27/3/03

$$\frac{3+x}{x^3-x}$$

$$x^3 - x = x(x^2-1)$$
$$= x(x-1)(x+1)$$

le radici sono $0, 1, -1$ tutte semplici

$$\downarrow$$

$$\frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} = \frac{A(x^2-1) + B(x^2+x) + C(x^2-x)}{x(x-1)(x+1)} =$$

$$\frac{(A+B+C)x^2 + (B-C)x - A}{x^3-x}$$

$$\begin{cases} A + B + C = 0 \\ B - C = 1 \\ -A = 3 \end{cases} \qquad \begin{cases} B+C = 3 \\ B-C = 1 \\ A = -3 \end{cases} \qquad \begin{cases} B = 2 \\ C = 1 \\ A = -3 \end{cases}$$

$$f(x) = -\frac{3}{x} + \frac{2}{x-1} + \frac{1}{x+1} \Rightarrow \int f(x)\,dx \overset{*}{=} \ln\left|\frac{(x-1)^2(x+1)}{x^3}\right| + c$$

Accoppiando z_j con $\overline{z}_j$ trovo

$$Q(x) = (x - x_1) \cdots (x - x_h) \Big(\underbrace{(x - a_1)^2 + b_1^2}_{} \Big) \cdots \Big(\underbrace{(x - a_\ell)^2 + b_\ell^2}_{} \Big)$$

ho radici

$a_1 \pm i b_1$ $\qquad$ $a_\ell \pm i b_\ell$

$$\frac{x+3}{x^3+x} = f(x) \qquad Q(x) = x(x^2+1) \implies$$

$$f(x) = \frac{A}{x} + \frac{Bx + C}{x^2+1} = \frac{A(x^2+1) + Bx^2 + Cx}{x(x^2+1)}$$

$$-\frac{3}{2}\,\frac{2x}{1+x^2} + \frac{1}{1+x^2}$$

$$\begin{cases} A + B = 0 \\ C = 1 \\ A = 3 \end{cases} \qquad \begin{aligned} A &= 3 \\ B &= -3 \\ C &= 1 \end{aligned} \qquad f(x) = \frac{3}{x} + \frac{-3x+1}{x^2+1}$$

$$\int f(x)\,dx = 3\ln(|x|) - \frac{3}{2}\ln(1+x^2) + \operatorname{arctg}(x) = 3\ln\left(\frac{|x|}{\sqrt{1+x^2}}\right) + \operatorname{arctg}(x) + c$$

$$\int \underbrace{\frac{x^2}{(1+x^2)(4+x^2)}}_{f(x)} dx$$

NO RADICI REALI

$$f(x) = \frac{Ax+B}{1+x^2} + \frac{Cx+D}{4+x^2} = \frac{A(4x+x^3)+B(4+x^2)+C(x+x^3)+D(1+x^2)}{(1+x^2)(4+x^2)}$$

$$\begin{array}{l} x^3 \to \\ x^2 \to \\ x \to \\ 1 \to \end{array} \begin{cases} A + C = 0 \\ B + D = 1 \\ 4A + C = 0 \\ 4B + D = 0 \end{cases}$$

$A = C = 0$

$B = -\frac{1}{3}$ $D = \frac{4}{3}$

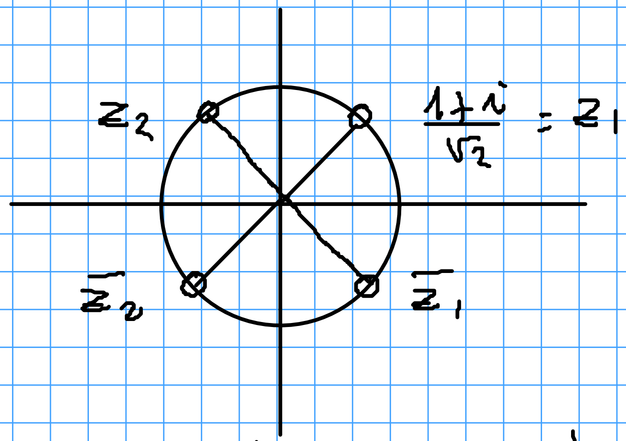
$$f(x) = -\frac{1}{3}\frac{1}{1+x^2} + \frac{4}{3}\frac{1}{4+x^2}$$

$$\int f(x) = -\frac{1}{3}\operatorname{artg}(x) + \frac{4}{3}\int \frac{1}{4+x^2}dx = -\frac{1}{3}\operatorname{artg}(x) + \frac{2}{3}\operatorname{artg}\left(\frac{x}{2}\right) + c$$

$$\frac{4}{3}\frac{1}{4}\int \frac{1}{1+\left(\frac{x}{2}\right)^2}dx = \frac{2}{3}\operatorname{artg}\left(\frac{x}{2}\right) + c$$

$$\int \frac{1}{1+x^4}\,dx \qquad 1+x^4=0 \text{ NON HA RADICI}$$

CONVIENE TROVARE LE RADICI COMPLESSE, cioè le radici 4e di (-1)



$$\frac{\pm 1 \pm i}{\sqrt{2}}$$

$$1+x^4$$
$$\parallel$$
$$(x-z_1)(x-\bar{z}_1)(x-z_2)(x-\bar{z}_2) = (x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)$$

$\Rightarrow$

$$\frac{1}{1+x^4} = \frac{Ax+B}{x^2-\sqrt{2}x+1} + \frac{Cx+D}{x^2+\sqrt{2}x+1} =$$

$$\frac{A(x^3+\sqrt{2}x^2+x)+B(x^2+\sqrt{2}x+1)+C(x^3-\sqrt{2}x^2+x)+D(x^2-\sqrt{2}x+1)}{1+x^4}$$

$$\begin{cases} A + C = 0 \\ \sqrt{2}A + B - \sqrt{2}C + D = 0 \\ A + \sqrt{2}B + C - \sqrt{2}D = 0 \\ B + D = 1 \end{cases} \qquad \begin{cases} C = -A \qquad D = 1 - B \\ \sqrt{2}A + \cancel{B} + \sqrt{2}A - \cancel{B} = -1 \\ \cancel{A} + \sqrt{2}B - \cancel{A} + \sqrt{2}B = \sqrt{2} \end{cases}$$

$$\begin{cases} C = -A \qquad D = 1 - B \\ A = -\dfrac{1}{2\sqrt{2}} \\ B = \dfrac{1}{2} \end{cases} \qquad \begin{cases} A = -\dfrac{1}{2\sqrt{2}} \qquad C = \dfrac{1}{2\sqrt{2}} \\ B = \dfrac{1}{2} \qquad D = \dfrac{1}{2} \end{cases}$$

$$f(x) = \frac{1}{2\sqrt{2}} \left(\frac{-x + \sqrt{2}}{\underbrace{x^2 - \sqrt{2}x + 1}_{1}} + \frac{x + \sqrt{2}}{\underbrace{x^2 + \sqrt{2}x + 1}_{2}} \right)$$

$$= \frac{1}{2} \frac{d}{dx} \left(x - \frac{\sqrt{2}}{2}\right)^2$$

$$\int 1 = \int \frac{-x + \sqrt{2}}{\left(x - \frac{\sqrt{2}}{2}\right)^2 + \frac{1}{2}} dx = \int \frac{-\left(x - \frac{\sqrt{2}}{2}\right) - \frac{\sqrt{2}}{2} + \sqrt{2}}{\left(x - \frac{\sqrt{2}}{2}\right)^2 + \frac{1}{2}} dx =$$

NON C'E' PIU' X

$$-\frac{1}{2} \int \frac{\frac{d}{dx}\left(x - \frac{\sqrt{2}}{2}\right)^2}{\left(x - \frac{\sqrt{2}}{2}\right)^2 + \frac{1}{2}} dx + \frac{\sqrt{2}}{2} \int \frac{dx}{\left(x - \frac{\sqrt{2}}{2}\right)^2 + \frac{1}{2}} =$$

$$-\frac{1}{2}\ln\left(\left(x-\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}\right)+\sqrt{2}\int\frac{dx}{\left(\sqrt{2}x-1\right)^2+1} \;(*) =$$

$$-\frac{1}{2}\ln\left(x^2-\sqrt{2}x+1\right)+\arctan\left(\sqrt{2}x-1\right)+\text{cost} \quad \Rightarrow \quad (1)$$

IN MODO ANALOGO SI TROVA (2) (VEDI LA RISPOSTA ESATTA SUI LUCIDI)

$(*)$
$$\left(x-\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}=\frac{1}{2}\left[2\left(x-\frac{\sqrt{2}}{2}\right)^2+1\right]=\frac{1}{2}\left[\left(\sqrt{2}x-\frac{\sqrt{2}\sqrt{2}}{2}\right)^2+1\right]$$

$$\int \frac{dx}{(x^4-1)^2}$$

$$Q(x) = \left[(x^2-1)(x^2+1)\right]^2 =$$
$$(x-1)^2(x+1)^2\,(x^2+1)^2$$

$$f(x) = \frac{1}{(x^4-1)^2}$$

$$f(x) = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} + \frac{Ex+F}{x^2+1} + \frac{Gx+H}{(x^2+1)^2} =$$

$$\frac{1}{den}\Big\{ A\,(x-1)(x^2+1)^2(x+1)^2 + B\,(x^2+1)^2(x+1)^2 +$$
$$C\,(x+1)(x^2+1)^2(x-1)^2 + D\,(x^2+1)^2(x-1)^2 +$$
$$(Ex+F)(x^2+1)(x^2-1)^2 + (Gx+H)(x^2-1)^2 \Big\} =$$

$$\frac{1}{den}\Big\{ A\,(x^2-1)(x+1)(x^4+2x^2+1) + B(x^4+2x^2+1)(x^2+2x+1) +$$
$$C\,($$

RINUNCIO !!

$$\int \frac{1}{(1+x^2)^2 x^2}\,dx$$

$$f(x) = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{(1+x^2)} + \frac{Ex+F}{(1+x^2)^2} =$$

$$\frac{1}{\text{den.}}\left\{ Ax\,(1+x^2)^2 + B\,(1+x^2)^2 + (Cx+D)\,x^2\,(1+x^2) + (Ex+F)\,x^2 \right\}$$

$$1+2x^2+x^4$$

$$= \frac{1}{\text{den}}\left\{ A\,(x+2x^3+x^5) + B(1+2x^2+x^4) + C\,(x^3+x^5) + D\,(x^2+x^4) + E\,x^3 + F\,x^2 \right\}$$

$$\begin{cases} A + C = 0 \\ B + D = 0 \\ 2A + C + E = 0 \\ 2B + D + F = 0 \\ A = 0 \\ B = 1 \end{cases}$$

$A=0$, $B=1$, $C=0$, $D=-1$, $E=0$, $F=-1$

$$f(x) = \frac{1}{x^2} - \frac{1}{x^2+1} - \frac{1}{(x^2+1)^2}$$

$$\int f(x)\,dx = -\frac{1}{x} - \operatorname{arctg}(x) - \int \frac{1}{(x^2+1)^2}\,dx$$

$$\int \frac{1}{(1+x^2)^2}\,dx = \int \frac{1+\cancel{x^2}}{(1+x^2)^2}\,dx - \int \frac{x^2}{(1+x^2)^2}\,dx = \int \frac{1}{1+x^2}\,dx -$$

$$-\int \frac{x}{2}\,\frac{2x}{(1+x^2)^2}\,dx = \operatorname{arctg}(x) - \left(\frac{x}{2}\cdot\left(-\frac{1}{x^2+1}\right) + \int \frac{1}{2}\,\frac{1}{1+x^2}\,dx\right) =$$

per parti

$$\operatorname{arctg}(x) + \frac{1}{2}\,\frac{x}{1+x^2} - \frac{1}{2}\operatorname{arctg}(x) = \frac{1}{2}\operatorname{arctg}(x) + \frac{1}{2}\,\frac{x}{1+x^2} + c$$

IN GENERALE

$$\boxed{\int \frac{1}{(1+x^2)^m}\,dx} = \int \left(\frac{1+x^2}{(1+x^2)^m} - \frac{x^2}{(1+x^2)^m}\right)dx =$$

$$\int \frac{1}{(1+x^2)^{m-1}}\,dx + \int \frac{x}{2}\,\frac{-2x}{(1+x^2)^m}\,dx = \int \frac{dx}{(1+x^2)^{m-1}} + \frac{x}{2}\,\frac{1}{(m-1)(1+x^2)^{m-1}}$$

(per parti

$$-\frac{1}{2(m-1)}\int\frac{dx}{(1+x^2)^{m-1}} = \left(1-\frac{1}{2(m-1)}\right)\int\frac{dx}{(1+x^2)^{m-1}} + \frac{1}{2(m-1)}\frac{x}{(x^2+1)^{m-1}}$$

FORMULA RICORSIVA, dopo m passi risolvo l'integrale.

$$\int \underbrace{\frac{1}{(1+x^2)^2 x^2}}_{f(x)} dx$$

Facciolo di nuovo l'integrale con la variante di Hermite

$$f(x) = \frac{A}{x} + \frac{Bx+C}{1+x^2} + \frac{d}{dx}\frac{Dx^2+Ex+F}{(1+x^2)x} =$$

$$\frac{A}{x} + \frac{Bx+C}{1+x^2} + \frac{(2Dx+E)(x+x^3) - (Dx^2+Ex+F)(1+3x^2)}{(1+x^2)^2 x^2} =$$

$$\frac{Ax(1+2x^2+x^4) + (Bx+C)(x^2+x^4) + D(2x^2+2x^4) + E(x+x^3) + -D(x^2+3x^4) - E(x+3x^3) - 3Fx^2 - F}{(1+x^2)^2 x^2}$$

$$x^5(A+B) + x^4(C+2D-3D) + x^3(2A+B+E-3E)$$
$$+\, x^2(C+2D-D-3F) + x(A+E-E) - F$$

$$\begin{cases} A+B = 0 \\ C-D = 0 \\ 2A+B-2E = 0 \\ C+D-3F = 0 \\ A = 0 \\ -F = 1 \end{cases}$$

$A=0 \quad B=0 \quad F=-1$

$C=D \quad E=0 \quad C=D=-\frac{3}{2}$

$C+D=-3$

$$f(x) = -\frac{3}{2}\frac{1}{1+x^2} + \frac{d}{dx}\frac{\overset{Dx^2}{-\frac{3}{2}x^2} \overset{Ex}{} \overset{+F}{-1}}{x(1+x^2)}$$

dunque si ritrova (ricontrollare!)

$$\int f(x)\,dx = -\frac{3}{2}\operatorname{artg}(x) - \frac{3x^2+2}{2x(x^2+1)} + \text{cost}.$$