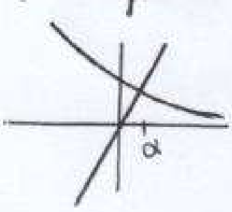
Soluzioni

1. • Il segno del numeratore si ottiene per via grafica, confrontando i grafici delle funzioni e^{-x}, $3x$. Quello del denominatore è ovvio.



$\operatorname{sgn} f(x)$ $\quad - - \times \text{———} 0 - - - -$

$-3 \qquad 0 \;\; \alpha \;\; 1$

Per $x \to -\infty$ $\quad f(x) \sim \frac{e^{-x}}{x} \to +\infty$ $\quad$ fz. non integrabile

per $x \to +\infty$ $\quad f(x) \sim \frac{-3x}{x} \to -3$ $\quad$ "

per $x \to -3$ $\quad f(x) \sim \frac{c}{x+3}$ $\quad$ "

• Studio $F(x)$

C.E. $\quad (-3, +\infty)$

LIM $\quad$ per $x \to -3^+$ $\quad F(x) \to -\infty$ $\quad$ as. verticale

$F(0) = 0$

per $x \to +\infty$ $\quad F(x) \to -\infty$

Guardiamo se ha un asintoto obliquo.

$$\lim_{x\to+\infty} \frac{F(x)}{x} \underset{H.}{=} \lim_{x\to+\infty} \frac{e^{-x}-3x}{x+3} = -3$$

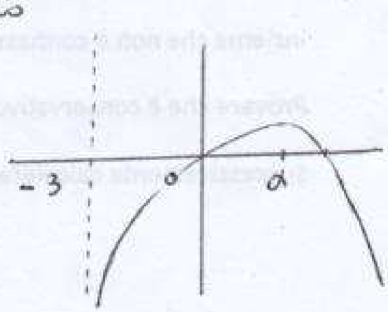
$$\lim_{x\to+\infty} F(x) + 3x = \lim_{x\to+\infty} \int_0^x \left(\frac{e^{-t}-3t}{t+3} + 3\right) dt$$

$$= \lim_{x\to+\infty} \int_0^x \frac{e^{-t}+9}{t+3}\, dt = \lim_{x\to+\infty} \int_0^x \frac{2}{t}\, dt = +\infty$$

Non c'è asintoto.

DRV $\quad F'(x) = \frac{e^{-x}-3x}{x+3}$ $\quad$ segno già studiato

DRV² $\quad F''(x) = -\frac{e^{-x}(x+4)+9}{(x+3)^2} < 0$ nel C.E.



2. $$\int \frac{\sqrt{x^2-4}}{x^2}\, x\,dx = \frac{1}{2} \int \frac{\sqrt{t-4}}{t}\, dt =$$

$x^2 = t$
$2x\,dx = dt$

$\sqrt{t-4} = z$
$t = z^2+4$
$dt = 2z\,dz$

$$= \int \frac{z^2}{z^2+4}\, dz = \int \left(1 - \frac{4}{z^2+4}\right) dz = z - 2 \operatorname{arctg} \frac{z}{2} + c$$

$$= \sqrt{x^2-4} - 2 \operatorname{arctg} \frac{\sqrt{x^2-4}}{2} + c.$$

3.

C.E. $x \in \mathbb{R}$, $y \geq 1$

$y = 1$ sol. costante

$$\int \frac{dy}{y\sqrt{y-1}} = \int x\,dx$$

$$\int \frac{dy}{y\sqrt{y-1}} = \int \frac{2}{t^2+1}\,dt =$$

$\sqrt{y-1} = t$

$y = 1+t^2$

$dy = 2t\,dt$

$$= 2\,\mathrm{arctg}\, t + c = 2\,\mathrm{arctg}\sqrt{y-1} + c$$

$$2\,\mathrm{arctg}\sqrt{y-1} = \frac{x^2-c}{2}$$

$$\downarrow$$

$$\sqrt{y-1} = \mathrm{tg}\,\frac{x^2-c}{4} \qquad \text{se } -\frac{\pi}{2} < \frac{x^2-c}{4} < \frac{\pi}{2}$$

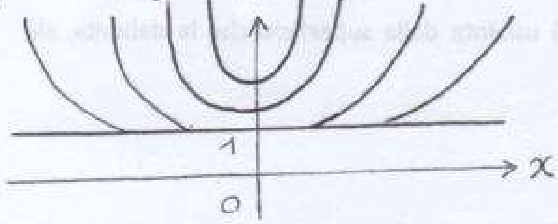
$$\downarrow$$

$$y = 1 + \mathrm{tg}^2\,\frac{x^2-c}{4} \qquad \text{se } 0 \leq \frac{x^2-c}{4} < \frac{\pi}{2} \quad \rightarrow \quad c \leq x^2 < c + 2\pi$$

-2π, 0, c

$c < |x| < \sqrt{c+2\pi}$

$|x| < \sqrt{c+2\pi}$



4. Consideriamo la fz. di variabile reale ottenuta ponendo $\frac{1}{n} = x$ e approssimiamola per $x \to 0$:

$$x^{\alpha}\,\frac{\sqrt[3]{1+x^4} - \sqrt[3]{1-x}}{x^{4/3}} = x^{\alpha - 4/3}\left(1 + \frac{1}{3}x^4 + o(x^4) - 1 + \frac{1}{3}x + o(x)\right) =$$

$$= \frac{1}{3}x^{\alpha - 1/3}$$

Dunque

$$x_n \sim \frac{1}{3n^{\alpha - \frac{1}{3}}} \rightarrow \begin{cases} 0 & \text{se } \alpha > \frac{1}{3} \\ 1/3 & \text{se } \alpha = \frac{1}{3} \\ +\infty & \text{se } \alpha < \frac{1}{3} \end{cases}$$

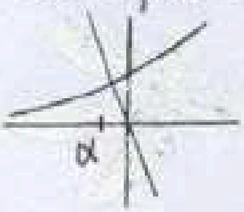
5

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = 0, \quad \lim_{x\to x_0} \frac{g(x)}{h(x)} = 0 \quad \overset{?}{\Rightarrow} \quad \lim_{x\to x_0} \frac{f(x)}{h(x)} = 0$$

$$\frac{f(x)}{h(x)} = \frac{f(x)}{g(x)}\,\frac{g(x)}{h(x)} \rightarrow 0 \cdot 0 = 0.$$

Soluzioni

1\. · Il segno del numeratore si ottiene per via grafica, confrontando i grafici delle funzioni e^x, $3x$. Quello del denominatore è ovvio.



$\operatorname{sgn} f(x)$ $\quad - - - \; 0$ ———— X $- - - - -$

$\qquad\qquad -1 \quad \alpha \quad 0 \qquad\quad 3$

Per $x \to +\infty$ $\quad f(x) \sim \frac{e^x}{-x} \to -\infty$ $\quad$ fz. non integrabile

per $x \to -\infty$ $\quad f(x) \sim \frac{3x}{-x} \to -3$ $\quad$ "

per $x \to 3$ $\quad f(x) \sim \frac{c}{3-x}$ $\quad$ "

· Studio $F(x)$

C.E. $\quad (-\infty, 3)$

LIM $\quad$ per $x \to 3^-$ $\quad F(x) \to +\infty$ $\quad$ as. verticale

$F(0) = 0$

per $x \to -\infty$ $\quad F(x) \to +\infty$

Guardiamo se ha un asintoto obliquo.

$$\lim_{x\to-\infty} \frac{F(x)}{x} = \lim_{x\to-\infty} \frac{e^x+3x}{3-x} = -3$$

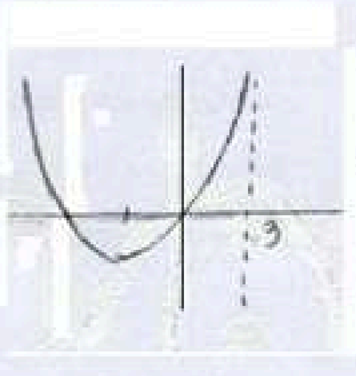
$$\lim_{x\to-\infty} F(x) + 3x = \lim_{x\to-\infty} \int_0^x \left(\frac{e^t+3t}{3-t} + 3\right) dt$$

$$= \lim_{x\to-\infty} \int_0^x \frac{e^t+9}{3-t}\, dt = \lim_{x\to-\infty} \int_0^x -\frac{9}{t}\, dt = +\infty$$

non c'è asintoto.

DRV $\quad F'(x) = \frac{e^x+3x}{3-x}$ $\quad$ segno già studiato

DRV² $\quad F''(x) = \frac{e^x(4-x)+9}{(3-x)^2} > 0$ nel C.E.



$$\int \frac{\sqrt{x^2-9}}{x^2}\, x\,dx = \frac{1}{2} \int \frac{\sqrt{t-9}}{t}\, dt =$$

$x^2 = t$, $\; 2x\,dx = dt$ $\qquad$ $\sqrt{t-9} = z$, $\; t = z^2+9$, $\; dt = 2z\,dz$

$$= \int \frac{z^2}{z^2+9}\, dz = \int \left(1 - \frac{9}{z^2+9}\right) dz = z - 3 \operatorname{arctg} \frac{z}{3} + c$$

$$= \sqrt{x^2-9} - 3 \operatorname{arctg} \frac{\sqrt{x^2-9}}{3} + c.$$

3.

C.E. $x \in \mathbb{R}$, $y \geq 1$

$y = 1$ sol. costante

$$\int \frac{dy}{y\sqrt{y-1}} = \int -x \, dx$$

$$\int \frac{dy}{y\sqrt{y-1}} = \int \frac{2}{t^2+1} \, dt = 2 \operatorname{arctg} t + c = 2 \operatorname{arctg} \sqrt{y-1} + c$$

$\sqrt{y-1} = t$
$y = 1 + t^2$
$dy = 2t \, dt$

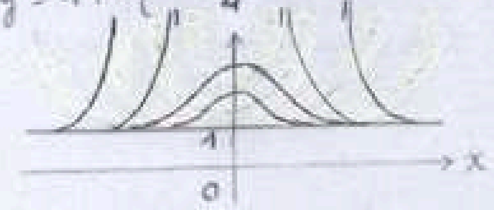
$$2 \operatorname{arctg} \sqrt{y-1} = \frac{-x^2+c}{2}$$

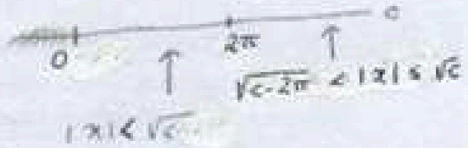
$\downarrow$

$$\sqrt{y-1} = \operatorname{tg} \frac{c-x^2}{4} \quad \text{se } -\frac{\pi}{2} < \frac{c-x^2}{4} < \frac{\pi}{2}$$

$\downarrow$

$$y = 1 + \operatorname{tg}^2 \frac{c-x^2}{4} \quad \text{se } 0 \leq \frac{c-x^2}{4} < \frac{\pi}{2} \longrightarrow c - 2\pi < x^2 \leq c$$





4.

Consideriamo la fz di variabile reale ottenuta ponendo $\frac{1}{n} = x$ e approssimiamola per $x \to 0$

$$x^\alpha \frac{\sqrt[3]{1+x^2} - \sqrt[3]{1+x^4}}{x^{4/3}} = x^{\alpha-4/3}\left(1 + \frac{1}{3}x^2 + o(x^2) - 1 - \frac{1}{3}x^4 + o(x^4)\right) =$$
$$= \frac{1}{3} x^{\alpha+2/3}$$

Dunque

$$x_n \sim \frac{1}{3n^{\alpha+2/3}} \to \begin{cases} 0 & \text{se } \alpha > -2/3 \\ 1/3 & \text{se } \alpha = -2/3 \\ +\infty & \text{se } \alpha < -2/3 \end{cases}$$

5.

$$\lim_{x \to x_0} \frac{g(x)}{f(x) + o(f(x))} = 0 \quad \overset{?}{\Rightarrow} \quad \lim_{x \to x_0} \frac{g(x)}{f(x)}$$

$$\frac{g(x)}{f(x)} = \frac{g(x)}{f + o(f)} \cdot \frac{f + o(f)}{f(x)} \to 0 \cdot 1 = 0.$$