Integrazione indefinita

$$\int \frac{x^2\sqrt{x} + 2x^4 - 3x^3}{x^5}\,dx = \int\left(x^{-\frac{5}{2}} + \frac{2}{x} - \frac{3}{x^2}\right)dx =$$

$$= \int x^{-\frac{5}{2}}\,dx + 2\int\frac{dx}{x} - 3\int\frac{dx}{x^2} = \frac{x^{-\frac{5}{2}+1}}{-\frac{5}{2}+1} + 2\log|x| - 3\,\frac{x^{-1}}{-1} + c$$

$$= -\frac{2}{3}\,\frac{1}{x\sqrt{x}} + 2\log|x| + \frac{3}{x} + c.$$

$$\int \frac{1+2\cos^2 x}{\operatorname{sen}^2 x}\,dx = \int\frac{1+2(1-\operatorname{sen}^2 x)}{\operatorname{sen}^2 x}\,dx = 3\int\frac{dx}{\operatorname{sen}^2 x} - 2\int dx =$$

$$= -3\operatorname{cotg} x - 2x + c$$

$$\int \frac{2x^2-1}{x^2+1}\,dx = \int\left(2 - \frac{3}{1+x^2}\right)dx = 2x - 3\operatorname{arctg} x + c$$

$$\begin{array}{rl|l} & 2x^2 - 1 & x^2+1 \\ & -2x^2 - 2 & 2 \\ \hline // & -3 & \end{array}$$

Se $\int f(x)\,dx = F(x) + c$

$$\int f(ax+b)\,dx = \frac{1}{a}\int f(t)\,dt = \frac{1}{a}F(t) + c = \frac{1}{a}F(ax+b) + c$$

$ax+b = t \quad a\,dx = dt \quad dx = \frac{1}{a}\,dt$

$$\int (ax+b)^\alpha\,dx = \frac{1}{a}\,\frac{(ax+b)^{\alpha+1}}{\alpha+1} + c\,, \quad \alpha \neq -1$$

$$\int \frac{dx}{ax+b} = \frac{1}{a}\log|ax+b| + c$$

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c \qquad \int K^{ax+b}\,dx = \frac{1}{a}\,\frac{K^{ax+b}}{\log K} + c \quad K>0$$

$$\int \operatorname{sen}(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + c$$

$$\int \cos(ax+b)\,dx = \frac{1}{a}\operatorname{sen}(ax+b) + c$$

$$\int \frac{dx}{\cos^2(ax+b)} = \frac{1}{a}\, tg(ax+b)+c \qquad \Bigg| \qquad \int \frac{dx}{sen^2(ax+b)} = -cotg(ax+b)+c$$

$$\int \frac{dx}{1+(ax+b)^2} = \frac{1}{a}\, arctg\,(ax+b)+c$$

$$\int \frac{dx}{\sqrt{1-(ax+b)^2}} = \frac{1}{a}\, arcsen(ax+b)+c$$

$$\int \sqrt{(3x-1)}\, dx = \int (3x-1)^{1/2} dx = \frac{1}{3}\,\frac{(3x-1)^{3/2}}{\frac{3}{2}}+c = \frac{2}{9}(3x-1)\sqrt{3x-1}+c$$

$$\int (2x+3)^3\, dx = \frac{1}{2}\,\frac{(2x+3)^4}{4}+c$$

$$\int \frac{dx}{2x+5} = \frac{1}{2}\log|2x+5|+c = \log\sqrt{|2x+5|}+c$$

$$\int \frac{3x-1}{2x+1}\,dx = \int\left(\frac{3}{2}-\frac{5}{2}\,\frac{1}{2x+1}\right)dx = \frac{3}{2}\int dx - \frac{5}{2}\int\frac{dx}{2x+1} =$$

$$\begin{array}{rr|l} 3x & -1 & 2x+1 \\ \hline -3x & -\frac{3}{2} & \frac{3}{2} \\ \hline // & -\frac{5}{2} & \end{array}$$

$$= \frac{3}{2}x - \frac{5}{2}\,\frac{1}{2}\log|2x+1|+c$$

$$\int \frac{x^3}{2x+1}\,dx = \int\left(\frac{x^2}{2}-\frac{x}{4}+\frac{1}{8}-\frac{1}{8}\,\frac{1}{2x+1}\right)dx =$$

$$\begin{array}{llll|l} x^3 & & & & 2x+1 \\ \hline -x^3 & -\frac{x^2}{2} & & & \frac{x^2}{2}-\frac{x}{4}+\frac{1}{8} \\ \hline // & -\frac{x^2}{2} & & & \\ & \frac{x^2}{2} & +\frac{x}{4} & & \\ \hline & // & \frac{x}{4} & & \\ & & -\frac{x}{4} & -\frac{1}{8} & \\ \hline & & // & -\frac{1}{8} & \end{array}$$

$$= \frac{1}{2}\int x^2 dx - \frac{1}{4}\int x\,dx + \frac{1}{8}\int dx + - \frac{1}{8}\int\frac{dx}{2x+1} =$$

$$= \frac{x^3}{6} - \frac{x^2}{8} + \frac{x}{8} - \frac{1}{16}\log|2x+1|+c$$

$$\int \operatorname{sen}(2x+5)\,dx = -\frac{1}{2}\cos(2x+5) + c$$

$$\int \frac{dx}{\cos^2 \frac{x}{2}} = \frac{1}{\frac{1}{2}}\operatorname{tg}\frac{x}{2} + c = 2\operatorname{tg}\frac{x}{2} + c$$

$$\int \operatorname{sen}^2 x\,dx = \int\left(\frac{1-\cos 2x}{2}\right)dx = \frac{1}{2}x - \frac{1}{2}\,\frac{1}{2}\operatorname{sen} 2x + c = \frac{x - \operatorname{sen}x\cos x}{2} + c$$

$$\cos 2x = \cos^2 x - \operatorname{sen}^2 x = 1 - 2\operatorname{sen}^2 x \quad ; \quad \operatorname{sen}^2 x = \frac{1-\cos 2x}{2}$$

$$\cos 2x = \cos^2 x - \operatorname{sen}^2 x = 2\cos^2 x - 1 \; ; \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$

$$\int \cos^2 x\,dx = \int\left(\frac{1+\cos 2x}{2}\right)dx = \frac{1}{2}x + \frac{1}{2}\,\frac{1}{2}\operatorname{sen} 2x = \frac{x + \operatorname{sen}x\cos x}{2} + c$$

$$\int 2^{3x-1}\,dx = \frac{1}{3}\,\frac{2^{3x-1}}{\log 2} + c$$

$$\int \frac{dx}{3+4x^2} = \int \frac{1}{3}\,\frac{dx}{1+\frac{4}{3}x^2} = \frac{1}{3}\int \frac{dx}{1+\left(\frac{2}{\sqrt{3}}x\right)^2} = \frac{1}{3}\,\frac{1}{\frac{2}{\sqrt{3}}}\operatorname{arctg}\left(\frac{2}{\sqrt{3}}x\right) + c$$

$$= \frac{\sqrt{3}}{6}\operatorname{arctg}\left(\frac{2}{\sqrt{3}}x\right) + c$$

$$\int \frac{dx}{\sqrt{4-9x^2}} = \int \frac{1}{2}\,\frac{dx}{\sqrt{1-\frac{9}{4}x^2}} = \frac{1}{2}\int \frac{dx}{\sqrt{1-\left(\frac{3}{2}x\right)^2}} = \frac{1}{\cancel{2}}\,\frac{\cancel{2}}{3}\operatorname{arcsen}\left(\frac{3}{2}x\right) + c$$

$$= \frac{1}{3}\operatorname{arcsen}\left(\frac{3}{2}x\right) + c$$

$$\int [f(x)]^{\alpha} f'(x)\,dx = \frac{1}{\alpha+1}[f(x)]^{\alpha+1} + c \qquad \alpha \neq -1$$

$$\int x(1+x^2)^2\,dx = \frac{1}{2}\int (1+x^2)^2\, 2x\,dx = \frac{1}{2}\,\frac{1}{3}(1+x^2)^3 + c$$

$$\boxed{D(1+x^2) = 2x} \qquad = \frac{1}{6}(1+x^2)^3 + c$$

$$\int \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int (1-x^2)^{-\frac{1}{2}}(-2x)\,dx$$

$$D(1-x^2) = -2x \qquad = -\frac{1}{\cancel{2}}\,\frac{(1-x^2)^{1/2}}{1/\cancel{2}} + c = -\sqrt{1-x^2} + c$$

$$\int \operatorname{sen} x \cos x \, dx = \frac{1}{2} \operatorname{sen}^2 x + c \qquad \boxed{D \operatorname{sen} x = \cos x}$$

$$\int \operatorname{sen} x \cos x \, dx = -\int \cos x \, (-\operatorname{sen} x) \, dx = -\frac{1}{2}\cos^2 x + c$$

$$\boxed{D \cos x = -\operatorname{sen} x}$$

$$\int \operatorname{sen} x \cos x \, dx = \frac{1}{2}\int \operatorname{sen} 2x \, dx = -\frac{1}{4}\cos 2x + c$$

osserviamo che $-\frac{1}{4}\cos 2x = -\frac{1}{4}(\cos^2 x - \operatorname{sen}^2 x) = -\frac{1}{2}\cos^2 x + \frac{1}{4}$

$$= -\frac{1}{4} + \frac{1}{2}\operatorname{sen}^2 x$$

Inoltre $\frac{1}{2}\operatorname{sen}^2 x = \frac{1}{2} - \frac{1}{2}\cos^2 x$. Pertanto le diverse primitive della funzione $\operatorname{sen} x \cos x$ differiscono fra loro per una costante.

$$\int \frac{f'(x)}{f(x)} \, dx = \log|f(x)| + c$$

$$\int \frac{x}{1+x^2} \, dx = \frac{1}{2}\int \frac{2x}{1+x^2} \, dx = \frac{1}{2}\log(1+x^2) + c \quad \Big| \, D(1+x^2) = 2x$$

$$\int \frac{3x^2-2x}{5-x^2+x^3} \, dx = \log|x^3-x^2+5| + c \quad \boxed{D(x^3-x^2+5) = 3x^2-2x}$$

$$\int \frac{dx}{x\log x} = \int \frac{\frac{1}{x}}{\log x} \, dx = \log|\log x| + c; \quad D\log x = \frac{1}{x}$$

$$\int \operatorname{tg} x \, dx = \int \frac{\operatorname{sen} x}{\cos x} \, dx = -\int \frac{(-\operatorname{sen} x)}{\cos x} \, dx = -\log|\cos x| + c$$

$$\boxed{D \cos x = -\operatorname{sen} x}$$

$$\int \operatorname{cotg} x \, dx = \int \frac{\cos x}{\operatorname{sen} x} \, dx = \log|\operatorname{sen} x| + c; \quad D \operatorname{sen} x = \cos x$$

$$\int \frac{e^x}{2e^x+5} \, dx = \frac{1}{2}\log(2e^x+5) + c \qquad D(2e^x+5) = 2e^x$$

$$\int \frac{\log^3 x}{x}\,dx = \int (\log^3 x)\left(\frac{1}{x}\right)dx = \frac{1}{4}\log^4 x + c$$

$$\int \operatorname{sen}^3 x\,dx = \int \operatorname{sen} x \operatorname{sen}^2 x\,dx = \int \operatorname{sen} x\,(1-\cos^2 x)\,dx$$

$$= \int (\cos^2 x - 1)(-\operatorname{sen} x)\,dx = \int (t^2-1)\,dt = \frac{t^3}{3} - t + c = \frac{\cos^3 x}{3} - \cos x + c$$

$\cos x = t \qquad (-\operatorname{sen} x)dx = dt$

$$\int \cos^3 x\,dx = \int \cos x \cos^2 x\,dx = \int \cos x\,(1-\operatorname{sen}^2 x)\,dx$$

$$= \int (1-t^2)\,dt = t - \frac{t^3}{3} + c = \operatorname{sen} x - \frac{1}{3}\operatorname{sen}^3 x + c$$

$\operatorname{sen} x = t \qquad \cos x\,dx = dt$

$$\int x \cos x^2\,dx = \int (\cos x^2)\,x\,dx = \frac{1}{2}\int \cos t\,dt = \frac{1}{2}\operatorname{sen} t + c =$$
$$= \frac{1}{2}\operatorname{sen} x^2 + c$$

$x^2 = t \quad 2x\,dx = dt\,;\ x\,dx = \frac{1}{2}dt$

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int e^t\,dt = 2e^{\sqrt{x}} + c$$

$\sqrt{x} = t \qquad \frac{1}{2\sqrt{x}}\,dx = dt \qquad \frac{dx}{\sqrt{x}} = 2\,dt$

$$\int \frac{dx}{x+\sqrt{x}} = \int \frac{1}{\sqrt{x}+1}\,\frac{dx}{\sqrt{x}} = 2\int \frac{dt}{1+t} = 2\log|1+t| + c$$
$$= 2\log(1+\sqrt{x}) + c$$

$$\int \frac{x\,dx}{1+x^4} = \int \frac{1}{2}\,\frac{2x\,dx}{1+(x^2)^2} = \frac{1}{2}\int \frac{dt}{1+t^2} = \frac{1}{2}\operatorname{arctg} t + c$$

$x^2 = t \quad 2x\,dx = dt \qquad = \frac{1}{2}\operatorname{arctg} x^2 + c$

Integrazione per parti

$$\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx$$

$$\int x^n \cos x\,dx \qquad \int x^n \operatorname{sen} x\,dx \qquad \int x^n e^x\,dx \qquad n \in \mathbb{N}$$

Si sceglie come fattore finito $f(x) = x^n$ e come fattore derivato rispettivamente $g'(x) = \cos x$ $g'(x) = \operatorname{sen} x$ $g'(x) = e^x$

$$\int (x^2 + x) \operatorname{sen} x \, dx = -(x^2+x)\cos x + \int (2x+1) \cos x \, dx$$

$f(x) = x^2 + x$ $\quad g(x) = -\cos x$

$f'(x) = 2x+1$ $\quad g'(x) = \operatorname{sen} x$

$f(x) = x$ $\quad g(x) = \operatorname{sen} x$

$f'(x) = 1$ $\quad g'(x) = \cos x$

$$= -(x^2+x)\cos x + 2\int x \cos x \, dx + \int \cos x \, dx$$

$$= -(x^2+x)\cos x + 2\left[x \operatorname{sen} x - \int \operatorname{sen} x \, dx\right] + \operatorname{sen} x$$

$$= -(x^2+x)\cos x + (2x+1)\operatorname{sen} x + 2\cos x + c$$

$$= (2 - x - x^2)\cos x + (2x+1)\operatorname{sen} x + c.$$

$$\int x^3 e^x \, dx = x^3 e^x - 3\int x^2 e^x \, dx =$$

$$= x^3 e^x - 3\left[x^2 e^x - 2\int x e^x \, dx\right] = (x^3 - 3x^2) e^x + 6\int x e^x \, dx$$

$$= (x^3 - 3x^2) e^x + 6\left[x e^x - \int e^x \, dx\right] = (x^3 - 3x^2 + 6x - 6) e^x + c$$

$\int P(x)\, e^x = Q(x)\, e^x$ con $Q(x)$ polinomio dello stesso grado di $P(x)$

$$\int x^3 e^x dx = (ax^3 + bx^2 + cx + d) e^x$$

È possibile determinare a, b, c, d usando il principio di identità dei polinomi.

Deve risultare

$$D\left[(ax^3 + bx^2 + cx + d) e^x\right] = x^3 e^x$$

$$(ax^3 + bx^2 + cx + d + 3ax^2 + 2bx + c) e^x = x^3 e^x$$

$$\left[ax^3 + (b+3a)x^2 + (c+2b)x + (c+d)\right] \cancel{e^x} = x^3 \cancel{e^x}$$

$$\begin{cases} a = 1 \\ b + 3a = 0 \\ c + 2b = 0 \\ d + c = 0 \end{cases} \qquad \begin{cases} a = 1 \\ b = -3a = -3 \\ c = -2b = 6 \\ d = -c = -6 \end{cases}$$

Quindi $\int x^3 e^x dx =$

$= (x^3 - 3x^2 + 6x - 6)\, e^x + c$

che coincide con la primitiva già determinata.

$\int x^n \log x \, dx \qquad \int x^n \operatorname{arctg} x \, dx \qquad \int x^n \arcsin x \, dx$

$n \in \mathbb{N}_0 \qquad \int x^n \arccos x \, dx$

x^n è da considerarsi il fattore derivato [se $n=0$ il fattore derivato è 1]

$$\int \operatorname{arctg} x \, dx = x \operatorname{arctg} x - \int \frac{x}{1+x^2} dx = x \operatorname{arctg} x - \frac{1}{2}\log(1+x^2) + c$$

$f(x) = \operatorname{arctg} x \qquad g(x) = x$

$f'(x) = \dfrac{1}{1+x^2} \qquad g'(x) = 1$

$f(x) = \arcsin x \qquad g(x) = x$

$f'(x) = \dfrac{1}{\sqrt{1-x^2}} \qquad g'(x) = 1$

$$\int \arcsin x \, dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx = x \arcsin x + \sqrt{1-x^2} + c$$

$$\int \log x \, dx = x \log x - \int \cancel{x} \frac{1}{\cancel{x}} dx = x(\log x - 1) + c$$

$f(x) = \log x \qquad g(x) = x$

$f'(x) = \dfrac{1}{x} \qquad g'(x) = 1$

$$\int x^3 \log^2 x \, dx = \frac{1}{4} x^4 \log^2 x - \int \frac{x^{\cancel{4}\,3}}{4_2} \frac{\cancel{2}}{\cancel{x}} \log x \, dx =$$

$f(x) = \log^2 x \qquad g(x) = \dfrac{x^4}{4}$

$f'(x) = \dfrac{2}{x}\log x \qquad g'(x) = x^3$

$f(x) = \log(x) \qquad g(x) = \dfrac{x^4}{4}$

$f'(x) = \dfrac{1}{x} \qquad g'(x) = x^3$

$$= \frac{1}{4} x^4 \log^2 x - \frac{1}{2} \int x^3 \log x \, dx =$$

$$= \frac{1}{4} x^4 \log^2 x - \frac{1}{2}\left[\frac{x^4}{4}\log x - \int \frac{x^{\cancel{4}\,3}}{4} \frac{1}{\cancel{x}} dx\right] =$$

$$= \frac{x^4}{4}\log^2 x - \frac{x^4}{8}\log x + \frac{1}{8}\frac{x^4}{4} + c = \frac{x^4}{4}\left(\log^2 x - \frac{1}{2}\log x + \frac{1}{8}\right) + c$$

Integrazione delle funzioni razionali fratte

$$\int \frac{x^3 + 3x + 3}{x^2 + 4} dx = \int \left(x + \frac{3-x}{x^2+4}\right) dx =$$

$$\begin{array}{rrr|l} x^3 & +3x & +3 & x^2+4 \\ -x^3 & -4x & & \hline x \\ \hline & -x & +3 & \end{array}$$

$$\int x\,dx + 3\int\frac{dx}{x^2+4} - \int\frac{x}{x^2+4}\,dx = \frac{x^2}{2} + \frac{3}{4}\int\frac{dx}{1+\left(\frac{x}{2}\right)^2} - \frac{1}{2}\int\frac{2x}{x^2+4}\,dx$$

$$= \frac{x^2}{2} + \frac{3}{\cancel{4}\,2}\,\overset{1}{\cancel{2}}\,\mathrm{arctg}\left(\frac{x}{2}\right) - \frac{1}{2}\log(x^2+4) + c = \frac{1}{2}x^2 + \frac{3}{2}\,\mathrm{arctg}\,\frac{x}{2} - \frac{1}{2}\log(x^2+4) + c$$

$$\int\frac{2x-1}{x^2-5x+6}\,dx$$

$$\frac{2x-1}{x^2-5x+6} = \frac{2x-1}{(x-2)(x-3)} = \frac{A}{(x-2)} + \frac{B}{(x-3)};$$

$$2x-1 = A(x-3) + B(x-2)$$

$x = 3$

$2\cdot 3 - 1 = A(3-3) + B(3-2)$

$B = 6 - 1 = 5$

$x = 2$

$2\cdot 2 - 1 = A(2-3) + B(2-2)$

$-A = 3 \qquad A = -3$

$$\frac{2x-1}{(x-2)(x-3)} = \frac{-3}{x-2} + \frac{5}{x-3}$$

$$2x-1 = (A+B)x - 3A - 2B$$

$$\begin{cases} A+B = 2 \\ -2A-3B = -1 \end{cases}$$

$$\begin{cases} A+B = 2 \\ 3A+2B = 1 \end{cases} \quad \begin{cases} 2A+2B = 4 \\ 3A+2B = 1 \end{cases}$$

$$\begin{cases} A = -3 \\ B = 2 - A = 5 \end{cases}$$

$$\frac{2x-1}{(x-2)(x-3)} = \frac{5}{x-3} - \frac{3}{x-2}$$

$$\int\frac{2x-1}{x^2-5x+6}\,dx = \int\left(\frac{5}{x-3} - \frac{3}{x-2}\right)dx = 5\log|x-3| - 3\log|x-2| + c$$

$$\int\frac{x+1}{x^2-10x+25}\,dx = \int\frac{dx}{x-5} + 6\int\frac{dx}{(x-5)^2} = \log|x-5| - \frac{6}{(x-5)} + c$$

$$\frac{x+1}{x^2-10x+25} = \frac{x+1}{(x-5)^2} = \frac{A}{(x-5)} + \frac{B}{(x-5)^2} = \frac{A(x-5)+B}{(x-5)^2}$$

$x + 1 = Ax + B - 5A$

$$\begin{cases} A = 1 \\ B - 5A = 1 \end{cases}$$

$$\begin{cases} A = 1 \\ B = 1 + 5A = 6 \end{cases}$$

$x + 1 = A(x-5) + B$

$x = 5 \qquad 5 + 1 = A(5-5) + B, \quad B = 6$

$x = 0 \qquad B - 5A = 1 \qquad \cancel{B = 1 + 5A}$

$A = \frac{B-1}{5} \qquad A = (6-1)/5 = 1$

$$\frac{x+1}{x^2-10x+25} = \frac{1}{x-5} + \frac{6}{(x-5)^2}$$

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