Esempio

$$\frac{n}{1+n^2} \simeq \frac{1}{n} \text{ , infatti}$$

$$\frac{\frac{n}{1+n^2}}{\frac{1}{n}} = \frac{n^2}{1+n^2} = \frac{1}{1+1/n^2} \to 1$$

$$\frac{n^3+\sin(n)}{n^2-n+2} \simeq n \text{ ; infatti}$$

$$\frac{\frac{n^3+\sin(n)}{n^2-n+2}}{n} = \frac{n^3+\sin(n)}{(n^2-n+2)n} =$$

$$\frac{1+\sin(n)/n^3}{1-\frac{1}{n}+\frac{2}{n^2}} \to 1$$

MEMENTO (!)

$$\frac{\sin(n)}{n^3} \to 0$$

(infinitesimo x limitato

Principio di sostituzione degli infiniti

$\{a_n\}$ ha ordine di infinito $\geq \{b_n\}$

$\{c_n\}$ " " " " $\geq \{d_n\}$

$$\Rightarrow \lim_{n \to \infty} \frac{a_n + b_n}{c_n + d_n} = \lim_{n \to +\infty} \frac{a_n}{b_n}$$

Dim. So che

$$\left|\frac{a_n}{b_n}\right| \to +\infty \iff \frac{b_n}{a_n} \to 0$$

$$\left|\frac{c_n}{d_n}\right| \to +\infty \iff \frac{d_n}{c_n} \to 0$$

Allora:

$$\frac{a_n + b_n}{c_n + d_n} = \frac{a_n}{c_n} \frac{1 + \frac{b_n}{a_n}}{1 + \frac{d_n}{c_n}} = \frac{a_n}{c_n} e_n$$

e_n

dove $e_n \to 1$. Ne segue la tesi

$$\lim_{n \to \infty} \frac{\sqrt{n^4+n^3+1} - n^2}{n} = \frac{1}{2}$$

$$\frac{\sqrt{n^4+n^3+1} - n^2}{n} =$$ (moltiplico sopra e sotto per $\sqrt{n^4+n^3+1} + n^2$

$$\frac{\cancel{n^4} + n^3 + 1 - \cancel{n^4}}{n\left(\sqrt{n^4+n^3+1} + n^2\right)} =$$

$$\frac{1}{\sqrt{1 + 1/n + 1/n^4} + 1} \to \frac{1}{2}$$

$$O(a_n) + O(a_n) = O(a_n)$$

Infatti se

$$b_n = O(a_n) \Rightarrow \frac{b_n}{a_n} \text{ limitata}$$

$$b'_n = O(a_n) \Rightarrow \frac{b'_n}{a_n} \text{ limitata}$$

$$\Rightarrow \frac{b_n + b'_n}{a_n} = \frac{b_n}{a_n} + \frac{b'}{a_n}$$

è limitata.

$$\Rightarrow b_n + b'_n = O(a_n)$$

$$\sigma(a_n) + \sigma(a_n) = \sigma(a_n)$$

<u>Dim.</u>

$$b_n = \sigma(a_n) \Rightarrow \frac{b_n}{a_n} \to 0$$

$$b'_n = \sigma(a_n) \Rightarrow \frac{b'_n}{a_n} \to 0 \qquad \Rightarrow$$

$$\frac{b_n + b'_n}{a_n} = \frac{b_n}{a_n} + \frac{b'_n}{a_n} \to 0$$

$$\Rightarrow b_n + b'_n = \sigma(a_n)$$

$$o(a_n) + O(a_n) = O(a_n)$$

(come le precedenti)

<u>Esempio</u>

$$\frac{1}{n+1} = O(1/n)$$

$$\frac{1}{n^2} = o(1/n)$$

$$\Rightarrow \quad \frac{1}{n+1} + \frac{1}{n^2} = O(1/n)$$

$$o(a_n) \cdot O(a'_n)$$

<u>Dim</u>

Se $b_n = o(a_n) \Rightarrow \frac{b_n}{a_n}$ limitato

se $b'_n = O(a'_n) \Rightarrow \frac{b'_n}{a_n} \to 0$

$$\Rightarrow \frac{b_n \cdot b'_n}{a_n \, a'_n} = \frac{b_n}{a_n} \cdot \frac{b'_n}{a'_n} \to 0$$

$$\Rightarrow b_n \, b'_n = o(a_n \cdot a'_n)$$

INVECE $O(a_n) \, O(a'_n) = O(a_n \, a'_n)$

<u>Esempi</u>

$$\frac{1}{n} \, \frac{1}{n^2+1} = O\left(\frac{1}{n}\right) O\left(\frac{1}{n^2}\right) = O\left(\frac{1}{n^3}\right)$$

$$\frac{1}{n^2} \, \frac{1}{n^2+1} = o\left(\frac{1}{n}\right) O\left(\frac{1}{n^2}\right) = o\left(\frac{1}{n^3}\right)$$

$$o(O(a_n)) = o(a_n)$$

Dim.

$$b_n = O(a_n) \Rightarrow \frac{b_n}{a_n} \text{ limitato}$$

$$c_n = o(b_n) \Rightarrow \frac{c_n}{b_n} \text{ limitato}$$

$$\Rightarrow \frac{c_n}{a_n} = \underbrace{\frac{c_n}{b_n}}_{\text{infinitesimo}} \cdot \underbrace{\frac{b_n}{a_n}}_{\text{limitato}} \to 0$$

$$\Rightarrow c_n = o(a_n)$$

Esempio

$$\frac{1}{n^2} = O\left(\frac{1}{n^2+1}\right), \quad \frac{1}{n^3} = o\left(\frac{1}{n^2}\right) \Rightarrow \frac{1}{n^3} = o\left(\frac{1}{n^2+1}\right)$$

$$a_n \simeq b_n \iff a_n = b_n + o(b_n)$$

Dim

$$a_n \simeq b_n \iff \frac{a_n}{b_n} \to 1 \iff$$

$$\frac{a_n}{b_n} - 1 \to 0 \iff \frac{a_n - b_n}{b_n} \to 0$$

$$\iff a_n - b_n = o(b_n) \iff$$

$$a_n = b_n + o(b_n)$$

REGOLA: Se $P(n) = a n^k + \text{grado} < k$

$Q(n) = b n^h + \text{grado} < h$

$$\Rightarrow \frac{P(n)}{Q(n)} \simeq \frac{a}{b} n^{k-h}$$

Esempio

$$\frac{n}{1+n^2} \simeq \frac{1}{n} \qquad \text{dunque}$$

$$\frac{n}{1+n^2} = \frac{1}{n} + o\left(\frac{1}{n}\right)$$

Andiamo a vedere l'$o\left(\frac{1}{n}\right)$

$$\frac{n}{1+n^2}-\frac{1}{n}=\frac{n^2-1-n^2}{(1+n^2)\,n}=$$

$$=\frac{-1}{(1+n^2)\,n}\simeq\frac{-1}{n^3};$$ dunque

$$\frac{n}{1+n^2}=\frac{1}{n}-\frac{1}{n^3}+o\left(\frac{1}{n^3}\right).$$ Riportiamo

$$\frac{n}{1+n^2}-\frac{1}{n}+\frac{1}{n^3}=\frac{n^4-n^2(1+n^2)+1+n^2}{(1+n^2)\,n^3}$$

$$=\frac{\cancel{n^4}-\cancel{n^2}-\cancel{n^4}+1+\cancel{n^2}}{(1+n^2)\,n^3}=\frac{1}{(1+n^2)\,n^3}\simeq\frac{1}{n^5}$$

Dunque

$$\frac{n}{1+n^2}=\frac{1}{n}-\frac{1}{n^3}+\frac{1}{n^5}+o\left(\frac{1}{n^5}\right)$$

<u>Nota</u> Se so che $a_n = \frac{1}{n} - \frac{1}{n^3} + o\left(\frac{1}{n^3}\right)$

allora ricavo che $n^3\left(a_n - \frac{1}{n}\right) \to -1$

infatti

$$n^3\left(a_n - \frac{1}{n}\right) = n^3\left(\frac{1}{n} - \frac{1}{n^3} + o\left(\frac{1}{n}\right) - \frac{1}{n}\right)$$

$$= n^3\left(-\frac{1}{n^3} + o\left(\frac{1}{n^3}\right)\right) = -1 + n^3 o\left(\frac{1}{n^3}\right)$$

$$= -1 + O(n^3)\, o\left(\frac{1}{n^3}\right) = -1 + o(1) \to -1$$

LIMITE IMPORTANTE

Se $|a_n| \to +\infty \Rightarrow \left(1+\frac{1}{a_n}\right)^{a_n} \to e$

<u>Caso 1</u> $a_n \to +\infty$. Ricordiamo che la parte intera di un numero x è l'unico intero $[x]$ con $[x] \leq x < [x]+1$

Se $a_n \to +\infty \Rightarrow [a_n] > a_n - 1 \to +\infty$

Inoltre

$$\left(1+\frac{1}{a_n}\right)^{a_n} \leq \left(1+\frac{1}{[a_n]}\right)^{[a_n]+1} = \left(1+\frac{1}{[a_n]}\right)^{[a_n]}\left(1+\frac{1}{[a_n]}\right)$$

e anche

$$\left(1+\frac{1}{a_n}\right)^{a_n} \geq \left(1+\frac{1}{[a_n]+1}\right)^{[a_n]} = \frac{\left(1+\frac{1}{[a_n]+1}\right)^{[a_n]+1}}{\left(1+\frac{1}{[a_n]+1}\right)}$$

dato che $\left(1+\frac{1}{[a_n]}\right)^{[a_n]} \to e$

e $\left(1+\frac{1}{[a_n]+1}\right)^{[a_n]+1} \to e$

(dato che sono entrambe estratte dalla successione $\left(1+\frac{1}{n}\right)^n \to e$)

e che $\left(1+\frac{1}{[a_n]}\right) \to 1$, $\left(1+\frac{1}{[a_n]+1}\right) \to 1$

si deduce (teor. due carabinieri)

che $\left(1+\frac{1}{a_n}\right)^{a_n} \to e$

II° caso: $a_n \to -\infty$.

Se $a_n \to -\infty$ possiamo scrivere:

$$\left(1+\frac{1}{a_n}\right)^{a_n} = \left(\frac{a_n+1}{a_n}\right)^{a_n} =$$

$$\left(\frac{a_n}{a_n+1}\right)^{-a_n} = \left(\frac{a_n+1-1}{a_n+1}\right)^{-a_n} =$$

$$\left(1-\frac{1}{a_n+1}\right)^{-a_n} = \left(1+\frac{1}{-a_n-1}\right)^{-a_n-1}\left(1-\frac{1}{a_n+1}\right)$$

$\to e \cdot 1 = e$ in quanto possiamo applicare il caso I alla successione $-a_n - 1 \to +\infty$.