Soluzioni

1\. (a) Deve essere $f = f_x = f_y$, cioè $\begin{cases} 2x = 0 \\ 3y(y-2z) = 0 \\ y^3+z^3-3y^2z+x^2+1=0 \end{cases}$

Si ottengono i punti:

$P_1 = (0,0,-1)$, $P_2 = \left(0, \frac{2}{\sqrt[3]{3}}, \frac{1}{\sqrt[3]{3}}\right)$.

Non sono singolari perché in entrambi i casi è $f_z \neq 0$.

(b) Derivando in forma implicita, per il primo punto si trova

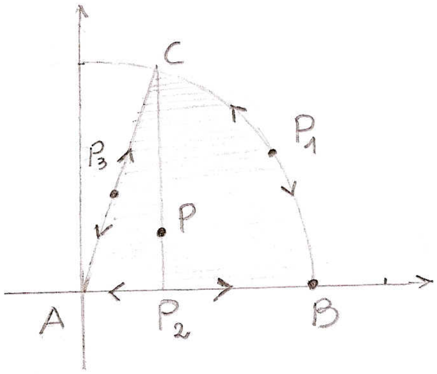
$z(x,y) = -1 - \frac{1}{3}x^2 - y^2 + o(x^2+y^2)$.

La fz. ha in $(0,0)$ un punto di massimo locale e dunque localmente la superficie sta sotto il piano tg. $z=-1$.

Per il secondo punto è $z(x,y) = \frac{1}{\sqrt[3]{3}} + \frac{1}{3\sqrt[3]{3}}x^2 + \frac{\sqrt[3]{3}}{3}\left(y - \frac{2}{\sqrt[3]{3}}\right)^2 + o\left(x^2 + \left(y-\frac{2}{\sqrt[3]{3}}\right)^2\right)$.

La fz. ha in $\left(0, \frac{2}{\sqrt[3]{3}}\right)$ un punto di minimo locale e dunque localmente la superficie sta sopra il piano tg. $z = \frac{1}{\sqrt[3]{3}}$.

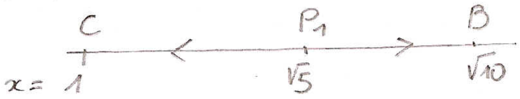
2\.



- Punti stazionari interni: $P=(1,1)$; $f(P)=0$
- Sull'arco di circonferenza:

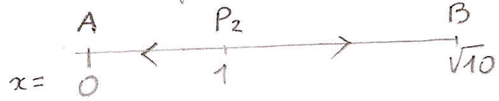
$\varphi(x) = 12-2x-2\sqrt{10-x^2}$

$x \in [1, \sqrt{10}]$



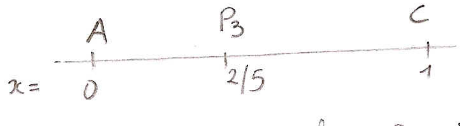
$f(C)=4$, $f(P_1) = 12-4\sqrt{5}$, $f(B) = 12-2\sqrt{10}$

- Sul segmento orizzontale: $\varphi(x) = x^2-2x+2$, $x \in [0,\sqrt{10}]$



$f(A)=2$, $f(P_2)=1$, $f(B)=12-2\sqrt{10}$

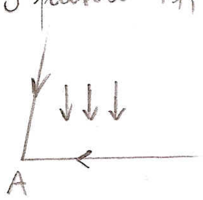
- Sul segmento obliquo: $\varphi(x) = 10x^2-8x+2$, $x\in[0,1]$



$f(A)=2$, $f(P_3) = \frac{2}{5}$, $f(C)=4$.

Dunque: $\min f = 0$ in $P=(1,1)$, $\max f = 12-2\sqrt{10}$ in $B=(\sqrt{10},0)$.

I punti P_1, P_2, P_3 possono essere di minimo locale; A, C di massimo locale.

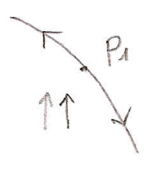


$\frac{\partial f}{\partial y}(A) = -2 < 0$

A pto max loc.

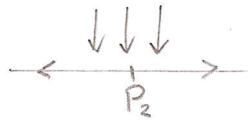
$\frac{\partial f}{\partial y}(C) = 4 > 0$

C pto max loc.



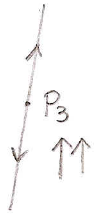
$\frac{\partial f}{\partial y}(P_1) = \sqrt{5} > 0$

P_1 di sella



$\frac{\partial f}{\partial y}(P_2) = -2 < 0$

P_2 di sella



$\frac{\partial f}{\partial y}(P_3) = \frac{2}{5} > 0$

P_3 di sella

3. (a)

Equazione della retta $x = t,\ y = t,\ z = 3t$ orientata nel verso delle x crescenti

Versore lungo cui si deriva : $V = (1, 1, 3)/\sqrt{11}$

$$\frac{\partial f}{\partial V}(1,-1,2) = \nabla f(1,-1,2)\cdot(1,1,3)/\sqrt{11} = -12/\sqrt{11}$$

3. (b)

$$\iint_A \frac{\partial f}{\partial x}\,dx\,dy = \int_{\partial A^+} f\,dy$$

$$\iint_A y\,dx\,dy = \int_0^{2\pi} sen\,\theta\,d\theta \int_1^r r^2\,dr = 0\ ;$$

$$\int_{\partial A^+} xy\,dy = \int_0^{2\pi} 4\,sen t \cos t\cdot 2\cos t\,dt - \int_0^{2\pi} sen t\cos t\cdot\cos t\,dt = 0.$$

4. $y = 0$ è soluzione costante. Posto $z = \sqrt{y} \to y = z^2$, $y' = 2zz'$ si ottiene

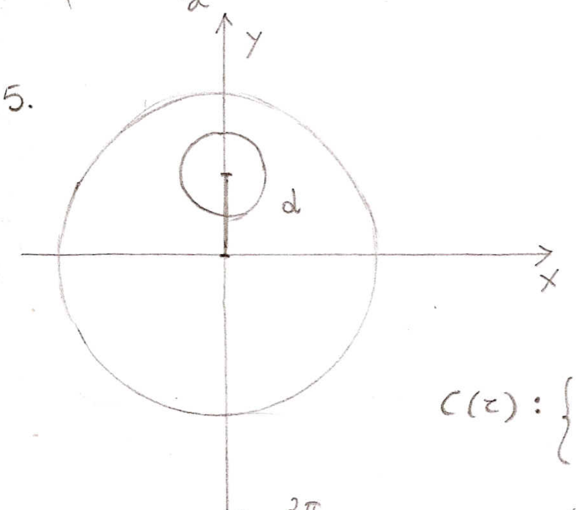
$$z' + \left(\frac{3x}{1+x^2} - \frac{1}{x}\right)z = \frac{1}{(1+x^2)^{3/2}}$$

$a(x) = \frac{3x}{1+x^2} - \frac{1}{x}$, $A(x) = \frac{3}{2}\lg(1+x^2) - \lg x$ (essendo $x > 0$).

$$\left(\frac{z\,(1+x^2)^{3/2}}{x}\right)' = \frac{1}{x} \Rightarrow z = \frac{x(\lg x + c)}{(1+x^2)^{3/2}} \Rightarrow y = \frac{x^2(\lg x + c)^2}{(1+x^2)^3}$$

$x > e^{-c}$

5.



Il baricentro sta sull'asse delle y.

$$\bar y = \frac{\iint_D y\,dx\,dy}{\text{area } D} = \frac{\iint_{C(R)} y\,dx\,dy - \iint_{C(r)} y\,dx\,dy}{\pi(R^2-r^2)} =$$

$$= -\iint_{C(r)} y\,dx\,dy \Big/ \pi(R^2-r^2)$$

$$C(r):\begin{cases} x = \rho\cos\theta \\ y = d + \rho\,sen\,\theta \end{cases} \qquad \begin{array}{l} 0 \le \theta \le 2\pi \\ 0 \le \rho \le r \end{array}$$

$$= \frac{-\int_0^r\int_0^{2\pi} \rho\,(d + \rho\,\cancel{\cos\theta})\,d\rho\,d\theta}{\pi(R^2-r^2)} = \frac{-2\pi d\int_0^r \rho\,d\rho}{\pi(R^2-r^2)} = -\frac{r^2 d}{R^2-r^2}$$