, from (38) we have

$$\dot{\eta} \geq -\frac{a}{L_{x,y}} + \frac{b}{L_{x,y}^2} \eta \geq -\frac{a}{L_{x,y}} + \frac{b\bar{\eta}}{2L_{x,y}^2} \geq -\frac{a^2}{2b\bar{\eta}},$$

which implies

$$\tau \geq t_2 - t_1 \geq \frac{b\bar{\eta}^2}{a^2} = \frac{2\bar{\eta}^2}{\left(1 + \frac{4\sqrt{2}}{\pi}\right)^2 \|g\|_\infty^2}. \quad \square$$

2.6 The graph case

We assume now that the curve can be parametrized as $\gamma(x, t) = (x, u(x, t))$, $x \in [0, 1]$ with the following periodic-type boundary conditions:

$$\begin{aligned} u(0, t) - u(0, 0) &= u(1, t) - u(1, 0) \\ u_x(0, t) &= u_x(1, t) \end{aligned} \quad (40)$$

for all $t \in [0, T]$. Notice that γ is not a closed curve, however it can be extended to a periodic infinite curve, that is, it is a closed curve on a suitable flat torus, and all the results of the previous sections also apply to this case. In this parametrization, Equation (4) becomes

$$u_t = \frac{u_{xx}}{1 + u_x^2} + g(x, u(x))\sqrt{1 + u_x^2}. \quad (41)$$

We say that γ solves (41) if $\gamma(x, t) = (x, u(x, t))$, where the function u solves (41).

Let us recall the following interpolation inequalities [16].

Proposition 2.16. *Let $u \in H^1([0, 1]) \cap L^p([0, 1])$, with $p \in [2, +\infty]$. We have*

$$\|u\|_{L^p} \leq C_p \|u_x\|_{L^2}^{\frac{p-2}{2p}} \|u\|_{L^2}^{\frac{p+2}{2p}} + B_p \|u\|_{L^2}. \quad (42)$$

where the constants C_p, B_p depend only on p .

The following result can be easily derived from Proposition 2.16 (see [3]). We recall that $L^p(\gamma)$ is the intrinsic L^p space on the curve γ [3].

Proposition 2.17. *Let z be a smooth function defined on the support of γ , where γ is a \mathcal{C}^1 curve, and let $p \in [2, +\infty]$. We have*

$$\|z\|_{L^p(\gamma)} \leq C_p \|z_s\|_{L^2(\gamma)}^{\frac{p-2}{2p}} \|z\|_{L^2(\gamma)}^{\frac{p+2}{2p}} + B_p \|z\|_{L^2(\gamma)}. \quad (43)$$

where the constants C_p, B_p depend on p but are independent of γ .

In particular, choosing $p = 4$, (43) becomes

$$\int_{\gamma} z^4 ds = \|z\|_{L^4(\gamma)}^4 \leq C \left(\|z_s\|_{L^2(\gamma)} \|z\|_{L^2(\gamma)}^3 + \|z\|_{L^2(\gamma)}^4 \right). \quad (44)$$

Lemma 2.18. *Let u be a smooth solution of (40)-(41), and let*

$$F(x) := \int_0^x \arctan(t) dt = x \arctan(x) - \log \sqrt{1 + x^2}.$$

We have

$$\partial_t \int_0^1 \sqrt{1 + u_x^2} dx \leq C \int_0^1 \sqrt{1 + u_x^2} dx \quad (45)$$

$$\partial_t \int_0^1 F(u_x) dx \leq C \int_0^1 (1 + u_x^2) dx \quad (46)$$

$$\partial_t \int_0^1 \left(\sqrt{1 + u_x^2} \right)^3 dx \leq C + C \left(\int_0^1 \left(\sqrt{1 + u_x^2} \right)^3 dx \right)^3, \quad (47)$$

where the constants $C > 0$ depend only on $\|g\|_{L^\infty}$.

Proof. Inequality (45) can be obtained exactly as (20).

In order to show (46), we compute

$$\begin{aligned}
\partial_t \int_0^1 F(u_x) dx &= \int_0^1 -u_t (\arctan u_x)_x dx \\
&= \int_0^1 -u_t^2 + g u_t \sqrt{1 + u_x^2} dx \\
&\leq \int_0^1 \frac{g^2}{4} (1 + u_x^2) dx
\end{aligned} \tag{48}$$

which leads to (46).

We now prove (47). Letting $e_1 = (1, 0) \in \mathbb{R}^2$ and $z := 1/(\tau \cdot e_1) = \sqrt{1 + u_x^2}$, from (7) we get

$$z_t = -(\kappa + g)_s z^2 \nu \cdot e_1. \tag{49}$$

We compute

$$\begin{aligned}
\partial_t \int_\gamma z^2 ds &= \int_\gamma 2z z_t - \kappa(\kappa + g) z^2 ds = \int_\gamma -2z^3(\kappa + g)_s \nu \cdot e_1 - \kappa(\kappa + g) z^2 ds \\
&= \int_\gamma (\kappa + g) (-3\kappa z^2 + 6z^2 z_s \nu \cdot e_1) ds = 3 \int_\gamma (\kappa + g) z_s \frac{1 + 2z^2(\nu \cdot e_1)^2}{\nu \cdot e_1} ds \\
&= 3 \int_\gamma -\frac{2z^2 - 1}{z^2 - 1} z_s^2 + g z_s \frac{2z^2 - 1}{\nu \cdot e_1} ds \\
&\leq 3 \int_\gamma -\frac{2z^2 - 1}{z^2 - 1} z_s^2 + (g z \sqrt{2z^2 - 1}) \left(\sqrt{\frac{2z^2 - 1}{z^2 - 1}} z_s \right) ds \\
&\leq 3 \int_\gamma -z_s^2 + \frac{\|g\|_\infty^2}{2} z^2 (2z^2 - 1) ds \leq 3 \int_\gamma -z_s^2 + \|g\|_\infty^2 z^4 ds \\
&\leq -3 \|z_s\|_{L^2(\gamma)}^2 + C \|g\|_\infty^2 \left(\|z_s\|_{L^2(\gamma)} \|z\|_{L^2(\gamma)}^3 + \|z\|_{L^2(\gamma)}^4 \right) \\
&\leq C \|g\|_\infty^4 \left(\int_\gamma z^2 ds \right)^3 + C \|g\|_\infty^2 \left(\int_\gamma z^2 ds \right)^2,
\end{aligned}$$

where we used (44) to estimate $\|z\|_{L^4}$. \square

Proposition 2.19. *Let $g(x, y) \in C^\infty([0, 1]^2)$, and let $u_0 \in C^\infty([0, 1])$, with $u_{0x}(0) = u_{0x}(1)$. Then, there exists $T > 0$ depending only on $\|u_0\|_{W^{1,\infty}}$ and $\|g\|_{L^\infty}$ such that Equations (40)-(41) admit a smooth solution $u \in C^\infty([0, 1] \times [0, T])$.*

Moreover $\|u(t, \cdot)\|_{H^1([0,1])} \leq K$ for every $t \in [0, T]$, where K depends only on $\|u_0\|_{W^{1,\infty}}$ and $\|g\|_{L^\infty}$.

Proof. By standard parabolic regularity theory [14], it is enough to show that the gradient u_x remains bounded for a time T as above. From (47) we get that there exists $T_1 > 0$, depending only on $\|u_0\|_{W^{1,\infty}}$ and $\|g\|_{L^\infty}$, such that $u_x(\cdot, t) \in L^3([0, 1])$ for all $t \in [0, T_1]$. Moreover, by Theorem 2.15 we also have $\kappa = u_{xx}(1 + u_x^2)^{-3/2} \in L^\infty([0, 1] \times [0, T_2])$, for some $T_2 > 0$

depending only on $\|u_0\|_{W^{1,\infty}}$ and $\|g\|_{L^\infty}$. As a consequence we get $u_x \in L^\infty([0, 1] \times [0, T])$ with $T = \min(T_1, T_2)$. \square

Lemma 2.20. *We have the continuous embedding*

$$H^1([0, T], L^2([0, 1])) \cap L^\infty([0, T], H^1([0, 1])) \hookrightarrow C^{\frac{1}{2}, \frac{1}{4}}([0, 1] \times [0, T]).$$

Proof. Let $u \in H^1([0, T], L^2([0, 1])) \cap L^\infty([0, T], H^1([0, 1]))$, and let $(x, t), (y, s) \in [0, 1] \times [0, T]$, with $x < y, t < s$. Since $u \in L^\infty([0, T], H^1([0, 1]))$, we have

$$|u(x, t) - u(y, t)| \leq \int_x^y |u_x| d\sigma \leq C\sqrt{x-y}. \quad (50)$$

Moreover, being also $u \in H^1([0, T], L^2([0, 1]))$, we have

$$\|u(\cdot, t) - u(\cdot, s)\|_{L^2}^2 = \int_0^1 |u(x, t) - u(x, s)|^2 dx \leq (s-t) \int_{[0,1] \times [0,T]} u_t^2 dx dt \leq C(s-t).$$

By (42) with $p = \infty$, this implies

$$\begin{aligned} \|u(\cdot, t) - u(\cdot, s)\|_{L^\infty} &\leq C \left(\|u_x(\cdot, t) - u_x(\cdot, s)\|_{L^2}^{\frac{1}{2}} \|u(\cdot, t) - u(\cdot, s)\|_{L^2}^{\frac{1}{2}} + \|u(\cdot, t) - u(\cdot, s)\|_{L^2} \right) \\ &\leq C(s-t)^{\frac{1}{4}}. \end{aligned} \quad (51)$$

The thesis follows from (50) and (51). \square

Proposition 2.21. *Let $u_0 \in W^{1,\infty}([0, 1])$, let $g_n \in C^\infty([0, 1]^2) \cap L^\infty([0, 1]^2)$, with $\|g_n\|_{L^\infty} \leq C$ for every n , and let $u_{0n} \in C^\infty([0, 1])$ be such that $\|u_{0n}\|_{W^{1,\infty}} \leq L$ for every n and u_{0n} converge to u_0 uniformly on $[0, 1]$.*

Then, letting $u_n \in C^\infty([0, 1] \times [0, T])$ be the solutions of (40)-(41) given by Proposition 2.19, with $g = g_n$ and with initial data u_{0n} , there exists $u \in H^1([0, T], L^2([0, 1])) \cap L^\infty([0, T], H^1([0, 1]))$ such that, up to a subsequence, $u_n \rightarrow u$ uniformly on $[0, 1] \times [0, T]$.

Proof. By Proposition 2.19 there exist $T > 0$, depending only on C and L , such that the solutions u_n are uniformly bounded in $L^\infty([0, T], H^1([0, 1]))$. Moreover, using the equality for (48), we obtain

$$\int_0^1 \frac{(u_n)_t^2}{2} dx \leq \frac{\|g_n\|_\infty^2}{2} \int_0^1 (1 + u_{nx}^2) dx - \partial_t \int_0^1 F((u_n)_x) dx$$

and integrating it in time we also get a uniform bound of u_n in $H^1([0, T], L^2([0, 1]))$. It then follows that the sequence u_n converges, up to a subsequence as $n \rightarrow +\infty$, to a limit function u in the weak topology of $H^1([0, T], L^2([0, 1])) \cap L^\infty([0, T], H^1([0, 1]))$. By Lemma 2.20, u_n are uniformly Hölder continuous and then we conclude by Arzelà-Ascoli theorem that, along a subsequence, $u_n \rightarrow u$ uniformly on $[0, 1] \times [0, T]$. \square

We are interested in studying solutions of (41) when g is only a L^∞ -function. We consider the simpler case in which g is independent of u , i.e. $g(x, y) = g(x)$. In this case we define the following notion of weak solution.

Definition 2.22. We say that a function $u \in H^1([0, T], L^2([0, 1])) \cap L^\infty([0, T], H^1([0, 1]))$ is a weak solution of (41) if

$$\int_{[0,1] \times [0,T]} \left(u_t \varphi + \arctan(u_x) \varphi_x - g(x) \sqrt{1+u_x^2} \varphi \right) dx dt = 0 \quad (52)$$

for all test functions $\varphi \in C_c^1([0, 1] \times (0, T))$, with periodic boundary conditions.

We have the following existence theorem for weak solutions to (41).

Theorem 2.23. Let $g(x, y) = g(x)$, with $g \in L^\infty([0, 1])$, and let $u_0 \in W^{2,\infty}([0, 1])$ satisfying $u_{0x}(0) = u_{0x}(1)$. Then, there exists $T > 0$ depending only on u_0 and $\|g\|_\infty$ such that Equation (41) admits a weak solution $u \in W^{1,\infty}([0, T], L^\infty([0, 1])) \cap L^\infty([0, T], W^{2,\infty}([0, 1]))$ with initial datum u_0 .

Proof. Let $g_n \in C^\infty([0, 1])$ be a sequence of smooth functions which converge to g weakly* in $L^\infty([0, 1])$. By Propositions 2.19 and 2.21 there exist $T > 0$, depending only on $\|u_0\|_{H^1}$ and $\|g\|_{L^\infty}$, and smooth solutions u_n of (41) which converge, up to a subsequence, to a limit function u in uniformly and in the weak topology of $H^1([0, T], L^2([0, 1])) \cap L^\infty([0, T], H^1([0, 1]))$.

Let us prove that u is a weak solution of (41). The main point is showing that u_{nx} converge to u_x almost everywhere, so that we can pass to the limit in (52). We compute

$$\begin{aligned} \partial_t \frac{u_t^2}{2} &= u_t u_{tt} = u_t \left(\frac{u_{xx}}{1+u_x^2} + g(x) \sqrt{1+u_x^2} \right)_t \\ &= \frac{u_t u_{txx}}{1+u_x^2} - 2 \frac{u_x u_{xx}}{(1+u_x^2)^2} \left(\frac{u_t^2}{2} \right)_x + g \frac{u_x}{\sqrt{1+u_x^2}} \left(\frac{u_t^2}{2} \right)_x \\ &\leq \frac{1}{1+u_x^2} \left(\frac{u_t^2}{2} \right)_{xx} + \left(g \frac{u_x}{\sqrt{1+u_x^2}} - 2 \frac{u_x u_{xx}}{(1+u_x^2)^2} \right) \left(\frac{u_t^2}{2} \right)_x. \end{aligned} \quad (53)$$

In particular, applying the same computation as (53) to u_n , we obtain that $\|u_{nt}\|_\infty$ is decreasing in time. Indeed if $M_n(t) = \sup_{x \in [0,1]} u_{nt}^2/2$, (53) gives that $M'_n(t) \leq 0$. Therefore $u \in W^{1,\infty}([0, T], L^\infty([0, 1]))$. Moreover, since g depends only on x we have

$$\begin{aligned} \partial_t \int_0^1 \frac{u_t^2}{2} dx &= \int_0^1 u_t u_{tt} dx = \int_0^1 -\arctan(u_x)_t u_{xt} + g \left(\sqrt{1+u_x^2} \right)_t u_t dx \\ &\leq \int_0^1 -\frac{u_{xt}^2}{1+u_x^2} + g u_x u_t \frac{u_{xt}}{\sqrt{1+u_x^2}} dx \\ &\leq \int_0^1 -\frac{u_{xt}^2}{1+u_x^2} + \frac{g^2 u_x^2 u_t^2}{2} dx \\ &\leq \frac{1}{2} \|g\|_\infty^2 \|u_t\|_\infty^2 \|u_x\|_{L^2}^2 = C \end{aligned}$$

where the constant $C > 0$ depends only on u_0 and $\|g\|_\infty$. We then get

$$\begin{aligned} \int_{[0,1]} (\arctan(u_x))_x^2 &= \int_{[0,1]} \left(u_t - g \sqrt{1+u_x^2} \right)^2 dx \leq C \quad \forall t \in [0, T] \\ \int_{[0,1] \times [0,T]} (\arctan(u_x))_t^2 dx dt &= \int_{[0,1] \times [0,T]} \frac{u_{xt}^2}{1+u_x^2} dx dt \leq C. \end{aligned} \quad (54)$$

As a consequence, the function $\arctan(u_{n_x})$ is uniformly bounded in $H^1([0, T], L^2([0, T])) \cap L^\infty([0, T], H^1([0, 1]))$. Therefore, the sequence $\arctan(u_{n_x})$ converges, up to a subsequence, to $\arctan(u_x)$ uniformly on $[0, 1] \times [0, T]$. Since \arctan is injective this implies that the sequence u_{n_x} converges to u_x a.e. on $[0, 1] \times [0, T]$, and we can pass to the limit in (52), obtaining that u is a weak solution of (41).

Finally, being $\arctan(u_x)$ continuous, possibly reducing T we have that u_x is also continuous (hence bounded) on $[0, 1] \times [0, T]$. In particular, recalling (41) the uniform bound on u_t implies an analogous bound on u_{xx} , that is $u \in L^\infty([0, T], W^{2,\infty}([0, 1]))$. \square

Remark 2.24. If u_0 is only in $H^1([0, 1])$, since the sequence u_n is uniformly bounded in $H^1([0, T], L^2([0, T]))$, reasoning as in Theorem 2.23 we get $u \in W_{\text{loc}}^{1,\infty}((0, T], L^\infty([0, 1])) \cap L_{\text{loc}}^\infty((0, T], W^{2,\infty}([0, 1]))$.

We conclude the section with a comparison and uniqueness result for solutions to (41).

Theorem 2.25. *Let $g(x, y) = g(x)$, with $g \in L^\infty([0, 1])$, and let u_1, u_2 be two solutions to (41) such that $u_1(x, 0) \leq u_2(x, 0)$ for all $x \in [0, 1]$. Then*

$$u_1 \leq u_2 \quad \text{on } [0, 1] \times [0, T].$$

In particular, there is a unique solution to (41), given an initial datum $u_0 \in W^{2,\infty}([0, 1])$.

Proof. Let

$$d(t) := \min_{x \in [0, 1]} u_2(x, t) - u_1(x, t).$$

Possibly replacing $u_1(\cdot, 0)$ with $u_1(\cdot, 0) - \delta$, we can assume that $d(0) = \delta > 0$. The thesis now follows if we can show that $d(t) \geq \delta$ for all $t \in [0, T]$. Let $w = u_2 - u_1$, so that

$$w_t = \frac{(u_2)_{xx}}{1 + (u_2)_x^2} - \frac{(u_1)_{xx}}{1 + (u_1)_x^2} + g(x) \left(\sqrt{1 + (u_2)_x^2} - \sqrt{1 + (u_1)_x^2} \right). \quad (55)$$

From (54) it follows

$$w_t \in L^2([0, T], H^1([0, 1])) \hookrightarrow L^2([0, T], C^\alpha([0, 1]))$$

for all $\alpha < 1/2$. Choose now $t \in [0, T]$ such that $w_t(\cdot, t) \in C^\alpha([0, 1])$ and notice that, for all $x \in [0, 1]$ such that $d(t) = w(x, t)$, we have

$$w_x = (u_2)_x - (u_1)_x = 0.$$

In particular, recalling (55), w is twice differentiable at x and we have

$$w_t = \frac{(u_2)_{xx}}{1 + (u_2)_x^2} - \frac{(u_1)_{xx}}{1 + (u_1)_x^2} = \frac{w_{xx}}{1 + (u_1)_x^2} \geq 0.$$

For almost every $t \in [0, T]$ we then get

$$\dot{d}(t) = \min_{x: d(t)=w(x,t)} w_t(x, t) \geq 0,$$

which gives the thesis. \square

Remark 2.26. We point out that in general we have $T < +\infty$ in Theorem 2.23, since the derivative u_x may blow up in finite time. This is related to the so-called fingering phenomenon, and we shall give an explicit example of such behavior. Let $u_0 = 0$, $g(x) = M$ for $x \in [0, 1/2)$, and $g(x) = -M$ for $x \in [1/2, 1]$, where the constant M is greater than 4. For all $t \geq 0$, we set

$$\begin{aligned} u^-(x, t) &:= \sqrt{\frac{x}{2} - x^2 - \frac{1}{4}} + (M - 4)t & x \in \left(0, \frac{1}{2}\right) \\ u^+(x, t) &:= \sqrt{\frac{3}{2}x - x^2 - \frac{1}{2}} + \frac{1}{4} - (M - 4)t & x \in \left(\frac{1}{2}, 1\right). \end{aligned}$$

A direct computation shows that u^-, u^+ are respectively a subsolution and a supersolution of (41) on their intervals of definition. By Theorem 2.25, for all $t \in [0, T]$ it follows that $u(\cdot, t) \geq u^-(\cdot, t)$ on $(0, 1/2)$, and $u(\cdot, t) \leq u^+(\cdot, t)$ on $(1/2, 1)$. Since u is continuous, this necessarily implies $T \leq 1/(4(M - 4))$. More precisely, if we extend the solution u on a maximal time interval $[0, T_{\max})$, we have $T_{\max} \leq 1/(4(M - 4))$ and the derivative $u_x(1/2, t)$ blows up (in absolute value) as $t \rightarrow T_{\max}$.

3 A homogenization problem

Given a smooth function g which is periodic on $[0, 1]^2$, we consider the following homogenization problem

$$u_t = \frac{u_{xx}}{1 + u_x^2} + g\left(\frac{x}{\varepsilon}, \frac{u}{\varepsilon}\right) \sqrt{1 + u_x^2} \quad (56)$$

with initial data $u(x, 0) = u_0(x)$, satisfying (40).

Notice that, after the parabolic rescaling $s = t/\varepsilon^2$, $y = x/\varepsilon$ and $v = u/\varepsilon$, problem (56) becomes

$$v_s = \frac{v_{yy}}{1 + v_y^2} + \varepsilon g(y, v) \sqrt{1 + v_y^2}. \quad (57)$$

In [6] existence of traveling wave solutions for (57) has been established. Moreover in [13] (see also [5]) the authors discuss the uniqueness of traveling fronts and characterize the asymptotic speed in some particular case.

A straightforward application of the results in Subsection 2.6 gives the first result about the convergence of the solutions to the perturbed problem (56).

Proposition 3.1. *Let $u_{0\varepsilon} \in C^\infty([0, 1])$ satisfying (40) and $\|u_{0\varepsilon}\|_{W^{1,\infty}} \leq L$, and let u_ε be the solutions to (56) with initial data $u_{0\varepsilon}$. Then, up to a subsequence,*

$$u_\varepsilon \rightarrow u \in H^1([0, T], L^2([0, 1])) \cap L^\infty([0, T], H^1([0, 1])) \quad \text{as } \varepsilon \rightarrow 0,$$

uniformly on $[0, 1] \times [0, T]$.

Proof. By Proposition 2.19 there exists $T > 0$ independent of ε and a family of smooth solution u_ε of (56), which are uniformly bounded in $H^1([0, T], L^2([0, 1])) \cap L^\infty([0, T], H^1([0, 1]))$. In particular, as in Proposition 2.21, we can pass to the limit, up to a subsequence as $\varepsilon \rightarrow 0$, and obtain that

$$u_\varepsilon \rightarrow u \in H^1([0, T], L^2([0, 1])) \cap L^\infty([0, T], H^1([0, 1])) \quad \text{uniformly on } [0, 1] \times [0, T].$$

□

There are two main open problems related to the homogenization of equation (56):

- 1) the characterization of u as the solution of an appropriate homogenized equation;
- 2) the convergence on large time intervals.

Concerning the second question, we expect the following result:

Conjecture 3.2. *The convergence in Proposition 3.1 is uniform on $[0, 1] \times [0, +\infty)$.*

Let us give a heuristic argument supporting our conjecture: due to the comparison principle and the periodicity of g , for all $N \in \mathbb{N}$ we have the estimate

$$|u_\varepsilon(x, t) - u_\varepsilon(x + N\varepsilon, t)| \leq ([L] + 1)N\varepsilon, \quad (58)$$

where $[L]$ denotes the integer part of L . Passing to the limit in (58) as $\varepsilon \rightarrow 0$, we get

$$|u(x, t) - u(y, t)| \leq L|y - x|,$$

that is the norm $\|u(\cdot, t)\|_{W^{1,\infty}}$ is non increasing in t . We expect this bound to be true also for the approximating sequence u_ε , which would imply that we can take $T = +\infty$.

Concerning the first question, we have only some partial results. We state a result when g depends only on x .

Theorem 3.3. *Let $g(x, y) = g(x) \in L^\infty([0, 1])$, and let $u_0 \in W^{2,\infty}([0, 1])$ with $u_{0x}(0) = u_{0x}(1)$. Then, there exists $T > 0$ depending only on u_0 and $\|g\|_\infty$ such that the solutions u_ε to (56) converge in $W^{1,\infty}([0, T], L^\infty([0, 1])) \cap L^\infty([0, T], W^{2,\infty}([0, 1]))$, as $\varepsilon \rightarrow 0$, to the unique solution u of*

$$u_t = \frac{u_{xx}}{1 + u_x^2} + \left(\int_0^1 g(x) dx \right) \sqrt{1 + u_x^2} \quad (59)$$

with initial datum u_0 and boundary conditions (40). In particular, $u \in C^\infty([0, 1] \times (0, T])$.

Proof. The proof is a straightforward adaptation of Theorem 2.23. □

Remark 3.4. When $g(x, y) = g(x) \in W^{1,\infty}([0, 1])$ and periodic, an analogous result can be obtained in general dimensions and for all times, i.e. the solutions of u_ε to (56) converge, as $\varepsilon \rightarrow 0$, to the unique solution u of (59) locally uniformly in $[0, 1] \times [0, +\infty)$. This can be proved by the so-called *perturbed test function method* (cf. [4]), which is by now a standard method in viscosity solutions theory applied to homogenization problems. More precisely, one considers the formal asymptotic expansion

$$u^\varepsilon(x, t) = u(x, t) + \varepsilon^2 \chi\left(\frac{x}{\varepsilon}, x, t\right) \quad (60)$$

where the corrector $\chi(\xi, x, t) = \psi(\xi, Du(x, t))$ is periodic in ξ and solves, for every fixed $p = Du(x, t)$, the cell problem

$$\operatorname{tr} \left[\left(\mathbf{I} - \frac{p \otimes p}{1 + |p|^2} \right) D_{\xi\xi}^2 \chi \right] = \left(\int_{[0,1]^n} g(x) dx - g(\xi) \right) \sqrt{1 + |p|^2} \quad \text{in } \mathbb{R}^n.$$

Plugging the expansion (60) in (56), and using the comparison principle for (viscosity) solutions to (59), one obtains that u solves (59).

In the general case, we can determine the limit equation satisfied by u only in a very specific case, that is when the initial data are plane-like.

Proposition 3.5. *Let $u_{0\varepsilon} \in C^\infty([0, 1])$ satisfying (40), $\|u_{0\varepsilon}\|_{W^{1,\infty}} \leq L$ and $u_{0\varepsilon}(x) \rightarrow \alpha x$ uniformly in $[0, 1]$. Then, if u_ε are the solution to (56), (40) with initial datum $u_{0\varepsilon}$, we have*

$$\lim_{\varepsilon \rightarrow 0} u_\varepsilon(x, t) = u(x, t) := \alpha x + c(\alpha)\sqrt{1 + \alpha^2}t,$$

uniformly in $[0, 1] \times [0, T]$, with

$$c(\alpha) := \begin{cases} 0 & \text{if } G(s) = 0 \text{ for some } s \in [0, 1] \\ \left(\int_0^1 \frac{1}{G(s)} ds\right)^{-1} & \text{otherwise,} \end{cases}$$

where

$$G(s) := \lim_{L \rightarrow \infty} \frac{1}{L} \int_0^L g(x, \alpha x + s) dx.$$

Proof. By Proposition 2.19 for every ε there exists a smooth solution to (56) in $[0, T]$, for T independent of ε . Moreover, by Proposition 3.1, these solutions u_ε converge uniformly on $[0, 1] \times [0, T]$, up to subsequences, to a function u .

By [6, Thm. 4.1] (see also [13]) for all $\alpha \in \mathbb{R}$ there exist global smooth solutions $\hat{u}_{\alpha,\varepsilon}$ of (56), with average slope α , which are *pulsating waves*, that is, there exist $\tau > 0$ and a vector $(v_1, v_2) \in \mathbb{Z}^2$, depending on (α, ε) and such that

$$\hat{u}_{\alpha,\varepsilon}(x, t + \tau) = \hat{u}_{\alpha,\varepsilon}(x - \varepsilon v_1, t) + \varepsilon v_2 \quad \forall (x, t) \in \mathbb{R}^2.$$

We let

$$c(\alpha, \varepsilon) = \frac{\varepsilon (v_1, v_2) \cdot \nu_\alpha}{\tau} \quad \text{where } \nu_\alpha = \left(-\frac{\alpha}{\sqrt{1 + \alpha^2}}, \frac{1}{\sqrt{1 + \alpha^2}}\right)$$

be the velocity of the wave in the normal direction ν_α and we set $c(\alpha, \varepsilon) = 0$ if $\hat{u}_{\alpha,\varepsilon}$ is a standing wave. In particular, in [6, Thm. 4.1] it is shown that \hat{u}_ε can be represented as

$$\hat{u}_{\alpha,\varepsilon}(x, t) = \alpha x + c(\alpha, \varepsilon)\sqrt{1 + \alpha^2}t + \mathcal{O}(\varepsilon) \quad \forall (x, t) \in \mathbb{R}^2, \quad (61)$$

where $|\mathcal{O}(\varepsilon)| \leq C\varepsilon$, for a constant C depending only on (the C^2 -norm of) g . Moreover, by [6, Cor. 2.5] the derivatives $(\hat{u}_{\alpha,\varepsilon})_x(x, t)$ are uniformly bounded for all $(x, t) \in \mathbb{R}^2$ and for all ε small enough.

Notice that, by [6, Prop. 4.4], for all $(x, t) \in \mathbb{R}^2$ we have

$$\begin{aligned} c(\alpha, \varepsilon) = 0 &\rightarrow (\hat{u}_{\alpha,\varepsilon})_t = 0 \\ c(\alpha, \varepsilon) > 0 &\rightarrow (\hat{u}_{\alpha,\varepsilon})_t > 0 \\ c(\alpha, \varepsilon) < 0 &\rightarrow (\hat{u}_{\alpha,\varepsilon})_t < 0. \end{aligned}$$

In particular, without loss of generality we can assume that

$$(\hat{u}_{\alpha,\varepsilon})_t \geq 0 \quad \forall (x, t) \in \mathbb{R}^2.$$

One can now argue as in [5, (4.9) and (4.10)] and conclude by maximum principle that

$$|(\hat{u}_{\alpha,\varepsilon})_x - \alpha| \leq C\varepsilon \quad \forall (x, t) \in \mathbb{R}^2, \quad (62)$$

for a constant C depends only on g . Integrating (56) on $[0, 1]$ and using (62), a direct computation as in [5, Prop. 6] gives $\lim_{\varepsilon \rightarrow 0} c(\alpha, \varepsilon) = c(\alpha)$.

Finally, by comparison principle for solutions to (56), we can use the functions $\hat{u}_{\alpha,\varepsilon}$ as barriers for u_ε , and obtain that

$$\lim_{\varepsilon \rightarrow 0} u_\varepsilon(x, t) = u(x, t) = \alpha x + c(\alpha)\sqrt{1 + \alpha^2} t,$$

uniformly in $[0, 1] \times [0, T]$. □

Remark 3.6. If we assume that $u_{0\varepsilon}(x) \rightarrow \alpha x$ in $C^1([0, 1])$, the result of Proposition 3.5 can be strengthened to

$$\lim_{\varepsilon \rightarrow 0} u_\varepsilon(x, t) = u(x, t) := \alpha x + c(\alpha)\sqrt{1 + \alpha^2} t, \quad \text{locally uniformly in } [0, 1] \times [0, +\infty).$$

Indeed, under this stronger assumption, for every ε small enough there exists a unique smooth solution u_ε to (56) in $[0, 1] \times (0, +\infty)$ [6, Thm. 2.7].

Remark 3.7. Notice that $c(\alpha) = \int_{[0,1]^2} g$ for all $\alpha \notin \mathbb{Q}$, so that the function $\alpha \rightarrow c(\alpha)$ is not necessarily continuous. This suggests that the homogenization limit of (2) should be a geometric evolution of the form

$$v = \bar{c}(\nu, \kappa) \nu$$

where the function \bar{c} is in general *discontinuous*.

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