$$|a+b| \le |a| + |b|$$

<u>Dim.</u> (1) $a \geq 0$, $b \geq 0$ $\qquad (\Rightarrow a+b \geq 0)$

$$|a+b| = a+b \le a+b$$

(2) $a \le 0 \quad b \geq 0$

(2a) $a+b \geq 0 \Longrightarrow$

$$|a+b| = a+b \le -a+b = |a|+|b|$$

(2b) $a+b \le 0$

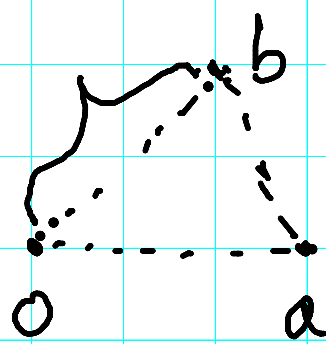
$$|a+b| = -a-b \le -a+b = |a|+|b|$$

(3) $a \geq 0 \quad b \le 0$ <u>come</u> il (2)

(4) $a \le 0 \quad b \le 0$ come l'(1)

$$|a+b| = -a-b = |a|+|b|$$

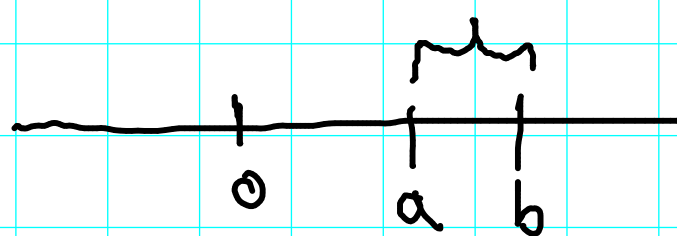
Dis. triangolo



$$|b| \leq |a| + |b-a|$$

$$\Updownarrow$$

$$|a+b| \leq |a| + |b|$$



$a_n \to \ell$, $\ell \in \mathbb{R}$

Allora $\{a_n\}$ limitata.

<u>Dim.</u> Prendo $\varepsilon = 1$, applico la def.

trovo $n_1 \in \mathbb{N}$: $\forall n \geq n_1$

$$\ell - 1 \leq a_n \leq \ell + 1$$

Rimangono $a_0\ a_1 \dots a_{n_1}$??

$$M = \max(a_0, a_1 \dots a_{n_1}, \ell+1)$$

$$m = \min(a_0, \dots, a_{n_1}, \ell-1) \implies$$

$\forall n \quad m \leq a_n \leq M$

<u>Permanenza del segno</u>: Se $a_n \to \ell > 0$

$\Rightarrow$ definitivamente $a_n > 0$

<u>Dim.</u> Applico la def. di limite con $\boxed{\varepsilon = \frac{\ell}{2}}$

$\Rightarrow$ definitivamente

$$0 < \frac{\ell}{2} = \ell - \varepsilon \leq a_n \leq \ell + \varepsilon = \frac{3}{2}\ell$$

Limite della somma (caso finito)

$$a_n \to \ell_1 \quad , \quad b_n \to \ell_2 \Rightarrow a_n + b_n \to \ell_1 + \ell_2$$

$$\ell_1, \ell_2 \in \mathbb{R}$$

Dim. Fisso $\varepsilon > 0$. Applico la def. di lim. per $\{a_n\}$, usando $\frac{\varepsilon}{2}$: trovo $n_1 \in \mathbb{N}$: $\forall n \geq n_1$

(I) $\quad \ell_1 - \frac{\varepsilon}{2} \leq a_n \leq \ell_1 + \frac{\varepsilon}{2}$

lo stesso per $\{b_n\}$: trovo $n_2 \in \mathbb{N}$: $\forall n \geq n_2$

(II) $\quad \ell_2 - \varepsilon/2 \leq b_n \leq \ell_2 + \varepsilon/2$

Allora se $n \geq \bar{n} := \max(n_1, n_2)$, valgono (I) e (II)

$$\ell_1 + \ell_2 - \varepsilon \leq a_n + b_n \leq \ell_1 + \ell_2 + \varepsilon$$

HO VERIFICATO LA DEF. DI LIMITE PER $\{a_n + b_n\}$

Limite di succ. monotone.

$$a_{n+1} \geq a_n \quad \forall n \quad \Rightarrow \quad a_n \to \sup_{n\in\mathbb{N}} a_n$$

Dim. $\ell := \sup a_n$. Caso $\ell \in \mathbb{R}$

ℓ VERIFICA:

(1) $\quad a_n \leq \ell \quad \forall n \in \mathbb{N}$

(2) $\quad \forall \varepsilon > 0 \ \exists n_1 \in \mathbb{N} : \quad a_{n_1} > \ell - \varepsilon$

Dato $\varepsilon > 0$ prendo n_1 come in (2); se $n \geq n_1$

$$\underbrace{\ell - \varepsilon \leq a_{n_1}}_{(2)} \underbrace{\leq a_n}_{a_n \text{ CRESCE}} \underbrace{\leq \ell}_{(1)} < \ell + \varepsilon$$

HO VERIFICATO CHE $a_n \to \ell$

Caso $\ell = +\infty$. In questo caso

(2') $\forall c \in \mathbb{R} \quad \exists n_1 : \quad a_{n_1} \geq c$

Dato $c \in \mathbb{R}$, prendo n_1 come in (2'); se $n \geq n_1$

$$\underbrace{a_n \geq a_{n_1}}_{a_n \text{ cresce}} \underbrace{\geq c}_{(2')}$$

$\Rightarrow \quad a_n \to +\infty$

Esempio Def. di e

$$e := \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

posso dare questa def. se dimostro che

$$\boxed{a_n := \left(1 + \frac{1}{n}\right)^n}$$

È CRESCENTE E LIMITATA

$$\boxed{a_n := \left(1+\frac{1}{n}\right)^n}$$

$(a_n)_{n\geq 1}$

Dimostriamo che

$$\boxed{a_n \geq a_{n-1}} \quad \forall n \geq 2$$

$$\left(1+\frac{1}{n}\right)^n \geq \left(1+\frac{1}{n-1}\right)^{n-1} \iff$$

$$\left(\frac{n+1}{n}\right)^n \geq \left(\frac{n}{n-1}\right)^{n-1} \iff$$

$$\frac{\left(\frac{n+1}{n}\right)^n}{\left(\frac{n}{n-1}\right)^n} \geq \left(\frac{n}{n-1}\right)^{-1} \iff$$

$$\left(\frac{n^2-1}{n^2}\right)^n \geq \frac{n-1}{n} \iff \underbrace{\left(1-\frac{1}{n^2}\right)^n \geq 1-\frac{1}{n}}$$

$$\left(1-\frac{1}{n^2}\right)^n \geq 1+n\left(-\frac{1}{n^2}\right) = 1-\frac{1}{n} \leftarrow$$

VERO PER BERNOULLI CON $a = -\frac{1}{n} > -1$

$$b_n := \left(1+\frac{1}{n}\right)^{n+1} \qquad b_n = a_n\left(1+\frac{1}{n}\right)$$

$$b_n \geq a_n$$

VEDIAMO CHE $b_n \leq b_{n-1}$ $\left(\Rightarrow 4 \geq b_n \geq a_n \ \forall n\right)$

$$\left(1+\frac{1}{n}\right)^{n+1} \leq \left(1+\frac{1}{n-1}\right)^{n} \iff$$

$$\left(\frac{n+1}{n}\right)^{n+1} \leq \left(\frac{n}{n-1}\right)^{n} \iff \left(\frac{n+1}{n}\right) \leq \frac{\left(\frac{n}{n-1}\right)^{n}}{\left(\frac{n+1}{n}\right)^{n}} \iff$$

$$\frac{n+1}{n} \leq \left(\frac{n^2}{n^2-1}\right)^{n} \iff$$

$$1+\frac{1}{n} \leq \left(1+\frac{1}{n^2-1}\right) \qquad ??$$

VERA: INFATTI PER BERNOULLI

$$\left(1+\frac{1}{n^2-1}\right)^{n} \geq 1+\frac{n}{n^2-1} \geq 1+\frac{n}{n^2} = 1+\frac{1}{n}$$

ALLA FINE

$$2=a_1 \leq a_n \leq a_{n+1} \leq b_{n+1} \leq b_n \leq b_1 = 4$$

$$a_n \to \ell_1 \qquad \text{(PER MONOTONIA)}$$
$$b_n \to \ell_2$$

$$2 \le \ell_1 \le \ell_2 \le 4$$

ma

$$\left.\begin{array}{ccc} b_n & = & a_n \left(1+\frac{1}{n}\right) \\ \downarrow & & \downarrow \quad \downarrow \quad \downarrow \\ \ell_2 & & \ell_1 = (1+0) \end{array}\right\} \Rightarrow \ell_1 = \ell_2$$

Il limite (comune) di a_n / b_n

si chiama e $(\in [2,4])$

$$\boxed{\left(1+\frac{1}{n}\right)^n \le e \le \left(1+\frac{1}{n}\right)^{n+1} \quad \forall n}$$

per esempio prendendo n=10 si trova 2.59 < e < 2.86 --
prendendo n=100 si trova 2.70 < e < 2.74
prendendo n=1000 viene 2.7169 < e < 2.7197
(tutto questo con la calcolatrice del mio computer ...)
In realtà ci sono metodi numerici più efficaci per calcolare
i valori approssimati di e